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TIME PERIODIC SOLUTIONS TO A HEAT EQUATION WITH LINEAR FORCING AND BOUNDARY CONDITIONS

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Abstract. In this study, we consider a heat equation with a variable-coefficient linear forcing term and a time-periodic boundary condition. Under some decay and smoothness assumptions on the coefficient, we establish the existence and uniqueness of a time-periodic solution satisfying the boundary condition. Furthermore, possible connections to the closed boundary layer equations were discussed. The difficulty with a perturbed leading order coefficient is demonstrated by a simple example.

1. Introduction

In this study, the existence and uniqueness of a time periodic solution for the following 1D heat equation with a linear forcing term were considered:

$$\begin{cases} Q_t - Q_{xx} = g(t,x)Q, \\ Q(t,0) = h(t), \\ Q(t,\infty) = 0, \end{cases} \tag{1.1}$$

where $(t,x) \in (\mathbb{R}/(2\pi\mathbb{Z})) \times [0,\infty)$ and g, h are the given real-valued smooth functions that are 2π-periodic in t. Furthermore, it is assumed that h has a mean of zero. There are several studies on the solvability of equations of the form (1.1), assuming that $h \equiv 0$. In this case, one can solve the equation under extremely general assumptions regarding the coefficient; see [3, 5, 6, 9, 12, 13] and references therein. Several abstract existence theorems for time-periodic differential equations can be found in [2, 4, 14].

However, the authors are not aware of any previous studies on the time-periodic solutions when h is non-trivial, unless the right-hand side of the equation for Q takes a special form. Specifically, the authors are interested in the case where h is nontrivial and g is small, such that (presumably) the solution approaches that of the homogeneous case where $g \equiv 0$. This interest in (1.1)

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stems from a boundary layer equation for the closed streamlines (see Section 4 below), where it is essential to have a nontrivial boundary condition on $\{x = 0\}$. Notably, when g vanishes, by taking the Fourier series in t, the following can be arrived at:

$$Q^* = \sum_{n\in\mathbb{Z}} e^{-\sqrt{\frac{n}{2}}(1+i)x} e^{int} h_n. \tag{1.2}$$

This is the unique smooth time periodic solution to (1.1), where h_n is the n-th Fourier coefficient of h. Unfortunately, it becomes incredibly difficult to solve (1.1) when g is nontrivial and/or the coefficients of Q_t or Q_{xx} are perturbed from constants; see the formal computations in Section 5 for a variable leading-order coefficient case below. The primary result of this study shows that when g is analytic in t and decays extremely fast in x, there exists a solution to (1.1) that is close to (1.2).

2. Main result

The primary result of the problem (1.1) is stated below:

Theorem 2.1. *For any $\rho > 4$, there exists a constant $\varepsilon_0 > 0$ such that the following holds true. Assume that the Fourier series $\{g_n\}_{n\in\mathbb{Z}}$ of g in t satisfies*

$$|g_n(x)| \le \varepsilon e^{-\rho|n|-x^2}$$

for certain $0 < \varepsilon < \varepsilon_0$ and the Fourier series of h satisfies

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|e^{|n|}|h_n| \le C.$$

Then there exists the unique solution Q to (1.1) that is C^∞–smooth in t, x and decaying exponentially fast as $x \to \infty$.

When proving the theorem, it shall be demonstrated that within a particular class of smooth functions equipped with a special norm $\|\cdot\|$, the solution is unique and is given by a perturbation of the homogeneous solution Q^* from (1.2). Furthermore, the solution satisfies $\|Q - Q^*\| = O(\varepsilon)$.

Initially, it can be shown that at least in a few cases, the equation with a variable coefficient in the leading order term can be reduced to the form (1.1). For this purpose, under the same boundary conditions, consider

$$Q_t - \psi(t)\phi(x)Q_{xx} = g(t,x)Q, \tag{2.1}$$

where ψ is periodic by 2π and ϕ is defined as $[0,\infty)$. It is assumed that ψ and ϕ are strictly positive, infinitely differentiable, and ϕ decays exponentially fast as $x \to \infty$. In this case, the variables in both t and x can be changed such that the coefficient of the second-order term is normalized. First, the change in variables is applied (assuming that $\phi > 0$)

$$\phi^{\frac{1}{2}}(x)\partial_x = \partial_{x'}, \qquad x'(x) = \int_0^x \frac{\mathrm{d}y}{\phi^{\frac{1}{2}}(y)},$$

such that

$$\partial_{x'x'} = \phi^{\frac{1}{2}}(x)\partial_x(\phi^{\frac{1}{2}}(x)\partial_x) = \phi(x)\partial_{xx} + \frac{\phi'(x)}{2\phi^{\frac{1}{2}}(x)}\partial_{x'},$$

and with $f := \frac{\phi'(x)}{2\phi^{\frac{1}{2}}(x)}$, (2.1) changes to

$$Q_t - \psi(t)(Q_{x'x'} - fQ_{x'}) = gQ. \tag{2.2}$$

Next, the change in variables

$$\frac{1}{\psi(t)}\partial_t = \partial_{t'}$$

converts (2.2) into

$$Q_{t'} - (Q_{x'x'} - fQ_{x'}) = \frac{g}{\psi}Q. \tag{2.3}$$

Here, ψ can be normalized such that $t' \in [0, 2\pi]$ for $t \in [0, 2\pi]$. Finally, the following is defined

$$\widetilde{Q}(t', x') = \exp\left(-\frac{1}{2}\int_0^{x'} f\right) Q(t', x')$$

such that this equation is derived

$$\widetilde{Q}_{t'} - \widetilde{Q}_{x'x'} = \left(-\frac{f_{x'}}{2} + \frac{f^2}{4} + \frac{g}{\psi}\right)\widetilde{Q}, \tag{2.4}$$

which is in the form of (1.1).

3. Proof of the main result

3.1. Expansion

For convenience, let g be replaced by εg (so that now g is in the order of 1), and consider the following expansion:

$$Q = \sum_{k\geq 0} \varepsilon^k Q^{(k)}, \tag{3.1}$$

where $Q^{(k)}$ is defined by the solution of

$$\begin{cases} Q_t^{(k)} - Q_{xx}^{(k)} = \frac{1}{2}g(t)\phi(x)Q^{(k-1)}, \\ Q^{(k)}(t,0) = Q^{(k)}(t,\infty) = 0 \end{cases} \tag{3.2}$$

for all $k \geq 1$. In the case of $k = 0$, $Q^{(0)}$ is considered to be the solution of

$$\begin{cases} Q_t^{(0)} - Q_{xx}^{(0)} = 0, \\ Q^{(0)}(t,0) = h(t), \\ Q^{(0)}(t,\infty) = 0. \end{cases} \tag{3.3}$$

This is simply (1.2) or $Q^{(0)} = Q^*$. A norm $\|\cdot\|$ has been constructed such that

$$\|Q^{(k)}\| \leq C\|Q^{(k-1)}\|$$

holds for an absolute constant $C > 0$. Once this is proven, Q defined by (3.1) satisfies

$$\|Q\| \leq \sum_{k\geq 0} \varepsilon^k \|Q^{(k)}\| \leq \|Q^{(0)}\| \sum_{k\geq 0} (C\varepsilon)^k < \infty,$$

as long as $\varepsilon < C^{-1}$. In the next section, the norm

$$\|F\| := \sup_{n\in\mathbb{Z}\backslash\{0\}} \sup_{x\in[0,\infty)} \left(e^{|n|+\sqrt{\frac{|n|}{2}}x} |f_n''(x)| + |n| e^{|n|+\sqrt{\frac{|n|}{2}}x} |f_n(x)| \right) \tag{3.4}$$

is proven to be suitable for the application, where

$$F = \sum_{n\in\mathbb{Z}\backslash\{0\}} f_n(x) e^{int}$$

is the Fourier series expansion of F in t. Notably, the solution $Q^{(0)}$ of (3.2) satisfies $\|Q^{(0)}\| < \infty$ owing to the assumption regarding h in Theorem 2.1.

3.2. An inhomogeneous problem

The following inhomogeneous problem is considered for F, where Q is regarded as a given function:

$$\begin{cases} F_t = F_{xx} + g(t,x)Q, \\ F(t,0) = F(t,\infty) = 0. \end{cases} \tag{3.5}$$

From this point, $g(t,x)$ is assumed to satisfy the condition in Theorem 2.1 with $\varepsilon = 1$. Q and F are represented by a Fourier series in t:

$$F(t,x) = \sum_{n\in\mathbb{Z}} f_n(x) e^{int}, \qquad Q(t,x) = \sum_{n\in\mathbb{Z}} q_n(x) e^{int},$$

with $f_0 = q_0 = 0$. Next, by expanding both sides of (3.5) in the Fourier series, the ordinary differential equation (ODE) satisfied by f_n is obtained:

$$\begin{cases} in f_n(x) = f_n''(x) + \sum_{k\in\mathbb{Z}} g_{n-k}(x) q_k(x), \\ f_n(0) = f_n(\infty) = 0. \end{cases} \tag{3.6}$$

The primary result regarding (3.5) is stated below:

Proposition 3.1. *Assume that*

$$e^{|n|+\sqrt{\frac{|n|}{2}}x} |q_n''(x)| + |n| e^{|n|+\sqrt{\frac{|n|}{2}}x} |q_n(x)| \leq 1 \tag{3.7}$$

for all $x \geq 0$ and $n \in \mathbb{Z}\backslash\{0\}$. Then, for a constant $C > 0$ depending only on $\rho > 4$, such that

$$e^{|n|+\sqrt{\frac{|n|}{2}}x} |f_n''(x)| + |n| e^{|n|+\sqrt{\frac{|n|}{2}}x} |f_n(x)| \leq C. \tag{3.8}$$

Proof. Only the case $n > 0$ is considered. The case $n < 0$ is similar and will be briefly discussed below. Note a particular solution of the ODE:

$$f''(x) = inf(x) + g(x) \tag{3.9}$$

is given by

$$f(x) = \frac{1}{2\alpha_n} \int_0^x (e^{\alpha_n(x-s)} + e^{-\alpha_n(x-s)}) G(s) ds, \tag{3.10}$$

where $\alpha_n = \sqrt{in} := \sqrt{\frac{n}{2}}(1+i)$ and

$$G(x) = -\alpha_n \int_x^\infty g(s) ds.$$

Further,

$$f'(x) = \frac{1}{\alpha_n} G(x) + \frac{1}{2} \int_0^x (e^{\alpha_n(x-s)} - e^{-\alpha_n(x-s)}) G(s) ds,$$

from which (3.9) followed by differentiating once again. Therefore, it can be observed that the unique solution to (3.6) is given by

$$\begin{aligned} f_n(x) = {} & \frac{e^{-\alpha_n x}}{2\alpha_n} \left(\int_0^\infty e^{-\alpha_n s} G_n(s) ds + \int_0^x e^{\alpha_n s} G_n(s) ds \right) \\ & - \frac{e^{\alpha_n x}}{2\alpha_n} \int_x^\infty e^{-\alpha_n s} G_n(s) ds, \end{aligned} \tag{3.11}$$

with

$$G_n(x) = \alpha_n \int_x^\infty \psi_n(s) ds, \qquad \psi_n(s) = \sum_{k \in \mathbb{Z} \setminus \{0\}} g_{n-k}(s) q_k(s). \tag{3.12}$$

First, ψ_n is estimated from the assumption regarding g,

$$|\psi_n(x)| \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} C|k|^{-1} e^{-\rho|n-k|} e^{-|k| - \sqrt{\frac{|k|}{2}} x} e^{-x^2}.$$

When $|k-n| > n/2$, the following is estimated:

$$\sum_{k:|k-n|>n/2} C e^{-\rho|n-k|} |k|^{-1} e^{-|k| - \sqrt{\frac{|k|}{2}} x} \leq C e^{-\frac{\rho}{2}|n|} \leq C|n|^{-1} e^{-2|n|},$$

using the assumption $\rho > 4$. However, when $n \leq k < 3n/2$,

$$\sum_{k:n \leq k < 3n/2} C|k|^{-1} e^{-\rho|n-k|} e^{-|k|} e^{-\sqrt{\frac{k}{2}} x} \leq C|n|^{-1} e^{-|n|} e^{-\sqrt{\frac{n}{2}} x},$$

as $\sqrt{\frac{k}{2}} \geq \sqrt{\frac{n}{2}}$ in this case. Finally, in the case where $n/2 \leq k < n$, using $\rho > 4$ again,

$$\sum_{k:n/2 \leq k < n} C|k|^{-1} e^{-\rho|n-k|} e^{-|k|} e^{-\sqrt{\frac{k}{2}} x} \leq C|n|^{-1} e^{-2|n|} \sum_{n/2 \leq k < n} e^{-|k|} \leq C|n|^{-1} e^{-2|n|}.$$

Combining the terms, the following estimate is derived:

$$|\psi_n(x)| \leq C|n|^{-1}(e^{-|n|}e^{-\sqrt{\frac{n}{2}}x} + e^{-2|n|})e^{-x^2}.$$

Subsequently, the equation can be bound using $s \geq x$

$$\begin{aligned} e^{|n|}e^{\sqrt{\frac{n}{2}}x}|G_n(x)| &\leq |\alpha_n| \int_x^\infty |\psi_n(s)|e^{|n|}e^{\sqrt{\frac{n}{2}}s}ds \\ &\leq C|n|^{-1}|\alpha_n| \int_0^\infty e^{-s^2}(1 + e^{\sqrt{\frac{n}{2}}s - |n|})ds \end{aligned}$$

and using

$$s^2 - \sqrt{\frac{n}{2}}s + |n| \geq \frac{7}{8}s^2$$

gives

$$\|e^{|n|}e^{\sqrt{\frac{n}{2}}x}|G_n(x)|\|_{L^\infty} \leq C|n|^{-1}|\alpha_n|.$$

Now, from the estimate

$$e^{|n|}e^{\sqrt{\frac{n}{2}}x}|G_n(x)| \leq |\alpha_n| \int_x^\infty |\psi_n(s)|e^{|n|}e^{\sqrt{\frac{n}{2}}s}ds,$$

using Fubini's theorem, the following equation is derived:

$$\|e^{|n|}e^{\sqrt{\frac{n}{2}}x}|G_n(x)|\|_{L^1} \leq |\alpha_n| \int_0^\infty \int_x^\infty |\psi_n(s)|e^{|n|}e^{\sqrt{\frac{n}{2}}s}dsdx \leq C|\alpha_n||n|^{-1}.$$

Further, the aforementioned estimates regarding G_n to bound $f_n(x)$ are applied, and from the solution formula of f_n, the following is estimated:

$$|e^{\sqrt{\frac{n}{2}}x}f_n(x)| \leq \frac{C}{|\alpha_n|}\int_0^\infty |e^{\alpha_n s}G_n(s)|ds + \frac{C}{|\alpha_n|}|e^{2\alpha_n x}\int_x^\infty e^{-\alpha_n s}G_n(s)ds|.$$

Using

$$\int_0^\infty |e^{\alpha_n s}G_n(s)|ds \leq C|\alpha_n||n|^{-1}e^{-|n|}$$

and

$$|e^{2\alpha_n x}\int_x^\infty e^{-\alpha_n s}G_n(s)ds| \leq Ce^{-|n|}|n|^{-1}|\alpha_n|\int_x^\infty e^{-2\sqrt{\frac{n}{2}}(s-x)}ds \leq Ce^{-|n|}|n|^{-1},$$

the following is obtained:

$$|n|e^{|n|}|f_n(x)e^{\sqrt{\frac{n}{2}}x}| \leq C.$$

Next, using (3.6) with the previous estimate, the following equation can be derived for f_n'':

$$e^{|n|}|f_n''(x)e^{\sqrt{\frac{n}{2}}x}| \leq C.$$

Thus far, the case where $n > 0$ has been considered. If $n < 0$, let $m = -n > 0$; the equation is derived in the same manner by considering $\alpha_m = \sqrt{i(-m)} =$

$\sqrt{\frac{m}{2}}(1-i)$. Next, the analogous inequalities and estimates are obtained using a straightforward technique. This completes the proof of the proposition. □

Using the aforementioned proposition, the existence of a $C^\infty_{t,x}$ smooth solution to (1.1) is immediate. Given the estimates for $|f_n(x)|$ and $|f_n''(x)|$, the estimates for the fourth derivative of $f_n(x)$ and further can be obtained by differentiating the equation for f_n in x. However, the smoothness in x implies that the solution is smooth in t. This shows that the solution obtained by the series expansion is infinitely differentiable in t and x, as claimed. To conclude Theorem 2.1, only the uniqueness of the smooth solution must be obtained.

3.3. Uniqueness

In the aforementioned equation, an inhomogeneous estimate was obtained

$$\|F\| \le C\|Q\|$$

for the solution of

$$F_t - F_{xx} = g(t,x)Q,$$

some absolute constant $C > 0$, with $\|\cdot\|$ defined as in (3.4) and g of size $O(1)$. In this estimate, it is crucial that F satisfies the boundary condition $F(t,0) = F(t,\infty) = 0$.

Now, two solutions are assumed: Q_1 and Q_2, with finite $\|\cdot\|$–norm, satisfying

$$(Q_i)_t - (Q_i)_{xx} = \varepsilon g(t,x)Q_i,$$

with the same boundary conditions on $x = 0$ and $x = \infty$. Denoting the difference by $F = Q_1 - Q_2$, the following is obtained: $F(t,0) = F(t,\infty) = 0$. By applying the previous inhomogeneous estimate using the equation for F, the following equation is obtained:

$$\|F\| \le C\varepsilon\|F\|.$$

When $\varepsilon > 0$ is smaller than C^{-1}, this implies that $\|F\| = 0$, or $Q_1 = Q_2$. This completes the proof of uniqueness.

4. Closed boundary layer equations

The issue (1.1) arises in the theory of the boundary layer of incompressible recirculating flows. This is briefly explained below. Let us suppose a stationary viscous incompressible fluid flow $\mathbf{u} = (u(x,y), v(x,y))$ in a bounded closed domain in $\mathbb{R}^2$. It is assumed that the flow circulates the wall boundary with a velocity $\mathbf{u_w} \neq \mathbf{0}$. Therefore, this becomes a moving boundary issue that is different from the conventional boundary layer where $\mathbf{u_w} = \mathbf{0}$ (see Figure 1).

Now, the "Prandtl–Batchelor theory" in [1, 11] provides the asymptotic of the incompressible Navier–Stokes flows for large Reynolds numbers (R). This theory states that as $R \to \infty$, the vorticity $\omega = v_x - u_y$ becomes a uniform constant in the entire domain, except for the (thin) boundary layer near the

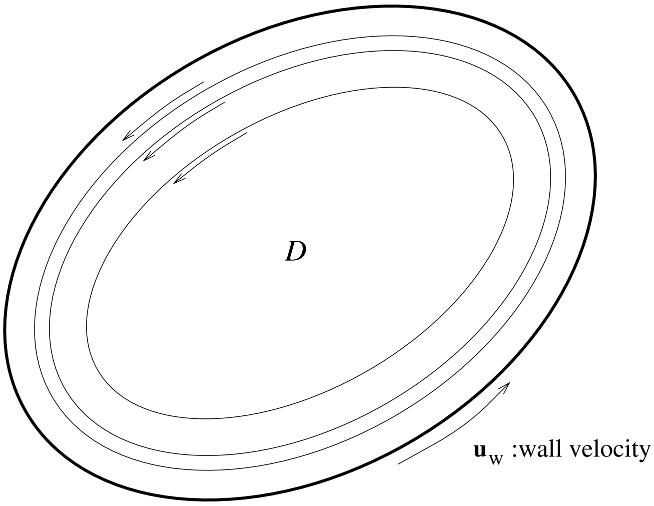

FIGURE 1. Circulating flow in a region D with nonzero wall velocity [7]

wall. In fact, the entire flow is then determined by a complicated interaction between the outer (almost inviscid) flow and inner (boundary layer) flow. (For details, see [7, 8].) Therefore, the space-periodic boundary layer flow is crucial for understanding the correct asymptotics and final state of the entire flow.

The steady two-dimensional boundary layer equation [10] is considered in the s, η-coordinates where, s is the arc length along the wall, and η is the distance normal to ∂D, measured from the wall toward the interior times $\sqrt{R}$. Let the corresponding velocity coordinates be u_s, u_η, that is, u_η is the component in the direction of the inward normal times $\sqrt{R}$. Then, the equations become

$$u_s \frac{\partial u_s}{\partial s} + u_\eta \frac{\partial u_s}{\partial \eta} - q_e(s) q_e'(s) - \frac{\partial^2 u_s}{\partial \eta^2} = 0, \quad \frac{\partial u_s}{\partial s} + \frac{\partial u_\eta}{\partial \eta} = 0. \tag{4.1}$$

Here, $q_e(s)$ denotes the limit inviscid flow speed at the outer edge of the boundary layer. These equations hold true for $0 \leq s < L,\ 0 \leq \eta < \infty$, where L is the perimeter of D. Thereafter the equations can be rewritten as (4.1) in the von Mises variables $s, \bar{\psi}$, where $\bar{\psi}$ is the scaled stream function defined by $-\nabla^\perp \psi = \mathbf{u}$ and $\bar{\psi} = \sqrt{R}\psi$. Setting $u_s = q(s, \bar{\psi})$, (4.1) is reduced to a single equation, in the form of

$$\frac{\partial q^2}{\partial s} - \frac{d q_e^2}{ds} - q \frac{\partial^2 q^2}{\partial \bar{\psi}^2} = 0. \tag{4.2}$$

The proper conditions for $q(s, \bar{\psi})$ in (4.2) are given by

$$q(s + L, \bar{\psi}) = q(s, \bar{\psi}), \quad q(s, 0) = q_w(s), \quad q(s, \infty) = q_e(s), \tag{4.3}$$

where $q_w(s)$ (the wall speed) is the same as u_s on ∂D.

For the circular eddy, the limit vorticity ω_0 can be evaluated by integrating (4.2) twice on $\bar{\psi}$. This is owing to the vanishing pressure term: $-q_e q_e'$ in (4.2). Consequently, the boundary condition at $\psi = \infty$ in (4.3) is specified, and the following problem may be posed for $Q := q^2 - q_e^2$:

$$Q_t - qQ_{xx} = 0, \quad 0 \le t \le 2\pi,\ 0 \le x \le \infty \tag{4.4}$$

$$Q(t+2\pi, x) = Q(t,x), \quad Q(t,0) = q_w^2 - q_e^2, \quad Q(t,\infty) = 0, \tag{4.5}$$

where the notation $t = s, x = \bar{\psi}$ to identify the variables in the previous sections was introduced.

It is a challenge to establish the existence of the solution of the equation owing to the periodicity requirement for Q in t (see Section 5). Therefore, the appropriate approximate models pertaining to the characteristics of the original boundary layer phenomenon are required. For instance, a simple possible choice is the average mean velocity at $x = 0$, $x = \infty$ to advect Q; that is,

$$\psi(t) = \frac{\sqrt{q_e^2 + Q(t,0)} + \sqrt{q_e^2 + Q(t,\infty)}}{2},$$

which corresponds to the equation in (2.1) for $\phi = 1$, $g = 0$. For another approximate model, $q_e = 1$ is taken for convenience and (4.2) is rewritten into $Q_t = \sqrt{1+Q}Q_{xx}$. Because the assumption is that $|Q|$ is small, it is reasonable to approximate

$$Q_{xx} = \frac{1}{\sqrt{1+Q}} Q_t \simeq \left(1 - \frac{Q}{2}\right) Q_t = Q_t - \frac{1}{2} Q_t Q.$$

Then, simply writing $Q_t = 2\varepsilon g(t,x)$, the following problem is obtained (1.1). Note that this model provides an approximate advection effect by substituting $Q_t Q$ with a linear term $2\varepsilon g(t,x)Q$ and correctly matches the boundary conditions at $x = 0$ and $x = \infty$ by considering the proper g.

5. Difficulties for a variable coefficient case

In Section 2, it was demonstrated that a certain variable coefficient case can be successfully reduced to the constant coefficient case. However, this is generally not so straightforward, as shown in the following example, which is a slight perturbation of the constant coefficient case. The perturbed linear problem is considered:

$$\begin{cases} (1 + g(t,x))Q_t - Q_{xx} = 0, \\ Q(t,0) = q_0(t), \\ Q(t,\infty) = 0 \end{cases} \tag{5.1}$$

with a periodic boundary condition in t. This equation is a variation of (1.1) and is regarded as an approximation of $Q_t = \sqrt{1+Q}Q_{xx}$ for small Q. Even with very simple data, $q_0(t) = \varepsilon e^{it}$ and $g(t,x) = e^{-\mu x + it}$, the existence of a smooth solution to (5.1) can be demonstrated to be highly nontrivial. Here,

$\Re(\mu) > 0$ and μ will be determined later (for easy computation). The following expansion is considered:

$$Q(t,x) = \sum_{n\geq 1} q_n(x)e^{int}. \tag{5.2}$$

From the boundary condition, the following is obtained: $q_1(0) = \varepsilon$, $q_1(\infty) = 0$, and $q_n(0) = q_n(\infty) = 0$ for $n \geq 2$. (It is demonstrated below that there is no need to include terms of the form e^{-int} with $n \geq 1$ in the expansion.) The following recurrence equations for q_n can be obtained by plugging (5.2) into (5.1):

$$i(nq_n + (n-1)e^{-\mu x}q_{n-1}) = q_n'', \quad n \geq 1. \tag{5.3}$$

Initially, when $n = 1$, the following solution can be directly derived

$$q_1(x) = \varepsilon e^{-\sqrt{i}x}, \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}.$$

Further, the following simple computation is used, where the operator $(\partial_x^2 - ai)^{-1}$ is well-defined by imposing the zero boundary conditions at $x = 0, \infty$.

Lemma 5.1. *For $a > 0$, the following is obtained*

$$(\partial_x^2 - ai)^{-1}e^{-\beta\sqrt{i}x} = \frac{1}{i}\frac{e^{-\beta\sqrt{i}x} - e^{-\sqrt{a}\sqrt{i}x}}{\beta^2 - a}.$$

Using the aforementioned lemma, in the case where $n = 2$,

$$(\partial_{xx} - 2i)q_2 = ie^{-\mu x}q_1 = i\varepsilon e^{(-\mu-\sqrt{i})x}.$$

From $\beta = 1 + \frac{\mu}{\sqrt{i}}$, the following is obtained:

$$q_2 = \frac{\varepsilon}{\beta^2 - 2}(e^{(-\mu-\sqrt{i})x} - e^{-\sqrt{2}\sqrt{i}x}).$$

Continuing,

$$(\partial_{xx} - 3i)q_3 = 2ie^{-\mu x}q_2 = \frac{2i\varepsilon}{\beta^2 - 2}(e^{(-2\mu-\sqrt{i})x} - e^{-(\sqrt{2}\sqrt{i}+\mu)x}).$$

Then,

$$q_3 = \frac{\varepsilon}{\beta^2 - 2}\left(A(\mu)(e^{(-2\mu-\sqrt{i})x} - e^{-\sqrt{3}\sqrt{i}x}) - B(\mu)(e^{-(\sqrt{2}\sqrt{i}+\mu)x} - e^{-\sqrt{3}\sqrt{i}x})\right),$$

where $A(\mu), B(\mu)$ are some constants.

To determine the rule for the sequence of functions q_n, let $\varepsilon = 1$ for simplicity, and the previous equation is rewritten in a better notation. First, let $\beta_1 = 1, a_1 = 1$ for the case $n = 1$. Then, for $n = 2$, let $\beta_2 = \beta_1 + \frac{\mu}{\sqrt{i}} = 1 + \frac{\mu}{\sqrt{i}}$, $a_2 = 2$. The solution can be written as:

$$q_2 = \frac{1}{{\beta_2}^2 - a_2}(e^{-\beta_2\sqrt{i}x} - e^{-\sqrt{a_2}\sqrt{i}x}).$$

Next, for $n = 3$, let $\beta_{31} = \beta_2 + \frac{\mu}{\sqrt{i}}$, $\beta_{32} = \sqrt{a_2} + \frac{\mu}{\sqrt{i}}$, $a_3 = 3$, and the equation is rewritten as

$$q_3 = \frac{2}{\beta_2^2 - a_2}\left[\frac{1}{\beta_{31}^2 - a_3}(e^{-\beta_{31}\sqrt{i}x} - e^{-\sqrt{a_3}\sqrt{i}x}) - \frac{1}{\beta_{32}^2 - a_3}(e^{-\beta_{32}\sqrt{i}x} - e^{-\sqrt{a_3}\sqrt{i}x})\right].$$

Therefore, now β_{nk} can be written for $k = 1, 2, \dots, 2^{n-2}$, and a_n for the corresponding coefficients of q_n in the form presented previously. From the previous computation, a rule for determining q_n from q_{n-1} is obtained. First, the number of terms of q_n is 2^{n-1}, of which only the coefficient of $e^{-\sqrt{n}\sqrt{i}x}$ is focused on, which has the slowest decay mode in x. From this consideration, the following can be derived:

$$\beta_{n1} = \beta_{(n-1)1} + \frac{\mu}{\sqrt{i}}, \quad \beta_{n1} = 1 + \frac{\mu}{\sqrt{i}}(n-1),$$

$$\beta_{n2} = \beta_{n4} = \beta_{n6} = \cdots = \beta_{n(2^{n-1})} = \sqrt{n-1} + \frac{\mu}{\sqrt{i}}$$

and

$$\beta_{n3} = \beta_{(n-1)2} + \frac{\mu}{\sqrt{i}} = \sqrt{n-2} + \frac{2\mu}{\sqrt{i}}, \quad \beta_{n5} = \beta_{(n-1)3} + \frac{\mu}{\sqrt{i}}, \dots.$$

Thus, a hierarchy of tables of coefficients of q_n for $n = 1, 2, \dots$ is achieved. To simplify further, let $\mu = \sqrt{i}$, which makes all the coefficients real and simple. More precisely, the following equation is obtained:

$$\beta_{n1} = n, \quad \beta_{n2} = \beta_{n4} = \beta_{n6} = \cdots = \beta_{n(2^{n-1})} = \sqrt{n-1} + 1$$

and

$$\beta_{n3} = \sqrt{n-2} + 2, \quad \beta_{n5} = \beta_{(n-1)3} + 1, \ \dots .$$

However, it is difficult to determine the coefficients of q_n (even asymptotically) during this recurrence procedure. Notably, each $\beta_{n(2k)} = \sqrt{n-1} + 1$, $k = 1, \dots, 2^{n-2}$ gives rise to the factor $1/(\beta_{n(2k)}^2 - n) = 1/(2\sqrt{n-1})$ of $e^{-\sqrt{n}\sqrt{i}x}$. This suggests that the series (5.2) may not converge uniformly. Unfortunately, each term on the right-hand side of the equation for $(\partial_{xx} - ni)q_n$ contributes to the coefficient of $e^{-\sqrt{n}\sqrt{i}x}$ in q_n, and there may be subtle cancellations that can make the series (5.2) convergent. However, it should be emphasized that even if the series converges, it cannot be proved using a direct norm estimate similar to that used in Section 3.

6. Conclusion

In this study, a linear parabolic differential equation and its time-periodic solution were considered under a non-trivial boundary condition. The existence and uniqueness were established under certain conditions on the boundary data and the coefficient. Moreover, a possible connection between the problem and a closed and recirculating boundary layer flow was investigated. Even for a linear problem, the existence of a time-periodic solution becomes highly nontrivial if the leading order coefficient is perturbed. The difficulty faced while trying to construct a Fourier series solution for a simple example was further demonstrated.

References

[1] G. K. Batchelor, *On steady laminar flow with closed streamlines at large Reynolds number*, J. Fluid Mech. **1** (1956), 177–190. https://doi.org/10.1017/S0022112056000123

[2] F. E. Browder, *Existence of periodic solutions for nonlinear equations of evolution*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1100–1103. https://doi.org/10.1073/pnas.53.5.1100

[3] M. Chen, X.-Y. Chen, and J. K. Hale, *Structural stability for time-periodic one-dimensional parabolic equations*, J. Differential Equations **96** (1992), no. 2, 355–418. https://doi.org/10.1016/0022-0396(92)90159-K

[4] N. Hirano, *Existence of multiple periodic solutions for a semilinear evolution equation*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 107–114. https://doi.org/10.2307/2047381

[5] R. Huang, Y. Wang, and Y. Ke, *Existence of non-trivial nonnegative periodic solutions for a class of degenerate parabolic equations with nonlocal terms*, Discrete Contin. Dyn. Syst. Ser. B **5** (2005), no. 4, 1005–1014. https://doi.org/10.3934/dcdsb.2005.5.1005

[6] Y. Ke, R. Huang, and J. Sun, *Periodic solutions for a degenerate parabolic equation*, Appl. Math. Lett. **22** (2009), no. 6, 910–915. https://doi.org/10.1016/j.aml.2008.06.047

[7] S.-C. Kim, *On Prandtl-Batchelor theory of a cylindrical eddy: asymptotic study*, SIAM J. Appl. Math. **58** (1998), no. 5, 1394–1413. https://doi.org/10.1137/S0036139996303282

[8] S.-C. Kim, *On Prandtl-Batchelor theory of a cylindrical eddy: existence and uniqueness*, Z. Angew. Math. Phys. **51** (2000), no. 5, 674–686. https://doi.org/10.1007/PL00001514

[9] G. M. Lieberman, *Time-periodic solutions of quasilinear parabolic differential equations. I. Dirichlet boundary conditions*, J. Math. Anal. Appl. **264** (2001), no. 2, 617–638. https://doi.org/10.1006/jmaa.2000.7145

[10] O. A. Oleinik and V. N. Samokhin, *Mathematical models in boundary layer theory*, Applied Mathematics and Mathematical Computation, 15, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[11] L. Prandtl, *Über flüssigkeitsbewegung bei sehr kleiner reibung*, Verhandl III, Intern. Math. Kongr. Heidelberg, Germany, 1904.

[12] T. I. Seidman, *Periodic solutions of a non-linear parabolic equation*, J. Differential Equations **19** (1975), no. 2, 242–257. https://doi.org/10.1016/0022-0396(75)90004-2

[13] O. Vejvoda, L. Herrmann, V. Lovicar, M. Sova, I. Straškraba, and M. Štědrý, *Partial differential equations: time-periodic solutions*, Martinus Nijhoff Publishers, The Hague, 1981.

[14] I. I. Vrabie, *Periodic solutions for nonlinear evolution equations in a Banach space*, Proc. Amer. Math. Soc. **109** (1990), no. 3, 653–661. https://doi.org/10.2307/2048204

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