J. Korean Math. Soc. **60** (2023), No. 2, pp. 395–406
https://doi.org/10.4134/JKMS.j220180
pISSN: 0304-9914 / eISSN: 2234-3008

RAMANUJAN CONTINUED FRACTIONS OF ORDER EIGHTEEN

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Abstract. As an analogy of the Rogers-Ramanujan continued fraction, we define a Ramanujan continued fraction of order eighteen. There are essentially three Ramanujan continued fractions of order eighteen, and we study them using the theory of modular functions. First, we prove that they are modular functions and find the relations with the Ramanujan cubic continued fraction $C(\tau)$. We can then obtain that their values are algebraic numbers. Finally, we evaluate them at some imaginary quadratic quantities.

1. Introduction

For the complex upper half plane $\mathfrak{H}$ and $\tau \in \mathfrak{H}$, let q be $\exp(2\pi i\tau)$. The Rogers-Ramanujan continued fraction $r(\tau)$ ([12])

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}$$

was studied by Ramanujan and Rogers. It generates the field $\mathcal{K}(\Gamma(5))$ of modular functions on $\Gamma(5)$, which is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(5)] = 60$.

There are more q-continued fractions, that appear in [12], such as the Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan's cubic continued fraction, and Ramanujan-Selberg continued fraction. These were studied from the perspective of modular functions after the modularity of $r(\tau)$ was proved ([2, 3, 5, 8]).

Moreover, there are more functions written as q-continued fractions that do not appear in Ramanujan's notebook. One is the continued fraction $X(\tau)$ of

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Received April 14, 2022; Accepted January 16, 2023.
2020 *Mathematics Subject Classification.* 11F03, 11R04, 11G16.
Key words and phrases. Ramanujan continued fraction, modular function, Klein forms.
This study was financially supported by NRF 2021R1F1A1055200.

order six [15]:

$$X(\tau) = \cfrac{q^{1/4}(1-q^2)}{1-q^{3/2}+\cfrac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)+\cfrac{(1-q^{5/2})(1-q^{13/2})}{q^{3/2}(1-q^{3/2})(1+q^6)}+\cdots}}$$
$$= q^{1/4}\prod_{n=1}^{\infty}\frac{(1-q^{6n-5})(1-q^{6n-1})}{(1-q^{6n-4})(1-q^{6n-2})},$$

and another is a continued fraction $U(\tau)$ of order twelve [11]:

$$U(\tau) = \cfrac{q(1-q)}{1-q^3+\cfrac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+\cfrac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})+\cdots}}}$$
$$= q\prod_{n=1}^{\infty}\frac{(1-q^{12n-1})(1-q^{12n-11})}{(1-q^{12n-5})(1-q^{12n-7})}.$$

Some properties of $X(\tau)$ and $U(\tau)$ were demonstrated, but Lee and the author studied them as modular functions ([9, 10]).

For $X(\tau)$, order *six* evidently corresponds to the period of the exponents of q in its infinite product expression; similarly, for $U(\tau)$, it is the same because the exponents of q is *twelve*. Surekha and Vanitha recently studied two functions, $I_1(\tau)$ and $I_2(\tau)$, written as q-continued fractions. They use the term "continued fractions of order sixteen" ([13, 14]).

To extend the definition of the order of the continued fraction, we call the function $r_N(\tau)$ *a continued fraction of order* N if it is written as a q-continued fraction and the period of the exponent of q is N in its infinite product expression.

When we obtain the infinite product expression of the continued fractions $I_1(\tau)$ and $I_2(\tau)$ of order sixteen, we can use the following identity from [12].

$$\prod_{n=1}^{\infty}\frac{(1-a^2q^{4n-1})(1-b^2q^{4n-1})}{(1-a^2q^{4n-3})(1-b^2q^{4n-3})}$$
$$= \cfrac{1}{1-ab+\cfrac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+\cfrac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)+\cdots}}} \tag{1.1}$$

with $|ab|<1$ and $|q|<1$.

In this paper, we study continued fractions of order eighteen. In (1.1), after changing q by $q^{9/2}$, and then taking $a = q^{(27-2j)/4}$ and $b = q^{(2j-9)/4}$, we can obtain the continued fractions forms of $r_{18,j}(\tau)$:

$$r_{18,j}(\tau) = q^{\frac{2j-9}{4}} \prod_{n=1}^{\infty} \frac{(1-q^{18n-(9+j)})(1-q^{18n-(9-j)})}{(1-q^{18n-(18-j)})(1-q^{18n-j})}. \tag{1.2}$$

In [4, Theorem 6.9], one can find that $r_{18,3}(\tau)$ is written in terms of Ramanujan's cubic continued fraction:

$$C(\tau) = q^{1/3} = \prod_{n=1}^{\infty} \frac{(1-q^{6n-5})(1-q^{6n-1})}{(1-q^{6n-3})^2} \tag{1.3}$$

as follows:

$$r_{18,3}(\tau)^4 = 1 + \frac{1}{C(3\tau)^3}, \tag{1.4}$$

we primarily study three of Ramanujan's continued fractions: $r_{18,1}(\tau)$, $r_{18,2}(\tau)$, and $r_{18,4}(\tau)$.

This paper is organized as follows. To prove our main results, we review parts of the modular function theory in Section 2. Using these, we prove the modularity of $r_{18,j}(\tau)$ $(j = 1, 2, 3, 4)$ in Theorem 3.1. In Section 3, we study three continued fractions: $r_{18,1}(\tau)$, $r_{18,2}(\tau)$ and $r_{18,4}(\tau)$. They are conjugates in the sense of the cubic polynomial with coefficients in $\mathcal{K}(\Gamma_0(18))$ (Theorem 3.3). We also evaluate $r_{18,3}(\tau)$ for $\tau = \sqrt{-1/27}$, $\sqrt{-2/27}$, $i/3$ and $\sqrt{-2}/3$ in Example 4.1, and we state $r_{18,1}(\sqrt{-1/27})$, $r_{18,2}(\sqrt{-1/27})$, and $r_{18,4}(\sqrt{-1/27})$ in Example 4.2. We use the MAPLE program to find results.

2. Preliminaries

In this section, we mention some facts that we require to obtain our results. Let $\mathfrak{H}^*$ be the union $\mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. For a positive integer N, the congruence subgroups $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$, $\Gamma^1(N)$, and $\Gamma^0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ are defined as the set of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$ modulo N, respectively.

For a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ acts on $\mathfrak{H}^*$ as $\gamma\tau = (a\tau + b)/(c\tau + d)$ for $\tau \in \mathfrak{H}^*$. Hence, the quotient space $\Gamma\backslash\mathfrak{H}^*$ can be considered as a compact Riemann surface. If s is an element of $\mathbb{Q} \cup \{\infty\}$, we call s a *cusp*. If there is an element $\gamma \in \Gamma$ such that $\gamma s_1 = s_2$, then two cusps s_1 and s_2 are equivalent under Γ.

For any cusp s of Γ, there is an element ρ of $\mathrm{SL}_2(\mathbb{Z})$ such that $\rho s = \infty$. Moreover, there is the smallest positive integer h_s satisfying $\rho^{-1} \left(\begin{smallmatrix} 1 & h_s \\ 0 & 1 \end{smallmatrix}\right) \rho \in \Gamma \cup (-1) \cdot \Gamma$. We call h_s the *width* of s. The width depends only on the equivalence class of the cusp s under Γ and is independent of the choice of ρ.

Now, we define a *modular function* $f(\tau)$ on Γ as the complex valued functions satisfying the following three conditions.

(1) $f(\tau)$ is meromorphic on $\mathfrak{H}$.
(2) $f(\tau)$ is invariant under Γ, that is, $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$.

(3) $f(\tau)$ is meromorphic at all cusps of Γ.

In the above definition, we explain the meaning of (3) as follows. For $s \in \mathbb{Q} \cup \{\infty\}$ and $\rho \in \mathrm{SL}_2(\mathbb{Z})$ with $\rho s = \infty$, we obtain an expansion of the type $f(\rho^{-1}\tau) = \sum_n a_n q^{n/h_s}$ because

$$f(\rho^{-1}(\tau + h_s)) = f\left(\left(\rho^{-1}\begin{pmatrix}1 & h_s\\ 0 & 1\end{pmatrix}\rho\right)\rho^{-1}\tau\right) = f(\rho^{-1}\tau).$$

Consider that a modular function $f(\tau)$ is written as $f(\tau) = \sum_n a_n q^{n/h_s}$ at s. Subsequently, there is an integer n_0 satisfying $\sum_{n \geq n_0} a_n q^{n/h_s}$ for some integer n_0 with $a_{n_0} \neq 0$. We call n_0 the *order* of $f(\tau)$ at the cusp s and write $\mathrm{ord}_s f(\tau)$ for n_0. We say that $f(\tau)$ has a zero (or a pole) at s if $\mathrm{ord}_s f(\tau)$ is positive (or negative). Now, we may identify a modular function on Γ as a meromorphic function on $\Gamma\backslash\mathfrak{H}^*$.

Denote the field of all modular functions on Γ by $\mathcal{K}(\Gamma)$. The field $\mathcal{K}(\Gamma)$ is identified with the field $\mathbb{C}(\Gamma\backslash\mathfrak{H}^*)$ of all meromorphic functions on $\Gamma\backslash\mathfrak{H}^*$. For a nonconstant function $f(\tau)$ in $\mathcal{K}(\Gamma)$, the field extension degree is the total degree of poles:

$$[\mathcal{K}(\Gamma) : \mathbb{C}(f(\tau))] = -\sum_{\substack{z \in \Gamma\backslash\mathfrak{H}^* \\ \mathrm{ord}_z f(\tau) < 0}} \mathrm{ord}_z f(\tau).$$

Here, we introduce the *Klein form*. In detail, refer to [7] or define it by (K4). Set $\tau \in \mathfrak{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. For $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2 - \mathbb{Z}^2$, the Klein form $\mathfrak{k}_{\mathbf{a}}(\tau)$ satisfies the following properties (K0)–(K5):

(K0) $\mathfrak{k}_{-\mathbf{a}}(\tau) = -\mathfrak{k}_{\mathbf{a}}(\tau)$.

(K1) $\mathfrak{k}_{\mathbf{a}}(\gamma\tau) = (c\tau + d)^{-1}\mathfrak{k}_{\mathbf{a}\gamma}(\tau)$.

(K2) For any $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$, we have

$$\mathfrak{k}_{\mathbf{a}+\mathbf{b}}(\tau) = \varepsilon(\mathbf{a}, \mathbf{b})\mathfrak{k}_{\mathbf{a}}(\tau),$$

where $\varepsilon(\mathbf{a}, \mathbf{b}) = (-1)^{b_1 b_2 + b_1 + b_2}\ e^{\pi i(b_2 a_1 - b_1 a_2)}$.

(K3) For $\mathbf{a} = (r/N, s/N) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ and any $\gamma \in \Gamma(N)$ with an integer $N > 1$,

$$\mathfrak{k}_{\mathbf{a}}(\gamma\tau) = \varepsilon_{\mathbf{a}}(\gamma)(c\tau + d)^{-1}\mathfrak{k}_{\mathbf{a}}(\tau),$$

where $\varepsilon_{\mathbf{a}}(\gamma) = -(-1)^{((a-1)r + cs + N)(br + (d-1)s + N)/N^2} e^{\pi i(br^2 + (d-a)rs - cs^2)/N^2}$.

(K4) Let $\tau \in \mathfrak{H}$, $z = a_1\tau + a_2$ with $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, and further, let $q = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z} = e^{2\pi i a_2}e^{2\pi i a_1\tau}$. Then

$$\mathfrak{k}_{\mathbf{a}}(\tau) = -\frac{1}{2\pi i}e^{\pi i a_2(a_1 - 1)}q^{a_1(a_1-1)/2}(1 - q_z)\prod_{n=1}^{\infty}\frac{(1 - q^n q_z)(1 - q^n q_z^{-1})}{(1 - q^n)^2},$$

and $\mathrm{ord}_q \mathfrak{k}_{\mathbf{a}}(\tau) = \langle a_1\rangle(\langle a_1\rangle - 1)/2$, where $\langle a_1\rangle$ denotes the number such that $0 \leq \langle a_1\rangle < 1$ and $a_1 - \langle a_1\rangle \in \mathbb{Z}$.

(K5) Let $f(\tau) = \prod_{\mathbf{a}} \mathfrak{k}_{\mathbf{a}}^{m(\mathbf{a})}(\tau)$ be a finite product of the Klein forms with $\mathbf{a} = (r/N, s/N) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for an integer $N > 1$, and let $k = -\sum_{\mathbf{a}} m(\mathbf{a})$.

Then $f(\tau)$ is a modular function on $\Gamma(N)$ if and only if $k=0$ and

$$\begin{cases} \sum_{\mathbf{a}} m(\mathbf{a})r^2 \equiv \sum_{\mathbf{a}} m(\mathbf{a})s^2 \equiv 0 \pmod{(2,N)N}, \\ \sum_{\mathbf{a}} m(\mathbf{a})rs \equiv 0 \pmod{N}. \end{cases}$$

To obtain the information of cusps on $\Gamma_0(N)$ we require the following lemma.

Lemma 2.1. *Let $a, c, a', c' \in \mathbb{Z}$ be such that $(a,c)=1$ and $(a',c')=1$. We understand that $\pm 1/0 = \infty$. We denote the set of all the inequivalent cusps on $\Gamma_0(N)$ by $S_{\Gamma_0(N)}$. Then*

(1) *a/c and a'/c' are equivalent under $\Gamma_0(N)$ if and only if there exist $\overline{s} \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $n \in \mathbb{Z}$ such that $(a',c') \equiv (\overline{s}^{-1}a + nc, \overline{s}c) \pmod{N}$.*

(2) *We may take $S_{\Gamma_0(N)}$ as the following set:*

$$S_{\Gamma_0(N)} = \left\{ \frac{a_{c,j}}{c} \in \mathbb{Q} : 0 < c \mid N,\ 0 < a_{c,j} \le N,\ (a_{c,j}, N) = 1, \right.$$
$$\left. a_{c,j} = a_{c,j'} \overset{\text{def.}}{\Longleftrightarrow} a_{c,j} \equiv a_{c,j'} \pmod{(c, N/c)} \right\}.$$

(3) *The width of the cusp $a/c \in S_{\Gamma_0(N)}$ is $N/(N,c^2)$.*

Proof. See [2, Corollary 4 (1)]. □

To determine the relation between two modular functions, we use the following lemma.

Lemma 2.2. *For any congruence subgroup Γ, let $f_1(\tau)$ and $f_2(\tau)$ be nonconstant functions such that $\mathbb{C}(f_1(\tau), f_2(\tau)) = \mathcal{K}(\Gamma)$ with the total degree D_k of poles of $f_k(\tau)$ for $k=1,2$. Let*

$$F(X,Y) = \sum_{\substack{0 \le i \le D_2 \\ 0 \le j \le D_1}} C_{i,j} X^i Y^j \in \mathbb{C}[X,Y]$$

be such that $F(f_1(\tau), f_2(\tau)) = 0$. For $k=1,2$, define the subsets $S_{k,0}$ and $S_{k,\infty}$ of the set S_Γ of all inequivalent cusps of Γ by

$$S_{k,0} = \{ s \in S_\Gamma : f_k(\tau) \text{ has zeros at } s \}$$

and

$$S_{k,\infty} = \{ s \in S_\Gamma : f_k(\tau) \text{ has poles at } s \}.$$

Then we have $C_{D_2,0} \ne 0$ if $S_{1,\infty} \cap S_{2,0} = \phi$.

Proof. See [6, Lemmas 3 and 6]. □

3. Continued fractions of order eighteen

By (1.2) and (K4), we write the continued fractions $r_{18,j}(\tau)$ $(j=1,2,3,4)$ of order eighteen in the products of Klein forms. For simplicity, we use $r_j(\tau)$ instead of $r_{18,j}(\tau)$.

$$(3.1) \quad \begin{aligned} r_1(\tau) &= \zeta_{72}^{25} \prod_{j=0}^{17} \frac{\mathfrak{k}_{(8/18,j/18)}(\tau)}{\mathfrak{k}_{(1/18,j/18)}(\tau)}, \quad & r_2(\tau) &= \zeta_{72}^{-13} \prod_{j=0}^{17} \frac{\mathfrak{k}_{(7/18,j/18)}(\tau)}{\mathfrak{k}_{(2/18,j/18)}(\tau)}, \\ r_3(\tau) &= \zeta_{24}^{7} \prod_{j=0}^{17} \frac{\mathfrak{k}_{(6/18,j/18)}(\tau)}{\mathfrak{k}_{(3/18,j/18)}(\tau)}, \quad & r_4(\tau) &= \zeta_{72}^{-17} \prod_{j=0}^{17} \frac{\mathfrak{k}_{(5/18,j/18)}(\tau)}{\mathfrak{k}_{(4/18,j/18)}(\tau)}, \end{aligned}$$

where $\zeta_N = \exp(2\pi i/N)$ is the Nth root of unity. Now, we are ready to show the modularity of $r_j(\tau)$ $(j = 1, 2, 3, 4)$.

Theorem 3.1. *For $j = 1, 2, 3$, and 4, the fourth powers $r_j(\tau)^4$ of the continued fractions of order eighteen are modular functions on $\Gamma_1(18)$.*

Proof. With some suitable roots of unity ε_j, we have

$$r_j(\tau)^4 = \varepsilon_j \prod_{l=0}^{17} \mathfrak{k}_{((9-j)/18, l/18)}(\tau)^4 / \mathfrak{k}_{(j/18, l/18)}(\tau)^4.$$

To check the modularity, we require the condition (K5) for $N = 18$. Then

$$\sum_{\mathbf{a}} m(\mathbf{a}) r^2 = 18 \times 4((9-j)^2 - j^2) \equiv 0 \pmod{36},$$

$$\sum_{\mathbf{a}} m(\mathbf{a}) s^2 = 0 \equiv 0 \pmod{36}$$

and

$$\sum_{\mathbf{a}} m(\mathbf{a}) rs = \sum_{l=0}^{17} (4(9-j)l - 4jl) \equiv 0 \pmod{18}.$$

Hence, for $j = 1, 2, 3$, and 4, $r_j(\tau)^4$ is a modular function on $\Gamma(18)$. From (1.2), $r_j(\tau)^4$ are written as q-expansion, and this means that they are invariant under the action of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, we prove that the fourth powers of all $r_j(\tau)$ are modular functions on $\langle \Gamma(18), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle = \Gamma_1(18)$. □

We study the modularity of three Ramanujan's continued fractions $r_1(\tau)$, $r_2(\tau)$, and $r_4(\tau)$ of order eighteen. Although none of their fourth powers are not modular functions on $\Gamma_0(18)$, and the genus of $\Gamma_1(18)$ is not zero, we can obtain certain modular functions on $\Gamma_0(18)$ comprising $r_1(\tau)$, $r_2(\tau)$ and $r_4(\tau)$.

Lemma 3.2. *Let $\gamma := \left(\begin{smallmatrix} 5 & -2 \\ 18 & -7 \end{smallmatrix}\right)$. Then,*

$$r_1(\gamma\tau) = \frac{1}{r_4(\tau)}, \quad r_2(\gamma\tau) = \frac{1}{r_1(\tau)}, \quad \textit{and} \quad r_4(\gamma\tau) = -r_2(\tau).$$

Proof. For simplicity, let $K_i(\tau)$ be the product of eighteen Klein forms:

$$K_i(\tau) = \prod_{j=0}^{17} \mathfrak{k}_{\left(\frac{i}{18}, \frac{j}{18}\right)}(\tau) \qquad (i = 1, \ldots, 8).$$

By (K0)–(K2), we have the action of γ on the set $\{K_j(\tau) : j = 1, 2, 4, 5, 7, 8\}$ upto the root of unity and factor $(18\tau - 7)^{-18}$:

$$K_1(\gamma\tau) = (18\tau - 7)^{-18} \zeta_{36}^{13} K_5(\tau), \qquad K_2(\gamma\tau) = (18\tau - 7)^{-18} \zeta_{18} K_8(\tau),$$
$$K_4(\gamma\tau) = (18\tau - 7)^{-18} \zeta_9 K_2(\tau), \qquad K_5(\gamma\tau) = (18\tau - 7)^{-18} \zeta_3^{-1} K_7(\tau),$$
$$K_7(\gamma\tau) = (18\tau - 7)^{-18} \zeta_9^{-1} K_1(\tau), \qquad K_8(\gamma\tau) = (18\tau - 7)^{-18} \zeta_4^{-3} K_4(\tau).$$

We already know the products of the Klein forms of $r_1(\tau)$, $r_2(\tau)$, and $r_4(\tau)$ in (3.1), and we have

$$r_1(\gamma\tau) = \zeta_{72}^{25}\frac{K_8(\gamma\tau)}{K_1(\gamma\tau)} = \zeta_{72}^{25}\frac{\zeta_4^{-3}K_4(\tau)}{\zeta_{36}^{13}K_5(\tau)} = \frac{1}{r_4(\tau)},$$

$$r_2(\gamma\tau) = \zeta_{72}^{-13}\frac{K_7(\gamma\tau)}{K_2(\gamma\tau)} = \zeta_{72}^{-13}\frac{\zeta_9^{-1}K_1(\tau)}{\zeta_{18}K_8(\tau)} = \frac{1}{r_1(\tau)},$$

and

$$r_4(\gamma\tau) = \zeta_{72}^{-17}\frac{K_5(\gamma\tau)}{K_4(\gamma\tau)} = \zeta_{72}^{-17}\frac{\zeta_3^{-1}K_7(\tau)}{\zeta_9 K_4(\tau)} = -r_2(\tau). \qquad \square$$

Theorem 3.3. *Consider a cubic polynomial $P(X)$ in X defined as follows*:

$$P(X) = (X - r_1(\tau)^4)\left(X - \frac{1}{r_2(\tau)^4}\right)\left(X - \frac{1}{r_4(\tau)^4}\right).$$

Then $P(X)$ is a cubic polynomial in $\mathcal{K}(\Gamma_0(18))[X]$, that is,

$$\begin{aligned} P(X) = {} & X^3 - (g^7 + 4g^6 + 10g^5 + 13g^4 + 11g^3 + 6g^2 + 4g + 3)X^2 \\ & + (g^6 + 4g^5 + 10g^4 + 14g^3 + 12g^2 + 8g + 3)X - \frac{(g+1)^4}{g^3+1}, \end{aligned}$$

where $g := g(\tau) = 1/C(3\tau)$, and $C(\tau)$ is in (1.3).

Proof. Note that $r_1(\tau)^4$ is a modular function on $\Gamma_1(18)$ by Theorem 3.1. Because $[\overline{\Gamma_0(18)} : \overline{\Gamma_1(18)}] = 3$, we may take $\gamma := \left(\begin{smallmatrix} 5 & -2 \\ 18 & -7 \end{smallmatrix}\right)$ such that $\Gamma_0(18) = \cup_{j=0}^{2}\Gamma_1(18)\gamma^j$. Assume that $f(\tau) = r_1(\tau)^4$. By Lemma 3.2, γ acts on $f(\tau)$ as follows:

$$f(\gamma\tau) = \frac{1}{r_4(\tau)^4} \quad \text{and} \quad f(\gamma^2\tau) = \frac{1}{r_2(\tau)^4}.$$

Then $P(X)$ is written as f and γ:

$$P(X) = (X - r_1(\tau)^4)\left(X - \frac{1}{r_2(\tau)^4}\right)\left(X - \frac{1}{r_4(\tau)^4}\right) = \prod_{j=0}^{2}\left(X - f(\gamma^j\tau)\right).$$

Now, all the coefficients of $P(X)$ are elementary symmetric functions and are invariant under the action of $\Gamma_0(18)$. Thus, they can be written as rational functions of the generator of the field of modular functions on $\Gamma_0(18)$. Note that $g = 1/C(3\tau)$ is a generator of $\mathcal{K}(\Gamma_0(18))$. To prove our theorem, we claim the following:

(1) $f(\tau) + f(\gamma\tau) + f(\gamma^2\tau) = g^7 + 4g^6 + 10g^5 + 13g^4 + 11g^3 + 6g^2 + 4g + 3$,
(2) $f(\tau)f(\gamma\tau) + f(\gamma\tau)f(\gamma^2\tau) + f(\tau)f(\gamma^2\tau) = g^6 + 4g^5 + 10g^4 + 14g^3 + 12g^2 + 8g + 3$,
(3) $f(\tau)f(\gamma\tau)f(\gamma^2\tau) = (g+1)^4/(g^3+1)$.

Before proving (1), (2), and (3), we calculate the orders of the functions at cusps on $\Gamma_0(18)$.

Table 1. Orders of functions at cusps

cusp s	∞	0	1/2	1/3	2/3	1/6	5/6	1/9
$\mathrm{ord}_s f(\tau)$	-7	0	1	0	0	$-1/3$	$-1/3$	0
$\mathrm{ord}_s f(\gamma\tau)$	5	0	1	0	0	$-1/3$	$-1/3$	0
$\mathrm{ord}_s f(\gamma^2\tau)$	1	0	1	0	0	$-1/3$	$-1/3$	0
$\mathrm{ord}_s(1/C(3\tau))$	-1	0	0	0	0	0	0	1

Note that at a point s,

$$\mathrm{ord}_s(h_1(\tau)+h_2(\tau)) \geq \min\{\mathrm{ord}_s h_1(\tau), \mathrm{ord}_s h_2(\tau)\}$$

and the equality holds if $\mathrm{ord}_s h_1(\tau) \neq \mathrm{ord}_s h_2(\tau)$.

Now, we prove our claims.

(1) Let $h_1(\tau) := f(\tau)+f(\gamma\tau)+f(\gamma^2\tau)$. At ∞, $\mathrm{ord}_\infty h_1(\tau)$ is $\min\{-7,5,1\} = -7$. When s is 1/6 or 5/6, $\mathrm{ord}_s h_1(\tau)$ is greater than $-1/3$. Because $h_1(\tau)$ is a modular function on $\Gamma_0(18)$, and the order of $h_1(\tau)$ is an integer, we have that $\mathrm{ord}_{1/6} h_1(\tau)$ and $\mathrm{ord}_{5/6} h_1(\tau)$ are nonnegative. That is, the only pole of $h_1(\tau)$ is ∞ and its order is -7. We apply Lemma 2.2 to find the relation between $h_1(\tau)$ and $g(\tau)$ by taking $f_1(\tau) = g(\tau)$ and $f_2(\tau) = h_1(\tau)$ in Lemma 2.2. Then $D_1 = 1$ and $D_2 = 7$, and we obtain a two variable polynomial $F(X,Y)$:

$$F(X,Y) = \sum_{\substack{0\leq i\leq 7\\ 0\leq j\leq 1}} C_{i,j} X^i Y^j$$

such that $F(g(\tau), h_1(\tau)) = 0$. Because $S_{1,\infty} = S_{2,\infty} = \{\infty\}$, we have that $S_{1,\infty} \cap S_{2,0} = \phi$ and $C_{7,0} \neq 0$. If we take $C_{7,0} = 1$, then we find

$$F(X,Y) = X^7+4X^6+10X^5+13X^4+11X^3+6X^2+4X+3-Y$$

by substituting the q-expansions of $g(\tau)$ and $h_1(\tau)$ to X and Y, respectively. This means that

$$h_1(\tau) = g^7+4g^6+10g^5+13g^4+11g^3+6g^2+4g+3.$$

(2) Let $h_2(\tau) := f(\tau)f(\gamma\tau)+f(\gamma\tau)f(\gamma^2\tau)+f(\gamma^2\tau)f(\tau)$. Similar to the case of $h_1(\tau)$, we have the following:

- at ∞, $\mathrm{ord}_\infty h_2(\tau) \geq \min\{-2,,6,-6\} = -6$, and we have $\mathrm{ord}_\infty h_2(\tau) = -6$,
- when $s = 1/6$ or $5/6$, $\mathrm{ord}_s h_2(\tau) \geq \min\{-2/3,-2/3,-2/3\}$. Because $\mathrm{ord}_s h_2(\tau)$ is an integer, we have $\mathrm{ord}_s h_2(\tau) \geq 0$, and
- when $s = 0, 1/2, 1/3, 2/3$ or $1/9$, $\mathrm{ord}_s h_2(\tau) \geq 0$.

Hence, ∞ is the only pole of $h_2(\tau)$ and $\mathrm{ord}_\infty h_2(\tau) = 6$. Consider a polynomial $F(X,Y) = \sum_{\substack{0\leq i\leq 6\\ 0\leq j\leq 1}} C_{i,j} X^i Y^j$ satisfying $F(g(\tau), h_2(\tau)) = 0$. By taking $f_1(\tau) = g(\tau)$ and $f_2(\tau) = h_2(\tau)$ in Lemma 2.2, we may take $C_{6,0} = 1$ and obtain the relation

$$h_2(\tau) = g^6+4g^5+10g^4+14g^3+12g^2+8g+3.$$

(3) Let $h_3(\tau)$ be the product of functions $f(\tau)$, $f(\gamma\tau)$, and $f(\gamma^2\tau)$. From Table 2, we obtain the exact order of $h_2(\tau)$ at all cusps:

$$\mathrm{ord}_s h_3(\tau) = \begin{cases} 3 & \text{if } s = 1/2, \\ -1 & \text{if } s = \infty, 1/6, 5/6, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we use Lemma 2.2 and set $f_1(\tau) = g(\tau)$ and $f_2(\tau) = h_3(\tau)$. Then $S_{1,\infty} = \{\infty\}$, $S_{2,\infty} = \{\infty, 1/6, 5/6\}$, $S_{1,0} = \{1/9\}$, and $S_{2,0} = \{1/2\}$. Thus, we obtain the polynomial

$$F(X,Y) = \sum_{\substack{0\le i\le 3\\ 0\le j\le 1}} C_{i,j} X^i Y^j$$

with $F(g(\tau), h_3(\tau)) = 0$. Because $S_{1,\infty} \cap S_{2,0} = \phi$, we take $C_{3,0} = 1$ and we obtain

$$F(X,Y) = 1 - Y + 3\,X + XY + 3\,X^2 - X^2Y + X^3.$$

Hence,

$$h_3(\tau) = \frac{g^3 + 3g^2 + 3g + 1}{g^2 - g + 1} = \frac{(g+1)^4}{g^3+1}. \qquad \square$$

4. Evaluations of $r_{18,1}(\tau)$, $r_{18,2}(\tau)$, $r_{18,3}(\tau)$, and $r_{18,4}(\tau)$

For an imaginary quadratic value τ, $1/C(\tau)$ is an algebraic integer ([3, Theorem 16]). By (1.4), we have that $r_3(\tau)$ is also an algebraic integer and $r_1(\tau)$, $r_2(\tau)$, $r_4(\tau)$ are algebraic numbers. In this section, we evaluate $r_j(\tau)$ by the value of $C(\tau)$ at some imaginary quadratic quantities τ.

Example 4.1. By (1.4), we obtain $r_3(\tau)^4 = 1 + 1/C(3\tau)^3$.

(1) From [1, Theorem 4.1(ii) and Corollary 4.3], $C(\sqrt{-1/3}) = 2^{-4/3}(\sqrt{3}-1)$, and we have $r_3(\sqrt{-1/27})^4 = 1 + 1/C(\sqrt{-1/3})^3 = 6(1+\sqrt{3})\left(\sqrt{3}-1\right)^{-3}$. By solving the quartic equation $X^4 - 6(1+\sqrt{3})\left(\sqrt{3}-1\right)^{-3}$, we have its four zeros:

$$\sqrt{3+2\sqrt{3}},\ -\sqrt{3+2\sqrt{3}},\ \sqrt{-3-2\sqrt{3}},\ \text{and}\ -\sqrt{-3-2\sqrt{3}}.$$

By comparing the approximation of $r_3(\sqrt{-1/27}) \approx 2.542459758$ with the above four values, we obtain that

$$r_3\left(\sqrt{-\frac{1}{27}}\right) = \sqrt{3+2\sqrt{3}},$$

because $\sqrt{3+2\sqrt{3}} \approx 2.542459757$.

(2) Similar to (1), we use the value

$$C\left(\sqrt{-\tfrac{2}{3}}\right) = \frac{\left(4+2\sqrt{2}\right)^{2/3}\sqrt{2}}{8} - \frac{\sqrt{2\left(4+2\sqrt{2}\right)^{4/3} + 16\sqrt[3]{4+2\sqrt{2}} - 16\sqrt[3]{4+2\sqrt{2}}\sqrt{2}}}{8}$$

which can be obtained from [1, Theorem 4.5(i)], and we have

$$r_3\left(\sqrt{-\frac{2}{27}}\right)^4 = \frac{1}{C(\sqrt{-2/3})^3} + 1 = \frac{3(3+\sqrt{2}-\sqrt{3})}{1+3\sqrt{2}-3\sqrt{3}}.$$

Solving the quartic equation $X^4 - 3(3+\sqrt{2}-\sqrt{3})(1+3\sqrt{2}-3\sqrt{3})^{-1}$ and comparing the approximation of $r_3(\sqrt{-2/27})$ to its zeros, we conclude that

$$r_3\left(\sqrt{-\frac{2}{27}}\right) = \left(\frac{3(3+\sqrt{2}-\sqrt{3})}{1+3\sqrt{2}-3\sqrt{3}}\right)^{1/4}.$$

(3) From [1, Theorem 4.5(iii)], $C(i) = (3^{3/4}\sqrt{2}-\sqrt{3}-1)/4$, in a similar way, we have

$$r_3\left(\frac{i}{3}\right) = \sqrt{3^{3/4}\sqrt{8}+6+3^{5/4}\sqrt{2}+3\sqrt{3}}.$$

(4) By the value $C(\sqrt{-2}) = \sqrt{2}(\sqrt{3}+\sqrt{2}-3)/4$ from [1, Theorems 4.6(i) and 4.10(i)], we have

$$r_3\left(\frac{\sqrt{-2}}{3}\right) = \sqrt{21+15\sqrt{2}+12\sqrt{3}+9\sqrt{6}}.$$

For an imaginary quadratic value τ, the evaluations of $r_1(\tau)$, $r_2(\tau)$, and $r_4(\tau)$ can be also obtained from $C(3\tau)$. First, we write the value of $C(3\tau)$ in terms of the radical using some values in [1]. With value $g = 1/C(3\tau)$ and cubic polynomial $P(X)$ from Theorem 3.3, we have the set of the three zeros of $P(X) = 0$ and it is exactly the same as $\{r_1(\tau)^4, 1/r_2(\tau)^4, 1/r_4(\tau)^4\}$. Comparing one of the zeros with the approximations of $r_1(\tau)^4$, $1/r_2(\tau)^4$, $1/r_4(\tau)^4$, we can determine which of them is in $\{r_1(\tau)^4, 1/r_2(\tau)^4, 1/r_4(\tau)^4\}$. In addition, if we consider their fourth roots, we obtain the explicit values of $r_j(\tau)$ for $j = 1, 2$, and 4 in terms of radicals.

Example 4.2. We can obtain $r_1(\sqrt{-1/27})$, $r_2(\sqrt{-1/27})$, and $r_4(\sqrt{-1/27})$ in terms of radical. Instead of these we write $r_1(\sqrt{-1/27})^4$, $1/r_2(\sqrt{-1/27})^4$, and $1/r_4(\sqrt{-1/27})^4$. Here, we use value $g = g(\sqrt{-1/27}) = 1/C(\sqrt{-1/3}) = 2^{1/3}(\sqrt{3}+1)$. The polynomial $P(X)$,

$$\begin{aligned}P(X) = X^3 &- \left(\tfrac{512\sqrt[3]{2}}{(\sqrt{3}-1)^7} + \tfrac{1024}{(\sqrt{3}-1)^6} + \tfrac{640\sqrt[3]{4}}{(\sqrt{3}-1)^5} + \tfrac{416\sqrt[3]{2}}{(\sqrt{3}-1)^4} + \tfrac{176}{(\sqrt{3}-1)^3} + \tfrac{24\sqrt[3]{4}}{(\sqrt{3}-1)^2} + \tfrac{8\sqrt[3]{2}}{\sqrt{3}-1} + 3\right)X^2 \\ &+ \left(\tfrac{256}{(\sqrt{3}-1)^6} + \tfrac{256\sqrt[3]{4}}{(\sqrt{3}-1)^5} + \tfrac{320\sqrt[3]{2}}{(\sqrt{3}-1)^4} + \tfrac{224}{(\sqrt{3}-1)^3} + \tfrac{48\sqrt[3]{4}}{(\sqrt{3}-1)^2} + \tfrac{16\sqrt[3]{2}}{\sqrt{3}-1} + 3\right)X \\ &- \left(\tfrac{16}{(\sqrt{3}-1)^3} + \tfrac{12\sqrt[3]{4}}{(\sqrt{3}-1)^2} + \tfrac{6\sqrt[3]{2}}{\sqrt{3}-1} + 1\right)\left(\tfrac{4\sqrt[3]{4}}{(\sqrt{3}-1)^2} - \tfrac{2\sqrt[3]{2}}{\sqrt{3}-1} + 1\right)^{-1},\end{aligned}$$

in Theorem 3.3 has three zeros as follows:

$$\begin{aligned}r_1\left(\sqrt{-1/27}\right)^4 = &\tfrac{4}{3}\sqrt[3]{\alpha} - \tfrac{3}{4}\frac{\beta}{\sqrt[3]{\alpha}} + \tfrac{3551}{3} + \tfrac{3004}{3}\sqrt[3]{2} + \tfrac{1544}{3}\sqrt[3]{4} + 684\sqrt{3} \\ &+ \tfrac{1732}{3}\sqrt[3]{2}\sqrt{3} + \tfrac{892}{3}\sqrt[3]{4}\sqrt{3},\end{aligned}$$

$$\begin{aligned}\frac{1}{r_2\left(\sqrt{-1/27}\right)^4} = & -\tfrac{2}{3}\sqrt[3]{\alpha}+\tfrac{3}{8}\tfrac{\beta}{\sqrt[3]{\alpha}}+684\sqrt{3}+\tfrac{1732}{3}\sqrt[3]{2}\sqrt{3}+\tfrac{3551}{3}\\ & +\tfrac{892}{3}\sqrt[3]{4}\sqrt{3}+\tfrac{1544}{3}\sqrt[3]{4}+\tfrac{3004}{3}\sqrt[3]{2}+\tfrac{i\sqrt{3}}{2}\left(\tfrac{4}{3}\sqrt[3]{\alpha}+\tfrac{3}{4}\tfrac{\beta}{\sqrt[3]{\alpha}}\right),\\ \frac{1}{r_4\left(\sqrt{-1/27}\right)^4} = & -\tfrac{2}{3}\sqrt[3]{\alpha}+\tfrac{3}{8}\tfrac{\beta}{\sqrt[3]{\alpha}}+684\sqrt{3}+\tfrac{1732}{3}\sqrt[3]{2}\sqrt{3}+\tfrac{3551}{3}\\ & +\tfrac{892}{3}\sqrt[3]{4}\sqrt{3}+\tfrac{1544}{3}\sqrt[3]{4}+\tfrac{3004}{3}\sqrt[3]{2}-\tfrac{i\sqrt{3}}{2}\left(\tfrac{4}{3}\sqrt[3]{\alpha}+\tfrac{3}{4}\tfrac{\beta}{\sqrt[3]{\alpha}}\right),\end{aligned}$$

where

$$\begin{aligned}\alpha = & \, 12342300669\sqrt[3]{4}+11233679274\sqrt{3}+7125830613\sqrt[3]{4}\sqrt{3}\\ & +15500248362\sqrt[3]{2}+8949072564\sqrt[3]{2}\sqrt{3}+19457303256\\ & +18\,i\sqrt{6596748678\sqrt[3]{4}+6051480246\sqrt{3}+3808634625\sqrt[3]{4}\sqrt{3}+8316687954\sqrt[3]{2}+10481471247+4801642029\sqrt[3]{2}\sqrt{3}},\\ \beta = & -\frac{17399648}{3}\sqrt[3]{2}-3998496\sqrt{3}-\frac{10045696}{3}\sqrt[3]{2}\sqrt{3}-2564032\sqrt[3]{4}\sqrt{3}\\ & -6925600-\frac{13323104}{3}\sqrt[3]{4}.\end{aligned}$$

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