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RELATIVE ROTA-BAXTER SYSTEMS ON LEIBNIZ ALGEBRAS

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Abstract. In this paper, we introduce relative Rota-Baxter systems on Leibniz algebras and give some characterizations and new constructions. Then we construct a graded Lie algebra whose Maurer-Cartan elements are relative Rota-Baxter systems. This allows us to define a cohomology theory associated with a relative Rota-Baxter system. Finally, we study formal deformations and extendibility of finite order deformations of a relative Rota-Baxter system in terms of the cohomology theory.

1. Introduction

In 1960, Baxter [3] introduced the notion of Rota-Baxter operators on associative algebras in his study of fluctuation theory in probability. Rota-Baxter operators have been found many applications, including in Connes-Kreimer's algebraic approach to the renormalization in perturbative quantum field theory [8].

The concept of Leibniz algebra was introduced by Bloh [4] and rediscovered by Loday [18,20] in the study of the algebraic K-theory. Leibniz algebras have been studied from different aspects. In particular, the integrals of Leibniz algebras are studied in [5,9] and deformation quantization of Leibniz algebras was considered in [12]. As the underlying structure of embedding tensor, Leibniz algebras also have application in higher gauge theories, see [17,26] for more details. Recently, relative Rota-Baxter operators on Leibniz algebras were studied in [28], which is the main ingredient in the study of the twisting theory and the bialgebra theory for Leibniz algebras. Moreover, relative Rota-Baxter operators on a Leibniz algebra can be seen as the Leibniz algebraic analogue of

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Poisson structures. Generally, Rota-Baxter operators can be defined on operads, which results in a split of operands [1, 24]. For more details on the Rota-Baxter operator, see [16].

The deformation of algebraic structures began with the seminal work of Gerstenhaber [14, 15] for associative algebras and followed by its extension to Lie algebras by Nijenhuis and Richardson [21, 22]. In general, the deformation theory of algebras over binary quadratic operads was developed by Balavoine [2]. Deformations of morphisms and $\mathcal{O}$-operators (also called relative Rota-Baxter operators) were developed in [10, 13] and [27, 29]. Rota-Baxter systems as a generalization of a Rota-Baxter operator were introduced by Brzeziński [6]. In a Rota-Baxter system, two operators are acting on the algebra and satisfy some Rota-Baxter type identities. Generalized Rota-Baxter systems in the presence of bimodule were introduced and their deformation theory was studied by Das [11].

It is well known that Rota-Baxter operators on Lie algebras are closely related to solutions of the classical Yang-Baxter equation, whereas the classical Yang-Baxter equation plays important role in many fields of mathematics and mathematical physics [7, 25]. In [28], Sheng and Tang introduced the classical Leibniz Yang-Baxter equation, classical Leibniz r-matrices and triangular Leibniz bialgebras. Furthermore, they proved that a solution of the classical Leibniz Yang-Baxter equation gives rise to a relative Rota-Baxter operator. Our main objective in this paper is the notion of the relative Rota-Baxter system on Leibniz algebras. A class of relative Rota-Baxter systems arise from Leibniz Yang-Baxter pairs which are defined as pairs of elements $r, s \in g \otimes g$ satisfying two equations similar to the classical Leibniz Yang-Baxter equation. Next, we construct a graded Lie algebra which characterizes relative Rota-Baxter systems as its Maurer-Cartan elements. Using this characterization, we define the cohomology associated with a relative Rota-Baxter system. Finally, we use this cohomology to study deformations of relative Rota-Baxter systems.

The paper is organized as follows. In Section 2, we first recall Leibniz algebras and their representations. Next, we introduce relative Rota-Baxter systems on Leibniz algebras with respect to representation and give some characterizations and new constructions. In Section 3, we emphasise relative Rota-Baxter systems with respect to the regular representation. In Section 4, we construct a graded Lie algebra whose Maurer-Cartan elements are relative Rota-Baxter systems, which leads us to define cohomology for a relative Rota-Baxter system. Finally, in Section 5, we consider formal deformations of relative Rota-Baxter systems.

Throughout this paper, $\mathbb{K}$ is a field of characteristic zero and all vector spaces, (multi)linear maps and tensor products are over $\mathbb{K}$.

2. Relative Rota-Baxter systems on Leibniz algebras with respect to representation

In this section, we first recall Leibniz algebras and representations [18, 20]. Next, we introduce relative Rota-Baxter systems on Leibniz algebras with respect to representation.

Definition 1. A Leibniz algebra is a vector space g together with a bilinear operation $[\cdot,\cdot]_g : g \otimes g \to g$ satisfying

$$[x,[y,z]_g]_g = [[x,y]_g,z]_g + [y,[x,z]_g]_g \quad \text{for } x,y,z \in g.$$

Definition 2. A representation of a Leibniz algebra $(g,[\cdot,\cdot]_g)$ is a triple (V,ρ^L,ρ^R), where V is a vector space, $\rho^L,\rho^R : g \to gl(V)$ are linear maps such that the following equalities hold: for all $x,y \in g$,

(1) $\rho^L([x,y]_g) = [\rho^L(x),\rho^L(y)]$,
(2) $\rho^R([x,y]_g) = [\rho^L(x),\rho^R(y)]$,
(3) $\rho^R(y) \circ \rho^L(x) = -\rho^R(y) \circ \rho^R(x)$.

Let $(g,[\cdot,\cdot])$ be a Leibniz algebra. Define the left multiplication $L : g \to gl(g)$ and the right multiplication $R : g \to gl(g)$ by $L_x y = [x,y]_g$ and $R_x y = [y,x]_g$ for all $x,y \in g$. Then (g,L,R) is a representation of $(g,[\cdot,\cdot]_g)$, called the regular representation. Define two linear maps $L^*,R^* : g \to gl(g^*)$ with $x \mapsto L^*_x$ and $x \mapsto R^*_x$, respectively, by

$$\langle L^*_x\xi,y\rangle = -\langle\xi,[x,y]_g\rangle,\ \langle R^*_x\xi,y\rangle = -\langle\xi,[y,x]_g\rangle \quad \text{for } x,y \in g,\ \xi \in g^*.$$

Then it has been shown in [28] that $(g^*,L^*,-L^*-R^*)$ is a representation. This is called the dual of the regular representation.

Definition 3. A quadratic Leibniz algebra is a Leibniz algebra $(g,[\cdot,\cdot]_g)$ equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 g^*$ such that the following invariant condition holds:

$$\omega(x,[y,z]_g) = \omega([x,z]_g + [z,x]_g, y) \quad \text{for } x,y,z \in g.$$

Proposition 2.1 ([28])**.** *Let $(g,[\cdot,\cdot]_g,\omega)$ be a quadratic Leibniz algebra. Then the map*

$$\omega^\natural : g \to g^*,\ \omega^\sharp(x)(y) = \omega(x,y) \quad \textit{for } x,y \in g$$

is an isomorphism from the regular representation (g,L,R) to its dual representation $(g^*,L^*,-L^*-R^*)$.

In the following, we introduce and study relative Rota-Baxter systems on Leibniz algebras with respect to representation.

Definition 4. (1) A relative Rota-Baxter system on $(g,[\cdot,\cdot]_g)$ with respect to the representation (V,ρ^L,ρ^R) consists of a pair (R,S) of linear maps $R,S : V \to g$ satisfying

$$[Ru,Rv]_g = R(\rho^L(Ru)v + \rho^R(Sv)u),$$

$$[Su, Sv]_g = S(\rho^L(Ru)v + \rho^R(Sv)u)$$

for $u, v \in V$.

(2) A Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ is a relative Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ with respect to the regular representation.

Example 2.2. A relative Rota-Baxter operator [28] on $(g, [\cdot,\cdot]_g)$ with respect to the representation (V, ρ^L, ρ^R) is a linear map $R : V \to g$ satisfying

$$[Ru, Rv]_g = R(\rho^L(Ru)v + \rho^R(Rv)u) \quad \text{for } u, v \in V.$$

Thus R is a relative Rota-Baxter operator if and only if the pair (R, R) is a relative Rota-Baxter system.

Example 2.3. Consider the 2-dimensional Leibniz algebra $(g, [\cdot,\cdot])$ given with respect to a basis $\{e_1, e_2\}$ by

$$[e_1, e_1] = 0,\ [e_1, e_2] = 0,\ [e_2, e_1] = e_1,\ [e_2, e_2] = e_1.$$

Let $\{e_1^*, e_2^*\}$ be the dual basis. Then $R = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right)$ is a relative Rota-Baxter system on $(g, [\cdot,\cdot])$ with respect to the representation $(g^*, L^*, -L^* - R^*)$ if and only if

$$\begin{aligned} [Re_i^*, Re_j^*] &= R(L^*_{Re_i^*} e_j^* - L^*_{Se_j^*} e_i^* - R^*_{Se_j^*} e_i^*), \\ [Se_i^*, Se_j^*] &= S(L^*_{Re_i^*} e_j^* - L^*_{Se_j^*} e_i^* - R^*_{Se_j^*} e_i^*),\ i, j = 1, 2. \end{aligned}$$

It is straightforward to deduce that

$$\begin{aligned} L_{e_1}(e_1, e_2) &= (e_1, e_2)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ L_{e_2}(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\ R_{e_1}(e_1, e_2) &= (e_1, e_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ R_{e_2}(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} L^*_{e_1}(e_1^*, e_2^*) &= (e_1^*, e_2^*)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ L^*_{e_2}(e_1^*, e_2^*) = (e_1^*, e_2^*)\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \\ R^*_{e_1}(e_1^*, e_2^*) &= (e_1^*, e_2^*)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\ R^*_{e_2}(e_1^*, e_2^*) = (e_1^*, e_2^*)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

We have

$$[Re_1^*, Re_1^*] = [a_{11}e_1 + a_{21}e_2, a_{11}e_1 + a_{21}e_2] = a_{21}(a_{11} + a_{21})e_1$$

and

$$\begin{aligned} & R(L^*_{Re_1^*} e_1^* - L^*_{Se_1^*} e_1^* - R^*_{Se_1^*} e_1^*) \\ = & -a_{21}(R(e_1^*) + R(e_2^*)) + b_{21}(R(e_1^*) + R(e_2^*)) + (b_{11} + b_{21})R(e_2^*) \\ = & -a_{21}(a_{11}e_1 + a_{21}e_2 + a_{12}e_1 + a_{22}e_2) + b_{21}(a_{11}e_1 + a_{21}e_2 + a_{12}e_1 + a_{22}e_2) \\ & + (b_{11} + b_{21})(a_{12}e_1 + a_{22}e_2) \\ = & ((b_{11} + b_{21})a_{12} + (a_{11} + a_{12})(b_{21} - a_{21}))e_1 \end{aligned}$$

$$
\begin{aligned}
&+((b_{11}+b_{21})a_{22}+(a_{21}+a_{22})(b_{21}-a_{21}))e_2,\\
&[Se_1^*, Se_1^*] = [b_{11}e_1+b_{21}e_2, b_{11}e_1+b_{21}e_2] = b_{21}(b_{11}+b_{21})e_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&S(L^*_{Re_1^*}e_1^* - L^*_{Se_1^*}e_1^* - R^*_{Se_1^*}e_1^*)\\
= &-a_{21}(R(e_1^*)+R(e_2^*)) + b_{21}(S(e_1^*)+S(e_2^*)) + (b_{11}+b_{21})S(e_2^*)\\
= &-a_{21}(a_{11}e_1+a_{21}e_2+a_{12}e_1+a_{22}e_2) + b_{21}(b_{11}e_1+b_{21}e_2+b_{12}e_1+b_{22}e_2)\\
&+(b_{11}+b_{21})(b_{12}e_1+b_{22}e_2)\\
= &((b_{11}+b_{21})b_{12}+(b_{11}+b_{12})b_{21}-(a_{11}+a_{12})a_{21})e_1\\
&+((b_{11}+b_{21})b_{22}+(b_{21}+b_{22})b_{21}-(a_{21}+a_{22})a_{21})e_2.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&a_{21}(a_{11}+a_{21}) = (b_{11}+b_{21})a_{12}+(a_{11}+a_{12})(b_{21}-a_{21}),\\
&(b_{11}+b_{21})a_{22}+(a_{21}+a_{22})(b_{21}-a_{21}) = 0,\\
&b_{21}(b_{11}+b_{21}) = (b_{11}+b_{21})b_{12}+(b_{11}+b_{12})b_{21}-(a_{11}+a_{12})a_{21},\\
&(b_{11}+b_{21})b_{22}+(b_{21}+b_{22})b_{21}-(a_{21}+a_{22})a_{21} = 0.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&a_{21}(a_{12}+a_{22}) = b_{22}(a_{11}+a_{12})+(b_{12}+b_{22})a_{12},\\
&b_{22}(a_{21}+a_{22})+(b_{12}+b_{22})a_{22} = 0,\\
&b_{21}(b_{12}+b_{22}) = b_{22}(b_{11}+b_{12})+(b_{12}+b_{22})b_{12},\\
&b_{22}(b_{21}+b_{22})+(b_{12}+b_{22})b_{22} = 0,\ a_{22}(a_{12}+a_{22}) = b_{22}(b_{12}+b_{22}) = 0,\\
&a_{22}(a_{11}+a_{21}) = a_{22}(a_{11}+a_{12}),\ -a_{22}(a_{21}+a_{22}) = 0,\\
&b_{22}(b_{11}+b_{21}) = a_{22}(b_{11}+b_{12}),\ -a_{22}(b_{21}+b_{22}) = 0.
\end{aligned}
$$

By summarizing the above observations, we have the following.

(1) If $a_{22} = b_{22} = 0$ and $a_{21} = b_{21}$, then $R = \left(\begin{smallmatrix} a_{11} & a_{12}\\ a_{21} & 0\end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} b_{11} & b_{12}\\ b_{21} & 0\end{smallmatrix}\right)$ is a relative Rota-Baxter system on $(g, [\cdot,\cdot])$ with respect to the representation $(g^*, L^*, -L^*-R^*)$ if and only if

$$
\begin{aligned}
&(b_{12}-a_{21})a_{12} = (b_{12}-a_{21})b_{12} = 0,\\
&a_{21}(a_{11}+a_{21}) = (b_{11}+b_{21})a_{12},\\
&b_{21}(b_{11}+b_{21}) = (b_{11}+b_{21})b_{12}+(b_{11}+b_{12})b_{21}-(a_{11}+a_{12})a_{21},
\end{aligned}
$$

(2) If $a_{22} = b_{22} \neq 0$ and $a_{21} \neq b_{21}$, then $R = \left(\begin{smallmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{smallmatrix}\right)$ is a relative Rota-Baxter system on $(g, [\cdot,\cdot])$ with respect to the representation $(g^*, L^*, -L^*-R^*)$ if and only if

$$
a_{11} = -a_{12} = -a_{21} = a_{22},\ b_{11} = -b_{12} = -b_{21} = b_{22}.
$$

We will give some more examples of Rota-Baxter systems on Leibniz algebras in the next section.

In the following, we give some characterizations of relative Rota-Baxter systems. Let (V, ρ^L, ρ^R) be a representation of a Leibniz algebra $(g, [\cdot,\cdot]_g)$. Then there is a Leibniz algebra structure on $g \oplus g \oplus V$ given by

$$[x_1 + x_2 + u, y_1 + y_2 + v] = [x_1, y_1]_g + [x_2, y_2]_g + \rho^L(x_1)v + \rho^R(y_2)u.$$

This is exactly the semidirect product if we consider the Leibniz algebra structure on $g \oplus g$ and define its representation on V by $\rho^L(x_1 + x_2)v = \rho^L(x_1)v$ and $\rho^R(x_1 + x_2)v = \rho^R(x_2)v$.

Proposition 2.4. *A pair (R, S) of linear maps from V to g is a relative Rota-Baxter system with respect to the representation (V, ρ^L, ρ^R) if and only if the pair $(\widetilde{R}, \widetilde{S})$ of maps*

$$\widetilde{R} : g \oplus g \oplus V \to g \oplus g \oplus V,\ \ x_1 + x_2 + u \mapsto R(u) + 0 + 0,$$
$$\widetilde{S} : g \oplus g \oplus V \to g \oplus g \oplus V,\ \ x_1 + x_2 + u \mapsto 0 + S(u) + 0,$$

is a Rota-Baxter system on the Leibniz algebra $g \oplus g \oplus V$.

Proof. For any $x_1, x_2, y_1, y_2 \in g$ and $u, v \in V$, we have

$$[\widetilde{R}(x_1 + x_2 + u), \widetilde{R}(y_1 + y_2 + v)] = [R(u), R(v)]_g + 0 + 0$$

and

$$\begin{aligned}&\widetilde{R}([\widetilde{R}(x_1 + x_2 + u), y_1 + y_2 + v] + [x_1 + x_2 + u, \widetilde{S}(y_1 + y_2 + v)])\\ =\ & R(\rho^L(Ru)v + \rho^R(Sv)u) + 0 + 0.\end{aligned}$$

Similarly, we have

$$[\widetilde{S}(x_1 + x_2 + u), \widetilde{S}(y_1 + y_2 + v)] = 0 + [S(u), S(v)]_g + 0$$

and

$$\begin{aligned}&\widetilde{S}([\widetilde{R}(x_1 + x_2 + u), y_1 + y_2 + v] + [x_1 + x_2 + u, \widetilde{S}(y_1 + y_2 + v)]\\ =\ & 0 + S(\rho^L(Ru)v + \rho^R(Sv)u) + 0.\end{aligned}$$

Hence (R, S) is a relative Rota-Baxter system if and only if $(\widetilde{R}, \widetilde{S})$ is a Rota-Baxter system. □

Recall that a Nijenhuis operator on a Leibniz algebra $(g, [\cdot,\cdot]_g)$ is a linear map $N : g \to g$ satisfying

$$[Nx, Ny]_g = N([N(x), y]_g + [x, N(y)]_g - N[x, y]_g) \ \text{ for } x, y \in g.$$

The following result relates to relative Rota-Baxter systems and Nijenhuis operators.

Proposition 2.5. *A pair (R, S) of linear maps from V to $\mathfrak{g}$ is a relative Rota-Baxter system if and only if*

$$N_{(R,S)} = \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix} : g \oplus g \oplus V \to g \oplus g \oplus V$$

is a Nijenhuis operator on the Leibniz algebra $g \oplus g \oplus V$.

Proof. For any $x_1, x_2, y_1, y_2 \in g$ and $u, v \in V$, by a simple calculation, we have

$$[N_{(R,S)}(x_1 + y_1 + u), N_{(R,S)}(x_2 + y_2 + v)] = [R(u), R(v)]_g + [S(u), S(v)]_g + 0$$

and

$$\begin{aligned} &N_{(R,S)}([N_{(R,S)}(x_1+y_1+u), x_2 + y_2 + v] + [x_1+y_1+u, N_{(R,S)}(x_2+y_2+v)] \\ &\quad - N_{(R,S)}[x_1 + y_1 + u, x_2 + y_2 + v]) \\ &= R(\rho^L(Ru)v + \rho^R(Sv)u) + S(\rho^L(Ru)v + \rho^R(Sv)u) + 0. \end{aligned}$$

It follows that $N_{(R,S)}$ is a Nijenhuis operator if and only if (R, S) is a relative Rota-Baxter system. □

Definition 5. Let (V, ρ^L, ρ^R) be a representation of a Leibniz algebra $(g, [\cdot,\cdot]_g)$. Suppose that dim (g) = dim (V). A pair (Φ, Ψ) of invertible linear maps from g to V is said to be an invertible 1-cocycle system if they satisfy

$$\begin{aligned} \Phi([x, y]_g) &= \rho^L(x)\Phi(y) + \rho^R(\Psi^{-1} \circ \Phi(y))\Phi(x), \\ \Psi([x, y]_g) &= \rho^L(\Phi^{-1} \circ \Psi(x))\Psi(y) + \rho^R(y)\Psi(x) \end{aligned}$$

for $x, y \in g$.

It follows from the above definition that (Φ, Φ) is an invertible 1-cocycle system if and only if $\Phi : g \to V$ is an invertible derivation.

Proposition 2.6. *Let* (V, ρ^L, ρ^R) *be a representation of a Leibniz algebra* $(g, [\cdot,\cdot]_g)$. *Suppose that* $\dim(g) = \dim(V)$. *A pair* (R, S) *of invertible linear maps from* V *to* g *is a relative Rota-Baxter system if and only if* (R^{-1}, S^{-1}) *is an invertible* 1*-cocycle system.*

Proof. For any $u, v \in V$ and $x, y \in g$, by taking $R(u) = x, R(v) = y$, the first identity of Definition 5 is equivalent to

$$R^{-1}[x, y]_g = \rho^L(x)R^{-1}y + \rho^R((S^{-1})^{-1} \circ R^{-1}(y))R^{-1}x.$$

Similarly, for any $u, v \in V$ and $x, y \in g$, by taking $S(u) = x,\ S(v) = y$, the second identity of Definition 5 is equivalent to

$$S^{-1}[x, y]_g = \rho^L((R^{-1})^{-1} \circ S^{-1}(x))S^{-1}y + \rho^R(y)S^{-1}x.$$

It follows that (R, S) of invertible linear maps from V to g is a relative Rota-Baxter system if and only if (R^{-1}, S^{-1}) is an invertible 1-cocycle system. □

Leibniz Yang-Baxter equation was introduced in [28] to understand relative Rota-Baxter operators on Leibniz algebras. Here we extend this to the context of relative Rota-Baxter systems.

Definition 6. Let $(g,[\cdot,\cdot]_g)$ be a Leibniz algebra. A Leibniz Yang-Baxter pair is a pair of elements $r,s\in g\otimes g$ such that $\tau(r)=r$, $\tau(s)=s$ satisfy the following equations

$$[r_{12},r_{13}]_g+[r_{13},r_{12}]_g-[r_{12},r_{23}]_g-[s_{13},r_{23}]_g=0,$$
$$[s_{12},r_{13}]_g+[r_{13},s_{12}]_g-[s_{12},s_{23}]_g-[s_{13},s_{23}]_g=0.$$

The brackets are defined as

$$[r_{12},r_{13}]_g=\sum[r_1,\widehat{r}_1]_g\otimes r_2\otimes\widehat{r}_2,\ [r_{13},r_{12}]_g=\sum[r_1,\widehat{r}_1]_g\otimes\widehat{r}_2\otimes r_2,$$
$$[r_{12},r_{23}]_g=\sum r_1\otimes[r_2,\widehat{r}_1]_g\otimes\widehat{r}_2,\ [s_{13},r_{23}]_g=\sum s_1\otimes r_1\otimes[s_2,r_2]_g,$$

where $r=\sum r_1\otimes r_2=\sum\widehat{r}_1\otimes\widehat{r}_2$ and $s=\sum s_1\otimes s_2=\sum\widehat{s}_1\otimes\widehat{s}_2$ and τ is the exchanging operator defined by $\tau(x\otimes y)=y\otimes x$ for any $x,y\in g$.

Proposition 2.7. *Let $(g,[\cdot,\cdot]_g,\omega)$ be a quadratic Leibniz algebra and $R,S: g^*\to g$ be two linear maps. Then (R,S) is a relative Rota-Baxter system on $(g,[\cdot,\cdot]_g)$ with respect to the representation $(g^*,L^*,-L^*-R^*)$ if and only if $(R\circ\omega^\natural,S\circ\omega^\natural)$ is a Rota-Baxter system on $(g,[\cdot,\cdot]_g)$.*

Proof. For any $x,y\in g$, we have

$$\begin{aligned}&R\circ\omega^\natural([R\circ\omega^\natural(x),y]_g+[x,S\circ\omega^\natural(y)]_g)\\=&R(\omega^\natural(L_{R\circ\omega^\natural(x)}y)+\omega^\natural(R_{S\circ\omega^\natural(y)}x))\\=&R(L^*_{R\circ\omega^\natural(x)}\omega^\natural(y)-L^*_{S\circ\omega^\natural(y)}\omega^\natural(x)-R^*_{S\circ\omega^\natural(y)}\omega^\natural(x)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&S\circ\omega^\natural([R\circ\omega^\natural(x),y]_g+[x,S\circ\omega^\natural(y)]_g)\\=&S(\omega^\natural(L_{R\circ\omega^\natural(x)}y)+\omega^\natural(R_{S\circ\omega^\natural(y)}x))\\=&S(L^*_{R\circ\omega^\natural(x)}\omega^\natural(y)-L^*_{S\circ\omega^\natural(y)}\omega^\natural(x)-R^*_{S\circ\omega^\natural(y)}\omega^\natural(x)).\end{aligned}$$

Thus it follows that $(R\circ\omega^\natural,S\circ\omega^\natural)$ is a Rota-Baxter system on $(g,[\cdot,\cdot]_g)$ if and only if

$$[R\circ\omega^\natural(x),R\circ\omega^\natural(y)]_g=R(L^*_{R\circ\omega^\natural(x)}\omega^\natural(y)-L^*_{S\circ\omega^\natural(y)}\omega^\natural(x)-R^*_{S\circ\omega^\natural(y)}\omega^\natural(x)),$$
$$[S\circ\omega^\natural(x),S\circ\omega^\natural(y)]_g=S(L^*_{R\circ\omega^\natural(x)}\omega^\natural(y)-L^*_{S\circ\omega^\natural(y)}\omega^\natural(x)-R^*_{S\circ\omega^\natural(y)}\omega^\natural(x)).$$

Since $\omega^\natural$ is an isomorphism, these identities hold if and only if (R,S) is a relative Rota-Baxter system on $(g,[\cdot,\cdot]_g)$ with respect to the representation $(g^*,L^*,-L^*-R^*)$. □

Corollary 2.8. *Let $(g,[\cdot,\cdot]_g,\omega)$ be a quadratic Leibniz algebra. Then $r,s\in g\otimes g$ is a Leibniz Yang-Baxter pair in g if and only if $(r^\natural\circ\omega^\natural,s^\natural\circ\omega^\natural)$ is a relative Rota-Baxter system on $(g,[\cdot,\cdot]_g)$, where $r^\natural:g^*\to g$ is defined by $\langle r^\natural(\xi),\eta\rangle=\langle r,\xi\otimes\eta\rangle$ for all $\xi,\eta\in g^*$, that is,*

$$[r^\natural\circ\omega^\natural(x),r^\natural\circ\omega^\natural(y)]_g=r^\natural\circ\omega^\natural([r^\natural\circ\omega^\natural(x),y]_g+[x,s^\natural\circ\omega^\natural(y)]_g),$$

$$[s^\natural \circ \omega^\natural(x), s^\natural \circ \omega^\natural(y)]_g = s^\natural \circ \omega^\natural([r^\natural \circ \omega^\natural(x), y]_g + [x, s^\natural \circ \omega^\natural(y)]_g).$$

3. Rota-Baxter systems

In this section, we mainly provide examples of Rota-Baxter systems on Leibniz algebras. As mentioned earlier, they are relative Rota-Baxter operators with respect to the regular representation.

Example 3.1. Consider the three-dimensional Leibniz algebra $(g, [\cdot,\cdot]_g)$ given with respect to a basis $\{e_1, e_2, e_3\}$ by

$$[e_1, e_1]_g = e_3.$$

Then $R = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $S = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ is a Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ if and only if

$$\begin{aligned} [Re_i, Re_j]_g &= R([Re_i, e_j]_g + [e_i, Se_j]_g), \\ [Se_i, Se_j]_g &= S([Re_i, e_j]_g + [e_i, Se_j]_g) \quad \text{for } i, j = 1, 2, 3. \end{aligned}$$

We have $[Re_1, Re_1]_g = [a_{11}e_1 + a_{21}e_2 + a_{31}e_3, a_{11}e_1 + a_{21}e_2 + a_{31}e_3] = a_{11}^2 e_3$ and

$$\begin{aligned} & R([Re_1, e_1]_g + [e_1, Se_1]_g) \\ = {} & R([a_{11}e_1 + a_{21}e_2 + a_{31}e_3, e_1]_g + [e_1, b_{11}e_1 + b_{21}e_2 + b_{31}e_3]_g) \\ = {} & (a_{11} + b_{11})Re_3 \\ = {} & (a_{11} + b_{11})a_{13}e_1 + (a_{11} + b_{11})a_{23}e_2 + (a_{11} + b_{11})a_{33}e_3. \end{aligned}$$

Thus, by $[Re_1, Re_1]_g = R([Re_1, e_1]_g + [e_1, Se_1]_g)$, we have

$$(a_{11} + b_{11})a_{13} = 0, \ (a_{11} + b_{11})a_{23} = 0, \ a_{11}^2 = (a_{11} + b_{11})a_{33}.$$

Similarly, by $[Se_1, Se_1]_g = S([Re_1, e_1]_g + [e_1, Se_1]_g)$, we have

$$(a_{11} + b_{11})b_{13} = 0, \ (a_{11} + b_{11})b_{23} = 0, \ b_{11}^2 = (a_{11} + b_{11})b_{33}.$$

By considering other choices of e_i and e_j, we obtain

$$\begin{aligned} & a_{11}a_{12} = b_{12}a_{33}, \ b_{12}a_{13} = 0, \ b_{12}a_{23} = 0, \\ & b_{11}b_{12} = b_{12}b_{33}, \ b_{12}b_{13} = 0, \ b_{12}b_{23} = 0, \\ & a_{11}a_{13} = b_{13}a_{33}, \ b_{13}a_{13} = 0, \ b_{13}a_{23} = 0, \\ & b_{11}b_{13} = b_{13}b_{33}, \ b_{13}b_{13} = 0, \ b_{13}b_{23} = 0, \\ & a_{12}a_{11} = a_{12}a_{33}, \ a_{12}a_{13} = 0, \ a_{12}a_{23} = 0, \\ & b_{12}b_{11} = a_{12}b_{33}, \ a_{12}b_{13} = 0, \ a_{12}b_{23} = 0, \\ & a_{13}a_{11} = a_{13}a_{33}, \ a_{13}a_{13} = 0, \ a_{13}a_{23} = 0, \\ & b_{13}b_{11} = a_{13}b_{33}, \ a_{13}b_{13} = 0, \ a_{13}b_{23} = 0, \\ & a_{12}^2 = 0, \ a_{13}^2 = 0, \ a_{12}a_{13} = 0, \ b_{12}^2 = 0, \ b_{13}^2 = 0, \ b_{12}b_{13} = 0. \end{aligned}$$

By summarizing the above observations, we have the following.

(1) If $a_{11} = b_{11} = a_{12} = b_{12} = a_{13} = b_{13} = 0$, then any $R = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ is a Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ with respect to the regular representation.

(2) If $a_{12} = b_{12} = a_{13} = b_{13} = 0$ and $a_{11} = b_{11} \neq 0$, $a_{23} = b_{23} = 0$, then any $R = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \frac{a_{11}}{2} \end{pmatrix}$, $S = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & \frac{b_{11}}{2} \end{pmatrix}$ is a Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ with respect to the regular representation.

We have seen that relative Rota-Baxter systems generalize relative Rota-Baxter operators. In the following, we show that they also generalize Rota-Baxter operators of arbitrary weight.

Definition 7. Let $(g, [\cdot,\cdot]_g)$ be a Leibniz algebra. A linear map $R : g \to g$ is said to be a Rota-Baxter operator of weight λ if R satisfies

$$[Rx, Ry]_g = R([Rx, y]_g + [x, Ry]_g + \lambda[x, y]_g) \quad \text{for } x, y \in g.$$

Proposition 3.2. *Let $(g, [\cdot,\cdot]_g)$ be a Leibniz algebra and $R : g \to g$ be a Rota-Baxter operator of weight λ. Then $(R, R + \lambda\mathrm{Id})$ and $(R + \lambda\mathrm{Id}, R)$ are Rota-Baxter systems on $(g, [\cdot,\cdot]_g)$.*

Proof. For any $x, y \in g$, we have

$$\begin{aligned} [Rx, Ry]_g &= R([Rx, y]_g + [x, Ry]_g + \lambda[x, y]_g) \\ &= R([Rx, y]_g + [x, (R + \lambda\mathrm{Id})y]_g) \\ &= R([(R + \lambda\mathrm{Id})x, y]_g + [x, Ry]_g) \end{aligned}$$

and

$$\begin{aligned} &[(R + \lambda\mathrm{Id})x, (R + \lambda\mathrm{Id})y]_g \\ &= [Rx, Ry]_g + \lambda[Rx, y]_g + \lambda[x, Ry]_g + [\lambda x, \lambda y]_g \\ &= R([Rx, y]_g + [x, Ry]_g + \lambda[x, y]_g) + \lambda[Rx, y]_g + \lambda[x, Ry]_g + [\lambda x, \lambda y]_g \\ &= (R + \lambda\mathrm{Id})([Rx, y]_g + [x, (R + \lambda\mathrm{Id})y]_g) \\ &= (R + \lambda\mathrm{Id})([(R + \lambda\mathrm{Id})x, y]_g + [x, Ry]_g). \end{aligned}$$

Hence, $(R, R+\lambda\mathrm{Id})$ and $(R+\lambda\mathrm{Id}, R)$ are Rota-Baxter systems on $(g, [\cdot,\cdot]_g)$. □

Let $(g, [\cdot,\cdot]_g)$ be a Leibniz algebra. A linear map $T : g \to g$ is said to be a left g-linear map (resp. a right g-linear map) if $T[x, y]_g = [x, Ty]_g$ (resp. $T[x, y]_g = [Tx, y]_g$) for any $x, y \in g$.

Lemma 3.3. *Let $(g, [\cdot,\cdot]_g)$ be a Leibniz algebra. Suppose that $R : g \to g$ is a left g-linear map and $S : g \to g$ is a right g-linear map. Then (R, S) is a Rota-Baxter system on $(g, [\cdot,\cdot]_g)$ if and only if*

$$[x, R \circ S(y)]_g = 0 = [S \circ R(x), y]_g \quad \textit{for } x, y \in g.$$

Proof. For any $x, y \in g$, we observe that

$$R([Rx, y]_g + [x, Sy]_g) = [Rx, Ry]_g + [x, R \circ S(y)]_g$$

and

$$S([Rx, y]_g + [x, Sy]_g) = [R \circ S(x), Ry]_g + [Sx, Sy]_g.$$

It follows from the above two identities that (R, S) is a Rota-Baxter system on $(g, [\cdot, \cdot]_g)$ if and only if

$$[x, R \circ S(y)]_g = 0 = [S \circ R(x), y]_g \quad \text{for } x, y \in g. \qquad \square$$

A Leibniz algebra $(g, [\cdot, \cdot]_g)$ is said to be nondegenerate if the bracket $[\cdot, \cdot]_g$ satisfies the following

$$[x, y]_g = 0 \quad \text{for all } y \text{ implies that } x = 0,$$
$$[x, y]_g = 0 \quad \text{for all } x \text{ implies that } y = 0.$$

Corollary 3.4. *Let $(g, [\cdot, \cdot]_g)$ be a nondegenerate Leibniz algebra. Let $R : g \to g$ be a left g-linear map and $S : g \to g$ be a right g-linear map. Then (R, S) is a Rota-Baxter system on $(g, [\cdot, \cdot]_g)$ if and only if*

$$R \circ S = S \circ R = 0.$$

Another class of Rota-Baxter systems arises from twisted Rota-Baxter operators. Let $(g, [\cdot, \cdot]_g)$ be a Leibniz algebra and $\sigma : g \to g$ be a Leibniz algebra morphism.

Definition 8. A linear map $R : g \to g$ is said to be a σ-twisted Rota-Baxter operator if R satisfies

$$(1) \qquad [Rx, Ry]_g = R([Rx, y]_g + [x, (\sigma \circ R)y]_g) \quad \text{for all } x, y \in g.$$

When $\sigma = \mathrm{Id}$, a σ-twisted Rota-Baxter operator is nothing but a Rota-Baxter operator.

Example 3.5. A differential Rota-Baxter Leibniz algebra of weight λ is a Leibniz algebra $(g, [\cdot, \cdot]_g)$ together with linear maps $R, \partial : g \to g$ satisfying the following set of identities

($dR1$) $[Rx, Ry]_g = R([Rx, y]_g + [x, Ry]_g + \lambda[x, y]_g)$,
($dR2$) $\partial[x, y]_g = [\partial x, y]_g + [x, \partial y]_g + \lambda[\partial x, \partial y]_g$,
($dR3$) $\partial \circ R = \mathrm{Id}$.

Let (g, R, ∂) be a differential Rota-Baxter Leibniz algebra of weight λ. It follows from ($dR2$) that the map

$$\sigma : g \to g, \ \sigma(x) = x + \lambda\partial(x) \quad \text{for } x \in g$$

is a Leibniz algebra morphism. On the other hand, ($dR3$) implies that

$$(\sigma \circ R)(x) = R(x) + \lambda x \quad \text{for } x \in g.$$

Furthermore, by $(dR2)$, we get

$$[Rx, Ry]_g = R([Rx, y]_g + [x, (\sigma \circ R)y]_g) \quad \text{for } x, y \in g.$$

Hence, R is a σ-twisted Rota-Baxter operator.

Proposition 3.6. *Let R be a σ-twisted Rota-Baxter operator on a Leibniz algebra $(g, [\cdot, \cdot]_g)$. Then $(R, \sigma \circ R)$ is a Rota-Baxter system on $(g, [\cdot, \cdot]_g)$.*

Proof. Note that condition Eq. (1) is the same as the first condition of a Rota-Baxter system. To prove the second one, we observe that

$$\begin{aligned}[(\sigma \circ R)x, (\sigma \circ R)y]_g &= \sigma[Rx, Ry]_g \\ &= (\sigma \circ R)([Rx, y]_g + [x, (\sigma \circ R)y]_g).\end{aligned}$$

This shows that $(R, \sigma \circ R)$ is a Rota-Baxter system on $(g, [\cdot, \cdot]_g)$. □

Example 3.7. Let $(W, [\cdot, \cdot]_W)$ be the Witt Lie algebra generated by basis elements $\{l_n\}_{n \in \mathbb{Z}}$ and the Lie bracket given by

$$[l_m, l_n]_W = (m - n)l_{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

View this Lie algebra as a Leibniz algebra. Let $q \in \mathbb{K}$ be a nonzero scalar that is not a root of unity. We define linear maps $\sigma, R : W \to W$ by

$$\sigma(l_n) = q^n l_n, \; R(l_n) = \frac{1 - q}{1 - q^n} l_n \quad \text{for } n \in \mathbb{Z}.$$

Then σ is a Leibniz algebra morphism. Moreover, it is easy to verify that R satisfies

$$[R(l_m), R(l_n)]_W = R([R(l_m), l_n]_W + [l_m, (\sigma \circ R)(l_n)]_W) \quad \text{for } m, n \in \mathbb{Z}.$$

Therefore, R is a σ-twisted Rota-Baxter operator. Hence, $(R, \sigma \circ R)$ is a Rota-Baxter system on W.

In [23] the authors introduced a notion of weak pseudotwistor on an associative algebra and showed that a weak pseudotwistor induces a new associative algebra structure. A Rota-Baxter system on an associative algebra gives rise to a weak pseudotwistor, hence a new associative algebra structure. This is not true for Rota-Baxter systems on Leibniz algebras. However, if we concentrate on Rota-Baxter operators, they induce a new Leibniz algebra structure via a Leibniz analogue of weak pseudotwistor. Let us first recall the new Leibniz algebra associated with a Rota-Baxter operator on a Leibniz algebra.

Let $(g, [\cdot, \cdot]_g)$ be a Leibniz algebra, and $R : g \to g$ be a Rota-Baxter operator, i.e., R satisfies

$$[Rx, Ry]_g = R([Rx, y] + [x, Ry]) \quad \text{for } x, y \in g.$$

Then the vector space g carries a new Leibniz algebra structure with bracket

$$[x, y]_R = [Rx, y] + [x, Ry] \quad \text{for } x, y \in g.$$

Here we give a new example of a Rota-Baxter operator on a Leibniz algebra induced from a dialgebra [19].

Definition 9. A dialgebra is a vector space D together with two bilinear operations $\dashv, \vdash: D \otimes D \to D$ satisfying the following identities

$$a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c),$$
$$(a \vdash b) \dashv c = a \vdash (b \dashv c),$$
$$(a \dashv b) \vdash c = (a \vdash b) \vdash c = a \vdash (b \vdash c) \quad \text{for } a, b, c \in D.$$

A dialgebra as above may be denoted by the triple $(D, \dashv, \vdash)$. Any associative algebra is a dialgebra with both the bilinear maps coinciding with the associative product. See Loday [19] for more examples of dialgebras.

It is known that a dialgebra $(D, \dashv, \vdash)$ induced a Leibniz algebra by

$$[a, b]_D := a \vdash b - b \dashv a \quad \text{for } a, b \in D.$$

The Leibniz algebra is called the Leibniz algebra induced from the dialgebra $(D, \dashv, \vdash)$.

Definition 10. Let $(D, \dashv, \vdash)$ be a dialgebra. A Rota-Baxter operator on D consists of a linear map $R : D \to D$ satisfying

$$R(a) * R(b) = R(R(a) * b + a * R(b))$$

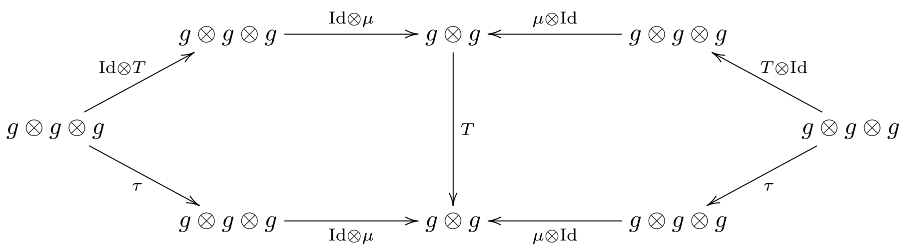
for all $a, b \in D$ and $* = \dashv, \vdash$.

The following proposition is easy to check.

Proposition 3.8. *Let $(D, \dashv, \vdash)$ be a dialgebra and R be a Rota-Baxter operator on it. Then R is a Rota-Baxter operator on the induced Leibniz algebra $(D, [\cdot, \cdot]_D)$.*

The Leibniz bracket $[\cdot, \cdot]_R$ induced from a Rota-Baxter operator R can be understood in terms of the weak pseudotwistor on a Leibniz algebra.

Definition 11. Let $(g, [\cdot, \cdot]_g)$ be a Leibniz algebra with the Leibniz bracket denoted by the product μ. A linear map $T : g \otimes g \to g \otimes g$ is said to be a weak pseudotwistor if there exist a linear map $\tau : g \otimes g \otimes g \to g \otimes g \otimes g$ with $(\eta_{12} \otimes \mathrm{Id}) \circ \tau = \tau \circ (\eta_{12} \otimes \mathrm{Id})$ and commuting the following diagram:



Here $\eta_{12} : g \otimes g \to g \otimes g$ is the flip map $\eta_{12}(x \otimes y) = y \otimes x$. The map τ is called a weak companion of T.

Proposition 3.9. *Let $(g, [\cdot, \cdot]_g)$ be a Leibniz algebra and $T : g \otimes g \to g \otimes g$ be a weak pseudotwistor. Then $(g, \mu \circ T)$ is a new Leibniz algebra structure on g.*

Proof. We have

$$\begin{aligned}
&(\mu \circ T) \circ (\mathrm{Id} \otimes (\mu \circ T)) \\
&= \mu \circ (\mathrm{Id} \otimes \mu) \circ \tau \\
&= \mu \circ (\mu \otimes \mathrm{Id}) \circ \tau + \mu \circ (\mathrm{Id} \otimes \mu) \circ (\eta_{12} \otimes \mathrm{Id}) \circ \tau \\
&= (\mu \circ T) \circ ((\mu \circ T) \otimes \mathrm{Id}) + \mu \circ (\mathrm{Id} \otimes \mu) \circ \tau \circ (\eta_{12} \otimes \mathrm{Id}) \\
&= (\mu \circ T) \circ ((\mu \circ T) \otimes \mathrm{Id}) + (\mu \circ T) \circ (\mathrm{Id} \otimes (\mu \circ T)) \circ (\eta_{12} \otimes \mathrm{Id}).
\end{aligned}$$

This shows that $\mu \circ T$ defines a Leibniz bracket on g. □

Proposition 3.10. *Let $(g, [\cdot,\cdot]_g)$ be a Leibniz algebra and $R : g \to g$ be a Rota-Baxter operator on it. Then the map $T : g \otimes g \to g \otimes g$ defined by*

$$T(x \otimes y) = R(x) \otimes y + x \otimes R(y)$$

is a weak pseudotwistor on g. Consequently, g carries a new Leibniz algebra structure with bracket $[x, y]_R = [Rx, y]_g + [x, Ry]_g$ for $x, y \in g$.

Proof. We define $\tau : g \otimes g \otimes g \to g \otimes g \otimes g$ by

$$\begin{aligned}
\tau(x \otimes y \otimes z) = {}& R(x) \otimes R(y) \otimes z + R(x) \otimes y \otimes R(z) \\
&+ x \otimes R(y) \otimes R(z) \quad \text{for } x, y, z \in g.
\end{aligned}$$

We will show that T is a weak pseudotwistor with a weak companion τ. First, observe that

$$\begin{aligned}
&((\eta_{12} \otimes \mathrm{Id}) \circ \tau)(x \otimes y \otimes z) \\
&= R(y) \otimes R(x) \otimes z + y \otimes R(x) \otimes R(z) + R(y) \otimes x \otimes R(z) \\
&= \tau(y \otimes x \otimes z) = (\tau \circ (\eta_{12} \otimes \mathrm{Id}))(x \otimes y \otimes z).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&(T \circ (\mathrm{Id} \otimes \mu \circ T))(x \otimes y \otimes z) \\
&= R(x) \otimes \mu(R(y) \otimes z + y \otimes R(z)) + x \otimes \mu(R(y) \otimes R(z)) \\
&= ((\mathrm{Id} \otimes \mu) \circ \tau)(x \otimes y \otimes z).
\end{aligned}$$

Similarly, we have

$$T \circ ((\mu \circ T) \otimes \mathrm{Id}) = (\mu \otimes \mathrm{Id}) \circ \tau.$$

Hence, the result follows. □

Remark 3.11. The notion of weak pseudotwistor on a Leibniz algebra is a generalization of weak pseudotwistor on an associative algebra introduced by Panaite and Oystaeyen [23]. In the associative context, a Rota-Baxter system induces a weak pseudotwistor on the underlying associative algebra. It is remarked that given a Leibniz algebra $(g, [\cdot,\cdot]_g)$ and a Rota-Baxter system (R, S) on g, the map

$$T : g \otimes g \to g \otimes g, \ T(x \otimes y) = R(x) \otimes y + x \otimes S(y)$$

is not a weak pseudotwistor on g with weak companion

$$\tau(x\otimes y\otimes z)=R(x)\otimes R(y)\otimes z+R(x)\otimes y\otimes S(z)+x\otimes S(y)\otimes S(z)$$

as $(\eta_{12}\otimes \mathrm{Id})\circ\tau\neq\tau\circ(\eta_{12}\otimes\mathrm{Id})$.

4. Maurer-Cartan characterization of relative Rota-Baxter systems

In the section, we construct a graded Lie algebra that characterizes relative Rota-Baxter systems as Maurer-Cartan elements. Using this characterization, we define the cohomology associated with a relative Rota-Baxter system. We first recall some results from [2].

A permutation $\sigma\in\mathbb{S}_n$ is called an $(i,n-i)$-shuffle if $\sigma(1)<\cdots<\sigma(i)$ and $\sigma(i+1)<\cdots<\sigma(n)$. If $i=0$ or n we assume $\sigma=\mathrm{Id}$. The set of all $(i,n-i)$-shuffles will be denoted by $\mathbb{S}_{(i,n-i)}$.

Let M be a vector space. We consider the graded vector space

$$C^*(M,M)=\oplus_{n\geq 1}C^n(M,M)=\oplus_{n\geq 1}\mathrm{Hom}(\otimes^n M,M)$$

of multilinear maps on M. The Balavoine bracket is a degree -1 bracket on the graded vector space $C^*(M,M)$ given by

$$[f,g]_B:=f\bar{\circ}g-(-1)^{pq}g\bar{\circ}f$$

for $f\in C^{p+1}(M,M)$, $g\in C^{q+1}(M,M)$. Here $f\bar{\circ}g\in C^{p+q+1}(M,M)$ is defined by

$$f\bar{\circ}g=\sum_{k=1}^{p+1}(-1)^{(k-1)q}f\circ_k g$$

with

$$\begin{aligned}&(f\circ_k g)(x_1,\dots,x_{p+q+1})\\=&\sum_{\sigma\in\mathbb{S}_{(k-1,q)}}(-1)^{\sigma}f(x_{\sigma(1)},\dots,x_{\sigma(k-1)},g(x_{\sigma(k)},\dots,x_{\sigma(k+q-1)},x_{k+q}),x_{k+q+1},\dots,x_{p+q+1}).\end{aligned}$$

Theorem 4.1 ([2]). *With the above notations, $(C^*(M,M),[\cdot,\cdot]_B)$ is a degree -1 graded Lie algebra. In other words $(C^{*+1}(M,M),[\cdot,\cdot]_B)$ is a graded Lie algebra. Its Maurer-Cartan elements are precisely the Leibniz algebra structures on M.*

Let (V,ρ^L,ρ^R) be a representation of a Leibniz algebra $(g,[\cdot,\cdot]_g)$. Consider the semidirect product Leibniz algebra structure on $g\oplus g\oplus V$. We denote the corresponding Leibniz product by $\widehat{\mu}$. Then $\widehat{\mu}$ is a Maurer-Cartan element in the graded Lie algebra $(C^{*+1}(g\oplus g\oplus V,g\oplus g\oplus V),[\cdot,\cdot]_B)$.

Consider the graded vector subspace $C^*(V,g)\subset C^*(g\oplus g\oplus V,g\oplus g\oplus V)$ given by

$$C^*(V,g):=\oplus_{n\geq 1}C^n(V,g):=\oplus_{n\geq 1}\mathrm{Hom}(V^{\otimes n},g\oplus g).$$

Theorem 4.2. *With the above notations, $(C^*(V,g),[[\cdot,\cdot]])$ is a graded Lie algebra, where the graded Lie bracket $[[\cdot,\cdot]]: C^m(V,g)\times C^n(V,g)\to C^{m+n}(V,g)$ is defined by*

$$[[(P,Q),(P',Q')]] := (-1)^m[[\widehat{\mu},(P,Q)]_B,(P',Q')]_B$$

for any $(P,Q)\in C^m(V,g)$, $(P',Q')\in C^n(V,g)$. Moreover, its Maurer-Cartan elements are relative Rota-Baxter systems on the Leibniz algebra $(g,[\cdot,\cdot]_g)$ with respect to the representation (V,ρ^L,ρ^R).

Let $Pr_1, Pr_2 : g\oplus g\to g$ denote the projection maps onto the first and second factor, respectively. Then the explicit description of the above graded Lie bracket is given by

$$\begin{aligned}
& Pr_1([[(P,Q),(P',Q')]](v_1,\ldots,v_{m+n}))\\
={}& \sum_{k=1}^{m}\sum_{\sigma\in\mathbb{S}_{(k-1,n)}}(-1)^{(k-1)n}(-1)^{\sigma}P(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},\rho^L(P'(v_{\sigma(k)},\ldots,v_{\sigma(k+n-1)}))v_{k+n},\ldots,v_{m+n})\\
&+\sum_{k=2}^{m}\sum_{\sigma\in\mathbb{S}_{(k-2,n,1)}}(-1)^{kn}(-1)^{\sigma}P(v_{\sigma(1)},\ldots,v_{\sigma(k-2)},\rho^R(Q'(v_{\sigma(k)},\ldots,v_{\sigma(k+n-2)}))v_{\sigma(k+n-1)},\\
&\qquad v_{k+n},\ldots,v_{m+n})\\
&+\sum_{k=1}^{n}\sum_{\sigma\in\mathbb{S}_{(k-1,m)}}(-1)^{(k+n-1)m}(-1)^{\sigma}P'(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},\rho^L(P(v_{\sigma(k)},\ldots,v_{\sigma(k+m-1)}))v_{\sigma(k+m)},\\
&\qquad \ldots,v_{m+n})\\
&\sum_{k=1}^{n}\sum_{\substack{\sigma\in\mathbb{S}_{(k-1,m,1)},\\ \sigma(k+m-1)=k+m}}(-1)^{(k+n-1)m+1}(-1)^{\sigma}P'(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},\rho^R(Q(v_{\sigma(k)},\ldots,v_{\sigma(k-1+m)}))v_{\sigma(k+m)},\\
&\qquad v_{k+m+1},\ldots,v_{m+n})\\
&+\sum_{\sigma\in\mathbb{S}_{(m,n-1)}}(-1)^{mn+1}(-1)^{\sigma}[P(v_{\sigma(1)},\ldots,v_{\sigma(m)}),P'(v_{\sigma(m+1)},\ldots,v_{\sigma(m+n-1)},v_{m+n}]_g\\
&+\sum_{k=1}^{m}\sum_{\sigma\in\mathbb{S}_{(k-1,n-1)}}(-1)^{(k-1)n}(-1)^{\sigma}[P'(v_{\sigma(k)},\ldots,v_{\sigma(k+n-2)}),P(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},v_{k+n},\ldots,v_{m+n})]_g
\end{aligned}$$

and

$$\begin{aligned}
& Pr_2([[(P,Q),(P',Q')]](v_1,\ldots,v_{m+n}))\\
={}& \sum_{k=1}^{m}\sum_{\sigma\in\mathbb{S}_{(k-1,n)}}(-1)^{(k-1)n}(-1)^{\sigma}Q(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},\rho^L(P'(v_{\sigma(k)},\ldots,v_{\sigma(k+n-1)}))v_{k+n},\ldots,v_{m+n})\\
&+\sum_{k=2}^{m}\sum_{\sigma\in\mathbb{S}_{(k-2,n,1)}}(-1)^{kn}(-1)^{\sigma}Q(v_{\sigma(1)},\ldots,v_{\sigma(k-2)},\rho^R(Q'(v_{\sigma(k)},\ldots,v_{\sigma(k+n-2)}))v_{\sigma(k+n-1)},v_{k+n},\\
&\qquad \ldots,v_{m+n})\\
&+\sum_{k=1}^{n}\sum_{\sigma\in\mathbb{S}_{(k-1,m)}}(-1)^{(k+n-1)m}(-1)^{\sigma}Q'(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},\rho^L(P(v_{\sigma(k)},\ldots,v_{\sigma(k+m-1)}))v_{\sigma(k+m)},\\
&\qquad \ldots,v_{m+n})
\end{aligned}$$

$$\sum_{k=1}^{n} \sum_{\substack{\sigma\in\mathbb{S}_{(k-1,m,1)},\\ \sigma(k+m-1)=k+m}} (-1)^{(k+n-1)m+1}(-1)^{\sigma}Q'(v_{\sigma(1)},\dots,v_{\sigma(k-1)},\rho^R(Q(v_{\sigma(k)},\dots,v_{\sigma(k-1+m)}))v_{\sigma(k+m)},$$
$$v_{k+m+1},\dots,v_{m+n})$$
$$+\sum_{\sigma\in\mathbb{S}_{(m,n-1)}} (-1)^{mn+1}(-1)^{\sigma}[Q(v_{\sigma(1)},\dots,v_{\sigma(m)}),Q'(v_{\sigma(m+1)},\dots,v_{\sigma(m+n-1)},v_{m+n}]_g$$
$$+\sum_{k=1}^{m}\sum_{\sigma\in\mathbb{S}_{(k-1,n-1)}} (-1)^{(k-1)n}(-1)^{\sigma}[Q'(v_{\sigma(k)},\dots,v_{\sigma(k+n-2)}),Q(v_{\sigma(1)},\dots,v_{\sigma(k-1)},v_{k+n},\dots,v_{m+n})]_g$$

for any $(P,Q)\in C^m(V,g)$, $(P',Q')\in C^n(V,g)$.

Proof. The graded Lie algebra $(C^*(V,g),[[\cdot,\cdot]])$ is obtained via the derived bracket [28]. First, consider the graded Lie algebra $(C^{*+1}(g\oplus g\oplus V,g\oplus g\oplus V),[\cdot,\cdot]_B)$. Since $\widehat{\mu}$ is the semidirect product Leibniz algebra structure on the vector space $g\oplus g\oplus V$, we deduce that $(C^{*+1}(g\oplus g\oplus V,g\oplus g\oplus V),[\cdot,\cdot]_B,d=[\widehat{\mu},\cdot]_B)$ is a differential graded Lie algebra. Obviously $C^{*+1}(V,g)$ is an abelian subalgebra. Therefore, by the derived bracket construction, we define a bracket on the shifted graded vector space $C^*(V,g)$ by

$$[[(P,Q),(P',Q']]:=(-1)^m[d((P,Q)),(P',Q')]_B=(-1)^m[[\widehat{\mu},(P,Q)],(P',Q')]$$

for any $(P,Q)\in C^m(V,g)$, $(P',Q')\in C^n(V,g)$. The derived bracket $[[\cdot,\cdot]]$ is closed on $C^*(V,g)$, which implies that $(C^*(V,g),[[\cdot,\cdot]])$ is a graded Lie algebra.

For $(R,S)\in C^1(V,g)$, we have

$$Pr_1([[(R,S),(R,S)]](u,v))=2([Ru,Rv]_g-R(\rho^L(Ru)v)-R(\rho^R(Sv)u)),$$
$$Pr_2([[(R,S),(R,S)]](u,v))=2([Su,Sv]_g-S(\rho^L(Ru)v)-S(\rho^R(Sv)u)).$$

Thus, (R,S) is a Maurer-Cartan element (i.e., $[[(R,S),(R,S)]]=0$) if and only if (R,S) is a relative Rota-Baxter systems on g with respect to the representation (V,ρ^L,ρ^R). The proof is finished. □

Thus, relative Rota-Baxter systems can be characterized as Maurer-Cartan elements in a graded Lie algebra. It follows from the above theorem that if (R,S) is a relative Rota-Baxter system, then $d_{(R,S)}:=[[(R,S),\cdot]]$ is a differential on $C^*(V,g)$ and makes the gLa $(C^\bullet(V,g),[[\cdot,\cdot]])$ into a differential graded Lie algebra.

The cohomology of the cochain complex $(C^\bullet(V,g),d_{(R,S)})$ is called the cohomology of the relative Rota-Baxter system (R,S). We denote the corresponding cohomology groups simply by $H^\bullet(V,g)$.

The following theorem describes the Maurer-Cartan deformation of a relative Rota-Baxter system.

Theorem 4.3. *Let (R,S) be a relative Rota-Baxter system on a Leibniz algebra $(g,[\cdot,\cdot]_g)$ with respect to a representation (V,ρ^L,ρ^R). For any pair (R',S') of linear maps from V to g, the pair of sums $(R+R',S+S')$ is a relative*

Rota-Baxter system if and only if (R', S') *is a Maurer-Cartan element in the differential graded Lie algebra* $(C^*(V,g), [[\cdot,\cdot]], d_{(R,S)})$*, i.e.,*

$$[[(R+R', S+S'), (R+R', S+S')]] = 0$$
$$\Leftrightarrow d_{(R,S)}(R', S') + \frac{1}{2}[[(R', S'), (R', S')]] = 0.$$

5. Deformations of relative Rota-Baxter systems

5.1. Formal deformations

Let $\mathbb{K}[[t]]$ be the ring of power series in one variable t. For any $\mathbb{K}$-linear space V, let $V[[t]]$ denote the vector space of formal power series in t with coefficients from V. If in addition, $(g, [\cdot,\cdot]_g)$ is a Leibniz algebra over $\mathbb{K}$, then there is a $\mathbb{K}[[t]]$-Leibniz algebra structure on $g[[t]]$ given by

$$[\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j]_g = \sum_{k=0}^{+\infty} \sum_{i+j=k} [x_i, y_j] t^k \quad \text{for all } x_i, y_j \in g.$$

Let (V, ρ^L, ρ^R) be a representation of the Leibniz algebra $(g, [\cdot,\cdot]_g)$. Then there is a representation $(V[[t]], \rho^L, \rho^R)$ of the $\mathbb{K}[[t]]$-Leibniz algebra $g[[t]]$. Here ρ^L and ρ^R are given by

$$\rho^L(\sum_{i=0}^{+\infty} x_i t^i)(\sum_{j=0}^{+\infty} v_j t^j) = \sum_{k=0}^{+\infty} \sum_{i+j=k} \rho^L(x_i)(v_j) t^k,$$
$$\rho^R(\sum_{i=0}^{+\infty} x_i t^i)(\sum_{j=0}^{+\infty} v_j t^j) = \sum_{k=0}^{+\infty} \sum_{i+j=k} \rho^R(x_i)(v_j) t^k \quad \text{for all } x_i \in g, v_j \in V.$$

Let (R, S) be a relative Rota-Baxter system on the Leibniz algebra $(g, [\cdot,\cdot]_g)$ with respect to the representation (V, ρ^L, ρ^R). We consider two power series

$$R_t = \sum_{i=0}^{+\infty} \mathfrak{R}_i t^i \text{ and } S_t = \sum_{j=0}^{+\infty} \mathfrak{S}_j t^j, \text{ where } \mathfrak{R}_i, \mathfrak{S}_j \in \mathrm{Hom}_{\mathbb{K}}(V, g).$$

That is, both R_t and S_t are in $\mathrm{Hom}_{\mathbb{K}}(V,g)[[t]]$. Extend them to $\mathbb{K}[[t]]$-linear maps from $V[[t]]$ to $g[[t]]$. We still denote them by the same symbols.

Definition 12. If $R_t = \sum_{i=0}^{+\infty} \mathfrak{R}_i t^i$ and $S_t = \sum_{j=0}^{+\infty} \mathfrak{S}_j t^j$ with $\mathfrak{R}_0 = R$, $\mathfrak{S}_0 = S$ satisfy

$$[R_t u, R_t v]_g = R_t(\rho^L(R_t u)v + \rho^R(S_t v)u),$$
$$[S_t u, S_t v]_g = S_t(\rho^L(R_t u)v + \rho^R(S_t v)u)),$$

we say that (R_t, S_t) is a formal deformation of the relative Rota-Baxter system (R, S).

By expanding these equations and comparing coefficients of various powers of t, we obtain for $k \geq 0$,

$$\sum_{k=0}^{+\infty} \sum_{i+j=k} [\mathfrak{R}_i u, \mathfrak{R}_j v]_g = \sum_{k=0}^{+\infty} \sum_{i+j=k} \mathfrak{R}_i(\rho^L(\mathfrak{R}_j u)v + \rho^R(\mathfrak{S}_j v)u),$$

$$\sum_{k=0}^{+\infty} \sum_{i+j=k} [\mathfrak{S}_i u, \mathfrak{S}_j v]_g = \sum_{k=0}^{+\infty} \sum_{i+j=k} \mathfrak{S}_i(\rho^L(\mathfrak{R}_j u)v + \rho^R(\mathfrak{S}_j v)u)).$$

Both of these identities hold for $k = 0$ as (R, S) is a relative Rota-Baxter system. For $k = 1$, we get

$$[Ru, \mathfrak{R}_1 v]_g + [\mathfrak{R}_1 u, Rv]_g = \mathfrak{R}_1(\rho^L(Ru)v + \rho^R(Sv)u) + R(\rho^L(\mathfrak{R}_1 u)v + \rho^R(\mathfrak{S}_1 v)u),$$

$$[Su, \mathfrak{S}_1 v]_g + [\mathcal{S}_1 u, Sv]_g = \mathfrak{S}_1(\rho^L(Ru)v + \rho^R(Sv)u) + S(\rho^L(\mathfrak{R}_1 u)v + \rho^R(\mathfrak{S}_1 v)u$$

for $u, v \in V$. These identities are equivalent to the single condition

$$[[(R, S), (\mathfrak{R}_1, \mathfrak{S}_1)]] = 0.$$

As a consequence, we get the following.

Proposition 5.1. *Let $(R_t = \sum_{i=0}^{+\infty} \mathfrak{R}_i t^i, S_t = \sum_{j=0}^{+\infty} \mathfrak{S}_j t^j)$ be a formal deformation of a relative Rota-Baxter system (R, S) on the Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation (V, ρ^L, ρ^R). Then $(\mathfrak{R}_1, \mathfrak{S}_1)$ is a 1-cocycle in the cohomology of the relative Rota-Baxter system (R, S), that is, $d_{(R,S)}(\mathfrak{R}_1, \mathfrak{S}_1) = 0$.*

Definition 13. Let (R, S) be a relative Rota-Baxter system on the Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation (V, ρ^L, ρ^R). The 1-cocycle $(\mathfrak{R}_1, \mathfrak{S}_1)$ is called the infinitesimal of the formal deformation $(R_t = \sum_{i=0}^{+\infty} \mathfrak{R}_i t^i, S_t = \sum_{j=0}^{+\infty} \mathfrak{S}_j t^j)$ of the relative Rota-Baxter system (R, S).

Definition 14. Two formal deformations (R_t, S_t) and (R'_t, S'_t) of a relative Rota-Baxter system (R, S) on the Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation (V, ρ^L, ρ^R) are said to be equivalent if there exist two elements $x, y \in g$ and linear maps $\phi_i, \varphi_i \in gl(g)$ and $\psi_i \in gl(V)$ for $i \geq 2$ such that for

$$\phi_t = Id_g + t(L_x - R_x) + \sum_{i=2}^{+\infty} \phi_i t^i, \ \varphi_t = Id_g + t(L_y - R_y) + \sum_{i=2}^{+\infty} \varphi_i t^i$$

$$\text{and} \quad \psi_t = Id_V + t(\rho^L(x) - \rho^R(y)) + \sum_{i=2}^{+\infty} \psi_i t^i,$$

the following conditions hold:

(i) $[\phi_t(z), \phi_t(w)]_g = \phi_t([z, w]_g), \ [\varphi_t(z), \varphi_t(w)]_g = \varphi_t([z, w]_g)$;
(ii) $\psi_t(\rho^L(z)u) = \rho^L(\phi_t(z))\psi_t(u)$;
(iii) $\psi_t(\rho^R(z)u) = \rho^R(\varphi_t(z))\psi_t(u)$;
(iv) $R'_t \circ \psi_t(u) = \phi_t \circ R_t(u), \ S'_t \circ \psi_t(u) = \varphi_t \circ S_t(u)$

for all $z, w \in g$ and $u \in V$.

By expanding the identities in (iv) and equating coefficients of t from both sides, we obtain

$$\begin{aligned}
&(\mathfrak{R}_1, \mathfrak{S}_1)(u) - (\mathfrak{R}'_1, \mathfrak{S}'_1)(u) \\
&= [R(u), x]_g - R(\rho^R(y)u) - [x, R(u)]_g + R(\rho^L(x)u) \\
&\quad + [S(u), y]_g - S(\rho^R(y)u) - [y, S(u)]_g + S(\rho^L(x)u) \\
&= (d_{(R,S)}(x, y))(u).
\end{aligned}$$

Thus, we have the following.

Theorem 5.2. *The cohomology class of the infinitesimal of a formal deformation depends only on the equivalence class of the deformation.*

5.2. Finite order deformations of a relative Rota-Baxter system

In this subsection, we introduce a cohomology class associated to any order n deformation of a relative Rota-Baxter system, and show that an order n deformation is extensible if and only if this cohomology class is trivial. Thus, we call this cohomology class the obstruction class of the order n deformation being extensible.

Definition 15. Let (R, S) be a relative Rota-Baxter system on a Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation (V, ρ^L, ρ^R). If the finite sums

$$R_t = \sum_{i=0}^{n} \mathfrak{R}_i t^i \quad \text{and} \quad S_t = \sum_{j=0}^{n} \mathfrak{S}_j t^j \quad \text{with} \quad \mathfrak{R}_0 = R,\ \mathfrak{S}_0 = S$$

as $\mathbb{K}[[t]]/(t^{n+1})$-module maps from $V[[t]]/(t^{n+1})$ to the Leibniz algebra $g[[t]]/(t^{n+1})$ satisfy

$$\begin{aligned}
[R_t u, R_t v]_g &= R_t(\rho^L(R_t u)v + \rho^R(S_t v)u), \\
[S_t u, S_t v]_g &= S_t(\rho^L(R_t u)v + \rho^R(S_t v)u)) \quad \text{for } u, v \in V,
\end{aligned}$$

we say that (R_t, S_t) is an order n deformation of the relative Rota-Baxter system (R, S).

Definition 16. Let (R_t, S_t) be an order n deformation of the relative Rota-Baxter system (R, S) on a Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation (V, ρ^L, ρ^R). If there exists a pair $(\mathfrak{R}_{n+1}, \mathfrak{S}_{n+1})$ of linear maps from V to g such that

$$(\widehat{R}_t = R_t + t^{n+1}\mathfrak{R}_{n+1}, \widehat{S}_t = S_t + t^{n+1}\mathfrak{S}_{n+1})$$

is a deformation of order $n+1$, we say that (R_t, S_t) is extensible.

Let (R_t, S_t) be an order n deformation of the relative Rota-Baxter system (R, S) on a Leibniz algebra $(g, [\cdot, \cdot]_g)$ with respect to a representation

(V,ρ^L,ρ^R). Define an element $Ob_{(R_t,S_t)}\in C^2(V,g)$ by

$$Ob_{(R_t,S_t)} = -\frac{1}{2}\sum_{i+j=n+1,i,j\geq 1}[[(\mathfrak{R}_i,\mathfrak{S}_i),(\mathfrak{R}_j,\mathfrak{S}_j)]]. \tag{2}$$

Proposition 5.3. *The* 2*-cochain* $Ob_{(R_t,S_t)}$ *is a* 2*-cocycle, that is,*

$$d_{(R,S)}(Ob_{(R_t,S_t)}) = 0.$$

Proof. We have

$$\begin{aligned}
&d_{(R,S)}(Ob_{(R_t,S_t)})\\
=&-\frac{1}{2}\sum_{i+j=n+1,i,j\geq 1}[[(R,S),[[(\mathfrak{R}_i,\mathfrak{S}_i),(\mathfrak{R}_j,\mathfrak{S}_j)]]]]\\
=&-\frac{1}{2}\sum_{i+j=n+1,i,j\geq 1}([[[[(R,S),(\mathfrak{R}_i,\mathfrak{S}_i)]],(\mathfrak{R}_j,\mathfrak{S}_j)]]\\
&\qquad\qquad -[[(\mathfrak{R}_i,\mathfrak{S}_i),[[(R,S),(\mathfrak{R}_j,\mathfrak{S}_j)]]]])\\
=&\frac{1}{4}\sum_{i_1+i_2+j=n,i_1,i_2,j\geq 1}[[[[(\mathfrak{R}_{i_1},\mathfrak{S}_{i_1}),(\mathfrak{R}_{i_2},\mathfrak{S}_{i_2})]],(\mathfrak{R}_j,\mathfrak{S}_j)]]\\
&-\frac{1}{4}\sum_{i+j_1+j_2=n,i,j_1,j_2\geq 1}[[(\mathfrak{R}_i,\mathfrak{S}_i),[[(\mathfrak{R}_{j_1},\mathfrak{S}_{j_1}),(\mathfrak{R}_{j_2},\mathfrak{S}_{j_2})]]]]\\
=&\frac{1}{2}\sum_{i+j+k=n+1,i,j,k\geq 1}[[[[(\mathfrak{R}_i,\mathfrak{S}_i),(\mathfrak{R}_j,\mathfrak{S}_j)]],(\mathfrak{R}_k,\mathfrak{S}_k)]]\\
=&\,0.
\end{aligned}$$

The proof is finished. □

Definition 17. Let (R_t,S_t) be an order n deformation of the relative Rota-Baxter system (R,S) on a Leibniz algebra $(g,[\cdot,\cdot]_g)$ with respect to a representation (V,ρ^L,ρ^R). The cohomology class $[Ob_{(R_t,S_t)}]\in H^2(V,g)$ is called the obstruction class for (R_t,S_t) being extensible.

As a consequence of Eq. (2) and Proposition 5.3, we obtain the following.

Theorem 5.4. *Let (R_t,S_t) be an order n deformation of the relative Rota-Baxter system (R,S) on a Leibniz algebra $(g,[\cdot,\cdot]_g)$ with respect to a representation (V,ρ^L,ρ^R). Then (R_t,S_t) is extensible if and only if the obstruction class $[Ob_{(R_t,S_t)}]$ is trivial.*

Corollary 5.5. *If $H^2(V,g)=0$, then every* 1*-cocycle in the cohomology of a relative Rota-Baxter system (R,S) is the infinitesimal of some formal deformation of (R,S).*

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