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GENERALIZED INTERTWINING LINEAR OPERATORS WITH ISOMETRIES

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Abstract. In this paper, we show that for an isometry on a Banach space the analytic spectral subspace coincides with the algebraic spectral subspace. Using this result, we have the following result. Let T be a bounded linear operator with property (δ) on a Banach space X. And let S be an isometry on a Banach space Y. Then every generalized intertwining linear operator $\theta : X \to Y$ for (S, T) is continuous if and only if the pair (S, T) has no critical eigenvalue.

1. Preliminaries

Throughout this paper we shall use the standard notions and some basic results on the theory of local spectral theory. Let X be a Banach space over the complex plane $\mathbb{C}$ and let $L(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X. Given an operator $T \in L(X)$, $\mathrm{Lat}(T)$ denotes the collection of all closed T-invariant linear subspaces of X, and for an $Y \in \mathrm{Lat}(T)$, $T|Y$ denotes the restriction of T on Y, and $\sigma(T)$, $\rho(T)$ denote the spectrum and the resolvent set of T, respectively.

Definition 1.1. *Let $T : X \to X$ be a linear operator on a Banach space X. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces Y of X which satisfy $(T - \lambda)Y = Y$ for all $\lambda \notin F$, let $E_T(F)$ denote the algebraic linear span of all such subspaces Y of X. Then $E_T(F)$ is called an algebraic spectral subspace of T.*

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We may define $E_T(F)$ as maximal among all linear subspaces Y of X for which $(T-\lambda)Y = Y$ for which $\lambda \notin F$. In general, the space $E_T(F)$ need not to be closed.

A linear subspace Z of X is called a *T-divisible subspace* if

$$(T-\lambda)Z = Z \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence $E_T(\emptyset)$ is precisely the largest T-divisible subspace.

Many important operators do not have non trivial divisible subspaces. For example, hyponormal operators on Hilbert spaces do not have non-trivial divisible subspaces.

In the next proposition, we collect a number of results on algebraic spectral subspaces. These results are found in [10].

PROPOSITION 1.2. *Let T be a linear operator on a Banach space X and let $F \subseteq \mathbb{C}$. Then the following assertions hold:*
(1) $E_T(F)$ is a hyperinvariant subspace.
(2) $E_T(F) = E_T(F \cap \sigma(T))$.
(3) If $x \in X$ satisfies $(T-\lambda)x \in E_T(F)$ for some $\lambda \in F$, then $x \in E_T(F)$.
(4) $E_T(\bigcap F_\alpha) = \bigcap E_T(F_\alpha)$ for any family of subsets $\{F_\alpha : \alpha \in A\}$ of $\mathbb{C}$.

The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T-\lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \to X$ defined by $f(\lambda) = (T-\lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies $(T-\lambda)f(\lambda) = x$ for all $\lambda \in \rho(T)$. Hence the resolvent set $\rho(T)$ is always a subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always a subset of $\sigma(T)$. The analytic solutions occurring in the definition of the local resolvent set may be thought as local extensions of the function $(T-\lambda)^{-1}x : \rho(T) \to X$. There is no uniqueness implied. Thus we need the following definition. An operator $T \in L(X)$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. Hence if T has the SVEP, then for each $x \in X$ there is the maximal analytic extension of $(T-\lambda)^{-1}x$ on $\rho_T(x)$. For a closed subset F of $\mathbb{C}$, $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ is said to be an *analytic* *spectral subspace* of T. It is easy to see that $X_T(F)$ is a hyperinvariant subspace of X, which generally is not closed. Analytic spectral subspaces have been fundamental in the recent progress of local spectral theory, for instance, in connection with functional models and invariant subspaces.

In the next proposition, we collect a number of results on analytic spectral subspaces. These results are found in [10].

PROPOSITION 1.3. *Let T be a bounded linear operator on a Banach space X and let $F \subseteq \mathbb{C}$. Then the following assertions hold:*
(1) $X_T(F) = X_T(F \cap \sigma(T))$.
(2) For all $\lambda \notin F$, $(T-\lambda)X_T(F) = X_T(F)$. That is, $X_T(F) \subseteq E_T(F)$.
(3) If $\{F_\alpha\}$ is a family of subsets of $\mathbb{C}$, then $X_T(\bigcap F_\alpha) = \bigcap X_T(F_\alpha)$.
(4) T has the SVEP if and only if $X_T(\emptyset) = \{0\}$.

An operator $T \in L(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$\sigma(T|Y) \subseteq U, \ \ \sigma(T|Z) \subseteq V \ \text{ and } \ Y + Z = X.$$

Decomposable operators are rich. For example, normal operators, spectral operators in the sense of Dunford, operators with totally disconnected spectrums and hence compact operators are decomposable.

An operator $T \in L(X)$ is said to have *property (δ)* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, and for each $x \in X$ there exist a pair of analytic functions $f : \mathbb{C} \setminus \overline{U} \to X$, $g : \mathbb{C} \setminus \overline{V} \to X$ such that

$$(T-\lambda)f(\lambda) = u, \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \overline{U},$$
$$(T-\lambda)g(\lambda) = v, \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \overline{V},$$

and

$$x = u + v.$$

If T has the SVEP, then property (δ) simply means that

$$X = X_T(\overline{U}) + X_T(\overline{V})$$

for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$. Albrecht and Eschmeier showed that T satisfies (δ) if and only if T is similar to a quotient of a decomposable operator. That is, if $T : X \to Y$ has property (δ) then there exist a Banach space $\widehat{X}$ and a continuous linear surjection $q : \widehat{X} \to X$ and a decomposable operator $\widehat{T} \in L(\widehat{X})$ with $Tq = q\widehat{T}$ [3].

Let θ be a linear operator from a Banach space X into a Banach space Y. The space

$$\mathfrak{S}(\theta) = \{y \in Y : \text{ there is a sequence } x_n \to 0 \text{ in } X \text{ and } \theta x_n \to y\}$$

is called the *separating space* of θ. It is easy to see that $\mathfrak{S}(\theta)$ is a closed linear subspace of Y. By the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$. The following lemma is found in [12].

Lemma 1.4. *Let X and Y be Banach spaces. If R is a continuous linear operator from Y to a Banach space Z, and if $\theta : X \to Y$ is a linear operator, then $(R\mathfrak{S}(\theta))^- = \mathfrak{S}(R\theta)$. In particular, $R\theta$ is continuous if and only if $R\mathfrak{S}(\theta) = \{0\}$.*

The next lemma states that a certain descending sequence of separating space which obtained from θ via a countable family of continuous linear operators is eventually constant. It is proved in [12].

Lemma 1.5 (Stability Lemma). *Let $\theta : X_0 \to Y$ be a linear operator between the Banach spaces X_0 and Y with separating space $\mathfrak{S}(\theta)$, and let $\langle X_i : i = 1, 2, \ldots \rangle$ be a sequence of Banach spaces. If each $T_i : X_i \to X_{i-1}$ is continuous linear operator for $i = 1, 2, \ldots,$ then there is an $n_0 \in \mathbb{N}$ for which $\mathfrak{S}(\theta T_1 T_2 \ldots T_n) = \mathfrak{S}(\theta T_1 T_2 \ldots T_{n_0})$ for all $n \geq n_0$.*

Given a topological space Ω and a topological vector space X, we denote by $\mathfrak{F}(\Omega)$ the collection of all closed subsets of Ω, and by $S(X)$ the collection of all closed linear subspaces of X. A mapping $\mathcal{E}(\cdot) : \mathfrak{F}(\Omega) \to S(X)$ is said to be a *precapacity* if $\mathcal{E}(\emptyset) = \{0\}$ and $\mathcal{E}(F) \subseteq \mathcal{E}(G)$ for all closed sets $F, G \subseteq \Omega$ with $F \subseteq G$. Given a precapacity $\mathcal{E}(\cdot) : \mathfrak{F}(\Omega) \to S(X)$, we say that $\mathcal{E}(\cdot)$ is *decomposable* if

$$X = \mathcal{E}(\overline{U}) + \mathcal{E}(\overline{V}) \quad \text{for every open cover } \ \{U, V\} \ \text{ of } \ \Omega,$$

and that $\mathcal{E}(\cdot)$ is *stable* if arbitrary intersections are preserved, that is,

$$\mathcal{E}(\bigcap F_\alpha) = \bigcap \mathcal{E}(F_\alpha)$$

for every family of closed subsets $\{F_\alpha : \alpha \in A\}$ of Ω. A stable map is called a *spectral capacity* if $\mathcal{E}(\cdot)$ satisfies the following condition:

$$X = \sum_\alpha \mathcal{E}(\overline{G_\alpha}) \ \text{ for every finite open cover } \ \{G_\alpha : \alpha \in A\} \ \text{ of } \ \mathbb{C}.$$

If Ω is second countable, then it follows easily from Lindelöf's covering theorem that a precapacity is stable whenever intersections of countable families of closed sets are preserved. We say that $\mathcal{E}(\cdot)$ is *order preserving* if it preserves the inclusion order. Clearly a stable map is order preserving. It is well known that T is decomposable if and only if there exists a spectral capacity $\mathcal{E}(\cdot)$ such that $\mathcal{E}(F) \in \operatorname{Lat}(T)$ and $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each closed set $F \subseteq \mathbb{C}$. In this case the spectral capacity of a closed subset F of $\mathbb{C}$ is uniquely determined and it is the analytic spectral subspace $X_T(F)$.

The following lemma, known as *localization of the singularities*, has appeared in various form. We adopted in [8].

Proposition 1.6. *Let X and Y be Banach spaces. Suppose that $\mathcal{E}_X : \mathcal{F}(\mathbb{C}) \to S(X)$ is an order preserving map such that $X = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$ whenever $\{U, V\}$ is an open cover of $\mathbb{C}$. And suppose that $\mathcal{E}_Y : \mathcal{F}(\mathbb{C}) \to S(Y)$ is a stable map. If $\theta : X \to Y$ is a linear operator for which $\mathfrak{S}(\theta|\mathcal{E}_X(F)) \subseteq \mathcal{E}_Y(F)$ for every $F \in \mathcal{F}(\mathbb{C})$, then there is a finite set $\Lambda \subseteq \mathbb{C}$ for which $\mathfrak{S}(\theta) \subseteq \mathcal{E}_Y(\Lambda)$.*

The following theorem is a variation of the Mittag-Leffler Theorem of Bourbaki. The theorem is found in [5].

Theorem 1.7 (Mittag-Leffler Theorem). *Let $\langle X_n : n = 0, 1, 2, \ldots \rangle$ be a sequence of complete metric spaces, and for $n = 1, 2, \ldots$, let $f_n : X_n \to X_{n-1}$ be a continuous map with $f_n(X_n)$ dense in X_{n-1}. Let $g_n = f_1 \circ \cdots \circ f_n$. Then $\bigcap_{n=1}^{\infty} g_n(X_n)$ is dense in X_0.*

2. Generalized intertwining linear operators with isometry

We denote by $\mathrm{C}^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, x_1, $x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. That is, with the topology generated by a family of pseudo-norm

$$|\varphi|_{K,m} = \max_{|p| \leq m} \sup_{z \in K} |D^p \varphi(z)|,$$

where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, p_1, $p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p \varphi = \frac{\partial^{|p|} \varphi}{\partial x_1{}^{p_1} \partial x_2{}^{p_2}}, \quad z = x_1 + ix_2.$$

An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : \mathrm{C}^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators was introduced by Colojoară and Foiaş [4]. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators.

It is well known that if T is invertible isometry then T is a generalized scalar operator. For a generalized scalar operator it is well known that $X_T(F) = E_T(F)$ for all closed sets $F \subseteq \mathbb{C}$.

The following proposition is in [10].

Proposition 2.1. *Let T be a bounded linear operator on a Banach space X. Suppose that $E_T(F)$ is closed for all closed sets $F \subseteq \mathbb{C}$. Then the identity $X_T(F) = E_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$*

Proposition 2.2. *Let T be an isometry on a Banach space X. Then for any closed set F of $\mathbb{C}$,*

$$X_T(F) = E_T(F).$$

Proof. If T is an invertible isometry, then T is a generalized scalar operator. Hence The identity $X_T(F) = E_T(F)$ holds for any closed set F of $\mathbb{C}$. Thus we may assume that T is a noninvertible isometry. By Proposition 2.1, it is enough to show that $E_T(F)$ is closed for any closed set F of $\mathbb{C}$. Let $F \subseteq \mathbb{C}$ be a given closed set. Suppose that there is a $\lambda \notin F$ with $|\lambda| < 1$. If $E_T(F) = \{0\}$ then the space $E_T(F)$ is closed. Hence we may assume that $E_T(F)$ is nontrivial. Let $W = \overline{E_T(F)}$. Since $T - \lambda$ is bounded below, $(T - \lambda)(W)$ is closed. Therefore, we have

$$(T - \lambda)(W) = W.$$

Hence $(T - \lambda)|W$ is invertible. And hence $\lambda \notin \sigma(T|W)$. It is well known that for the spectrum of a noninvertible isometry is the entire unit disk. Since $|\lambda| < 1$, $T|W$ can not be a noninvertible isometry. Hence $T|W$ is an invertible isometry. Thus $E_{T|W}(F)$ is closed in W. Since W is closed, $E_{T|W}(F)$ is closed in X. It is clear that

$$E_{T|W}(F) = E_T(F) \cap W = E_T(F).$$

Therefore, $E_T(F)$ is closed in X. If there is no $\lambda \notin F$ with $|\lambda| < 1$, then $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq F$. Since T is noninvertible isometry,

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subseteq F.$$

Therefore we have $E_T(F) = X$. Hence $E_T(F)$ is closed in X.
In any case $E_T(F)$ is closed for all closed $F \subseteq \mathbb{C}$. □

Hence for an isometry T, the above proposition allows us to combine the analytic tools associated with the space $X_T(F)$ with the algebraic tools associated with the space $E_T(F)$. Since an isometry has the single-valued extension property, $X_T(\emptyset) = E_T(\emptyset) = \{0\}$. Hence, if T is an isometry then T has no non-trivial divisible subspaces.

Let T and S be bounded linear operators on Banach spaces X and Y, respectively. A linear operator $\theta : X \to Y$ is said to be an *intertwining linear operator* for the pair (S, T) if $S\theta = \theta T$. Let $C(S, T)$ denote the commutator, $C(S, T)\theta = S\theta - \theta T$. For a natural number n, define $C(S, T)^n$ to be the n-th composition. That is,

$$C(S,T)^n\theta = C(S,T)^{n-1}(S\theta - \theta T) = \sum_{k=0}^{n} \binom{n}{k} S^k\theta(-T)^{n-k}.$$

Then we shall say that θ is a *generalized intertwining linear operator* for (S, T) if

$$\|C(S,T)^n\theta\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

For this to make sense, $C(S, T)^n\theta$ is continuous for some n, hence for all sufficiently large n, is assumed.

The following lemma is found in [11].

Lemma 2.3. *Let T and S be bounded linear operators on Banach spaces X and Y, respectively. And let $\theta : X \to Y$ be a linear operator. If $F \subseteq \mathbb{C}$ satisfies $C(S, T)^n\theta E_T(F) \subseteq E_S(F)$ for some $n \in \mathbb{N}$, then actually we have*

$$\theta E_T(F) \subseteq E_S(F).$$

Proposition 2.4. *Let T be a bounded linear operator on a Banach space X. And let S be an isometry on a Banach space Y. Then every generalized intertwining linear operator $\theta : X \to Y$ for (S, T) necessarily satisfies the following:*

$$\theta X_T(F) \subseteq Y_S(F) \quad \text{for all closed subsets } F \text{ of } \mathbb{C}.$$

Proof. Since θ is a generalized intertwining linear operator for (S, T), there is $k \in \mathbb{N}$ such that $C(S, T)^k\theta$ is continuous. By the assumption we have

$$\|C(S,T)^n C(S,T)^k\theta\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we may apply the proof of [4, Theorem 2.3.3] (this theorem remains valid if the operator S on the range space Y is only assumed to have the single valued extension property and closed space $Y_S(F)$ for all closed $F \subseteq \mathbb{C}$, this condition is certainly fulfilled in the case of an isometry). Then for a given closed set $F \subseteq \mathbb{C}$ we have

$$C(S,T)^k \theta X_T(F) \subseteq Y_S(F).$$

Let $R = T|X_T(F)$ and consider $E_R(F)$. Since $R-\lambda$ is surjective for any $\lambda \notin F$,

$$X_T(F) = E_R(F).$$

Hence by the assumption,

$$C(S,T)^k \theta E_R(F) = C(S,T)^k \theta|X_T(F)(E_R(F)) \subseteq Y_S(F) \subseteq E_S(F),$$

so that by the above lemma we have

$$\theta|X_T(F)(E_R(F)) \subseteq E_S(F).$$

Since S is isometry, $Y_S(F) = E_S(F)$, by Proposition 2.2, that is

$$\theta X_T(F) \subseteq Y_S(F).$$

This completes the proof. □

Let $T \in L(X)$ and $S \in L(Y)$. A complex number $\lambda \in \mathbb{C}$ is said to be a *critical eigenvalue* for the pair (S,T) if $(T-\lambda)X$ is of infinite codimension in X and λ is an eigenvalue of S.

The following proposition is well known and tells us existence of discontinuous intertwining linear operators for (S,T). The proof of next proposition is in [12]

PROPOSITION 2.5. *Let T and S be bounded linear operators on Banach spaces X and Y, respectively. If (S,T) has a critical eigenvalue, then there is a discontinuous linear operator $\theta : X \to Y$ with $S\theta = \theta T$.*

Now we state and prove the following main theorem.

THEOREM 2.6. *Let T be a bounded linear operator with property (δ) on a Banach space X. And let S be an isometry on a Banach space Y. Then the following statements are equivalent:*
(a) Every generalized intertwining linear operator $\theta : X \to Y$ for (S,T) is necessarily continuous.
(b) The pair (S,T) has no critical eigenvalues.

Proof. (a) $\Rightarrow$ (b) By Proposition 2.5, it is clear.
(b) $\Rightarrow$ (a) Assume that the condition (b) is fulfilled, and consider an arbitrary generalized intertwining linear operator $\theta : X \to Y$ for (S, T). To prove the continuity of θ, it suffices to construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Indeed if we do so, all injective factors $S - \lambda$ of $p(S)$ may be removed from $p(S)$; what is left still annihilate $\mathfrak{S}(\theta)$. Thus we have obtained a polynomial p, all of whose roots are eigenvalues of S. Since θ is a generalized intertwining linear operator for (S, T), there is a natural number n such that $C(S, T)^n\theta$ is continuous.

(case 1) $n = 1$

Suppose that $C(S, T)\theta$ is continuous, And suppose that there is a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$ and all roots of p are eigenvalues of S. Since $S\theta - \theta T$ is continuous, $p(S)\theta - \theta p(T)$ is also continuous. By the assumption $p(S)\theta$ is continuous, so we have $\theta p(T)$ is continuous. Let λ be a root of p. Since (S, T) has no critical eigenvalues, $(T - \lambda)X$ is of finite codimension in X. This means that $p(T)X$ has finite codimension in X. Hence the open mapping theorem implies that $p(T)X$ is closed and that $p(T)$ is an open mapping from X onto $p(T)X$. Therefore, θ is continuous.

(case 2) $n > 1$

Suppose that $C(S, T)^n\theta$ is continuous for $n > 1$. And suppose that there is a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$ and all roots of p are eigenvalues of S. We define

$$\theta_k = C(S, T)^{n-k}\theta$$

for $k = 0, 1, \dots, n$. Then $\theta_0 = C(S, T)^n\theta$ is continuous and $\theta_n = \theta$. Moreover, for all polynomial q and all $k = 1, \dots, n$, we have

$$\theta_{k-1}q(T) = C(S, T)(\theta_k q(T)) \quad \text{and} \quad p(S)\mathfrak{S}(\theta_k q(T)) = \{0\},$$

since the continuity of $p(S)\theta$ obviously forces $p(S)\theta_k q(T)$ to be continuous as well. Hence we may successively apply the proof of (case 1) of this case to obtain polynomials $p_1, p_2, \dots p_n$ whose roots are all eigenvalues of S and $\theta_k p_1(T) \cdots p_k(T)$ is continuous for $k = 1, \dots, n$. Let $r = p_1 \cdots p_n$. Then all roots of r are eigenvalues of S. Hence $\theta r(T)$ is continuous. Since the pair (S, T) has no critical eigenvalues, $r(T)X$ has finite codimension in X. By the open mapping theorem $r(T)X$ is closed in X and $r(T)$ is an open mapping from X onto $r(T)X$. Therefore, θ is continuous.

In any case, if we have a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$, then the continuity of θ is ensured. Now, we will construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Since T has property (δ), there is a Banach space $\widehat{X}$ and a continuous linear surjection $q : \widehat{X} \to X$ and a decomposable operator $\widehat{T} \in L(\widehat{X})$ with $Tq = q\widehat{T}$. Hence it is clear that

$$(C(S,T)^n\theta)q = C(S,\widehat{T})^n(\theta q).$$

Since $\theta : X \to Y$ is a generalized intertwining linear operator for (S,T), $\theta q : \widehat{X} \to Y$ is a generalized intertwining linear operator for $(S,\widehat{T})$. Let $\widehat{\theta} = \theta q$. Then we observe that it suffices to consider the case that $C(S,\widehat{T})\widehat{\theta}$ is continuous: indeed, the general case can be easily deduced by this special case and the argument of the proof of (case 2). Since $\widehat{T}$ has the single-valued property and S is isometry, from Proposition 2.4, we infer that

$$\widehat{\theta}(\widehat{X}_{\widehat{T}}(F)) \subseteq Y_S(F)$$

for all closed subsets F of $\mathbb{C}$. Since $\widehat{X}_{\widehat{T}}(F)$ is the spectral capacity and $Y_S(F)$ is stable, by Proposition 1.6, there is a finite set Λ of $\mathbb{C}$ such that $\mathfrak{S}(\theta) \subseteq Y_S(\Lambda)$. An application of the Stability Lemma to the sequence $S - \lambda$ for $\lambda \in \Lambda$ yields a polynomial p for which

$$((S-\lambda)p(S)\mathfrak{S}(\widehat{\theta}))^- = (p(S)\mathfrak{S}(\widehat{\theta}))^- \quad \text{for every } \lambda \in \Lambda.$$

Applying Mittag-Leffler Theorem, there exists a dense subspace

$$W \subseteq (p(S)\mathfrak{S}(\widehat{\theta}))^-$$

for which $(S-\lambda)W = W$ for every $\lambda \in \Lambda$. This means that

$$W \subseteq E_S(\mathbb{C} \setminus \Lambda)$$

by the definition of algebraic spectral subspaces. From the continuity of $C(S,\widehat{T})\widehat{\theta}$ we deduce that $p(S)\mathfrak{S}(\widehat{\theta}) \subseteq \mathfrak{S}(\widehat{\theta})$ Hence $W \subseteq \mathfrak{S}(\widehat{\theta}) \subseteq E_S(\Lambda)$, and we obtain that

$$W \subseteq E_S(\Lambda) \cap E_S(\mathbb{C} \setminus \Lambda) = E_S(\emptyset).$$

Since S is an isometry, S has no non-trivial divisible subspace, we have $W = \{0\}$. Consequently, $p(S)\widehat{\theta} = \{0\}$. Hence $p(S)\widehat{\theta} = p(S)\theta q$ is continuous. Since q is a continuous linear surjection, by the open mapping theorem, $p(S)\theta$ is also continuous. Therefore, θ is continuous. $\square$

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