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BERNSTIEN AND TURÁN TYPE INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. The goal of this paper is to extend some inequalities of Bernstein as well as Turán type to polar derivative of a polynomial.

1. Introduction and Preliminaries

According to a well-known classical result due to Bernstein [3, 9], if $p(z)$ is a polynomial of degree n, then concerning the estimate of the maximum of $|p^{'}(z)|$ on the unit circle $|z| = 1$,

$$\max_{|z|=1} |p^{'}(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

A simple deduction from the Maximum Modulus Principle [10] for the estimate of $|p(z)|$ on a larger circle $|z| = R > 1$, we have

$$\max_{|z|=R>1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

Both (1.1) and (1.2) are sharp and equality hold if $p(z)$ has all its zeros at the origin.

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It was proved by Frappier et al. [5] that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p^{'}(z)| \leq n \max_{1\leq k\leq n}|p(e^{ik\pi n})|. \tag{1.3}$$

It is evident that inequality (1.3) is a refinement of (1.1), since the maximum of $|p(z)|$ on $|z|=1$ may be larger than the maximum of $|p(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simple example $p(z)=z^n+ia, a>0$.

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then Erdös conjectured and later Lax [7] proved that

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.4}$$

On the other hand, when $p(z)$ has all its zeros in $|z|\leq 1$, Turán [12] proved

$$\max_{|z|=1}|p^{'}(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.5}$$

As a generalization of inequality (1.5), Dubinin [4] proved that if $p(z)=\sum_{j=0}^{n}c_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, then for $|z|=1$

$$\max_{|z|=1}|p^{'}(z)| \geq \frac{1}{2}\left\{n+\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right\}\max_{|z|=1}|p(z)|. \tag{1.6}$$

As a refinement of inequality (1.4) analogous to (1.3) Aziz [2] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for every real α

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{2}(M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.7}$$

where

$$M_\alpha = \max_{1\leq k\leq n}|p(e^{i(\alpha+2k\pi)n})| \tag{1.8}$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$. Further, under the same hypotheses, Aziz [2] proved for $R>1$

$$\max_{|z|=1}|p(Rz)-p(z)| \leq \frac{R^n-1}{2}(M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.9}$$

where M_α is as defined in (1.8).

As a refinement of inequality (1.7), Wali and Shah [13] proved the following theorem.

Theorem 1.1. *If* $p(z) = c_n \prod_{j=1}^{n}(z - z_j)$ *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for every given real* α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2}\Big\{(M_\alpha^2 + M_{\alpha+\pi}^2) - \frac{2}{n}\Big(\sum_{j=1}^{n} \frac{|z_j| - 1}{|z_j| + 1}\Big)|p(z)|^2\Big\}^{\frac{1}{2}}, \tag{1.10}$$

where M_α *is as defined in* (1.8).

As a generalization of inequality (1.6), in the same paper, Wali and Shah [13] proved:

Theorem 1.2. *If* $p(z) = c_n \prod_{j=1}^{n}(z - z_j)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq 1$, *then*

$$\max_{|z|=1} |p'(z)| \geq \frac{1}{2}\Big[n + \sum_{j=1}^{n} \frac{1 - |z_j|}{1 + |z_j|}\Big] \max_{|z|=1} |p(z)|. \tag{1.11}$$

We now define for a polynomial $p(z)$ of degree n, the polar derivative of $p(z)$ with respect to a real or complex number β as

$$D_\beta p(z) = np(z) + (\beta - z)p'(z).$$

This polynomial $D_\beta p(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\beta \to \infty} \frac{D_\beta p(z)}{\beta} = p'(z).$$

Aziz [1] was among the first who extended some of the above inequalities to polar versions by replacing the derivative of the polynomial with the polar derivative of the polynomial. He, in fact, extended inequality (1.4) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every real or complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta p(z)| \leq \frac{n}{2}(|\beta| + 1) \max_{|z|=1} |p(z)|. \tag{1.12}$$

Dividing both sides of (1.12) by $|\beta|$ and letting $|\beta| \to \infty$, we get inequality (1.4).

Shah [11] extended Turán's inequality (1.5) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every real or complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta p(z)| \geq \frac{n}{2}(|\beta| - 1) \max_{|z|=1} |p(z)|. \tag{1.13}$$

Over the last four decades many different authors produced a large number of results concerning the polar derivative of polynomials. More information on classical results and polar derivatives can be found in the books of Milovanović et al. [9] and Marden [8].

2. Main results

It is clearly of interest to find the corresponding extensions of Theorems 1.1 and 1.2 to polar derivative of the polynomial.

In this paper, we first extend Theorem 1.1 to polar derivative. For the proof of the theorems, we require the following lemmas.

The next lemma is a special case of a result due to Govil and Rahman [6].

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$,*

$$|p^{'}(z)| + |q^{'}(z)| \le n \max_{|z|=1} |p(z)|, \tag{2.1}$$

where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$.

The following lemma is due to Aziz [2].

Lemma 2.2. *If $p(z)$ is a polynomial of degree n and $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then for $|z| = 1$ and for every real α,*

$$|p^{'}(z)|^2 + |q^{'}(z)|^2 \le \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2), \tag{2.2}$$

where M_α is as define in (1.8).

Theorem 2.3. *If $p(z) = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every given real α and for every real or complex number β with $|\beta| \ge 1$,*

$$\begin{aligned}\max_{|z|=1} |D_\beta p(z)| \le n \max_{|z|=1} |p(z)| + (|\beta| - 1)\Bigg[\frac{n}{2}\Bigg\{(M_\alpha^2 + M_{\alpha+\pi}^2) \\ - \frac{2}{n}\sum_{j=1}^{n}\left(\frac{|z_j| - 1}{|z_j| + 1}\right)|p(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg],\end{aligned} \tag{2.3}$$

where M_α is as defined in (1.8).

Proof. If $p(z)$ is a polynomial of degree n, then for $|z| = 1$,

$$|q^{'}(z)| = |np(z) - zp^{'}(z)|, \tag{2.4}$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

Now, for every real or complex number β, the polar derivative of $p(z)$ with respect to β is

$$D_\beta p(z) = np(z) + (\beta - z)p^{'}(z),$$

which further implies for $|z| = 1$,

$$\begin{aligned} |D_\beta p(z)| &\leq |np(z) - zp^{'}(z)| + |\beta||p^{'}(z)| \\ &= |q^{'}(z)| + |\beta||p^{'}(z)| \text{ by } (2.4) \\ &\leq n\max_{|z|=1}|p(z)| + (|\beta| - 1)|p^{'}(z)|. \end{aligned} \tag{2.5}$$

If $p(z) = 0$ for $|z| = 1$, then the result follows from Theorem 1.1. So we assume that $p(z) \neq 0$ for $|z| = 1$. Since $p(z) = c_n\prod_{j=1}^{n}(z - z_j)$, $|z_j| \geq 1$, we have

$$Re\frac{zp^{'}(z)}{p(z)} = Re\sum_{j=1}^{n}\frac{z}{z - z_j}.$$

Hence, we have, for $z \neq z_j$ on $|z| = 1$,

$$Re\frac{z}{z - z_j} \leq \frac{1}{1 + |z_j|}.$$

Thus,

$$Re\frac{zp^{'}(z)}{p(z)} \leq \sum_{j=1}^{n}\frac{1}{1 + |z_j|}. \tag{2.6}$$

Also, equality (2.4) for $|z| = 1$, gives

$$\begin{aligned} \left|\frac{z(q^{'}(z))}{p(z)}\right|^2 &= \left|n - z\frac{p^{'}(z)}{p(z)}\right|^2 \\ &= n^2 + \left|\frac{zp^{'}(z)}{p(z)}\right|^2 - 2nRe\Big(\frac{zp^{'}(z)}{p(z)}\Big) \\ &\geq n^2 + \left|\frac{zp^{'}(z)}{p(z)}\right|^2 - 2n\Big(\sum_{j=1}^{n}\frac{1}{1 + |z_j|}\Big), \end{aligned}$$

$$|q^{'}(z)|^2 \geq n^2|p(z)|^2 + |zp^{'}(z)|^2 - 2n|p(z)|^2\Big(\sum_{j=1}^{n}\frac{1}{1 + |z_j|}\Big)$$

or

$$2|p^{'}(z)|^2 \leq |p^{'}(z)|^2 + |q^{'}(z)|^2 - n\Big\{n - 2\sum_{j=1}^{n}\frac{1}{1 + |z_j|}\Big\}|p(z)|^2. \tag{2.7}$$

Now using Lemma 2.2 to (2.7) we get,

$$\begin{aligned} 2|p^{'}(z)|^2 &\leq \frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2)-n\Big\{n-2\sum_{j=1}^{n}\frac{1}{1+|z_j|}\Big\}|p(z)|^2 \\ &= \frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2)-2n\Big\{\frac{n}{2}-\sum_{j=1}^{n}\frac{1}{1+|z_j|}\Big\}|p(z)|^2 \\ &= \frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2)-2n\Big\{\sum_{j=1}^{n}\Big(\frac{1}{2}-\frac{1}{1+|z_j|}\Big)\Big\}|p(z)|^2 \\ &\leq \frac{n^2}{2}\Big\{(M_\alpha^2+M_{\alpha+\pi}^2)-\frac{2}{n}\sum_{j=1}^{n}\Big(\frac{|z_j|-1}{|z_j|+1}\Big)|p(z)|^2\Big\} \end{aligned}$$

or

$$|p^{'}(z)|\leq\frac{n}{2}\Big\{(M_\alpha^2+M_{\alpha+\pi}^2)-\frac{2}{n}\sum_{j=1}^{n}\left(\frac{|z_j|-1}{|z_j|+1}\right)|p(z)|^2\Big\}^{\frac{1}{2}}. \tag{2.8}$$

Combining inequalities (2.5) and (2.8), we get the desired result. This completes the proof of Theorem 2.3. □

Remark 2.4. Dividing both sides of (2.3) by $|\beta|$ and letting $|\beta|\to\infty$, we get inequality (1.10) of Theorem 1.1.

Further, we extend Theorem 1.2 to polar derivative analogue.

Theorem 2.5. *If $p(z)=c_n\prod_{j=1}^{n}(z-z_j)$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, then for $|z|=1$ and for every real or complex number β with $|\beta|\geq 1$,*

$$\max_{|z|=1}|D_\beta p(z)|\geq\left[(|\beta|-1)\left\{\frac{n}{2}+\frac{1}{2}\sum_{j=1}^{n}\frac{1-|z_j|}{1+|z_j|}\right\}-n\right]\max_{|z|=1}|p(z)|. \tag{2.9}$$

Proof. We have, on $|z|=1$

$$\begin{aligned} |D_\beta p(z)| &= \left|np(z)+(\beta-z)p^{'}(z)\right| \\ &\geq \left|n\,|p(z)|+|(\beta-z)|\left|p^{'}(z)\right|\right| \end{aligned}$$

or equivalently

$$\begin{aligned} |D_\beta p(z)| &\geq |(\beta-z)|\left|p^{'}(z)\right|-n\,|p(z)| \\ &\geq (|\beta|-1)\left|p^{'}(z)\right|-n\,|p(z)|\,. \end{aligned} \tag{2.10}$$

Now, since $p(z)$ has all its zeros in $|z| \leq 1$, we can write

$$p(z) = \sum_{j=1}^{n} c_j z^j = c_n \prod_{j=1}^{n} (z - z_j),$$

where $|z_j| \leq 1$, for all $j = 1, 2, ..., n$. Thus,

$$Re \frac{zp^{'}(z)}{p(z)} = Re \sum_{j=1}^{n} \frac{z}{z - z_j}.$$

Since $|z_j| \leq 1$, we have similarly as in the proof of Theorem 2.3, for $|z| = 1$

$$Re \frac{z}{z - z_j} \geq \frac{1}{1 + |z_j|}.$$

This implies

$$\begin{aligned} \left| \frac{p^{'}(z)}{p(z)} \right| &\geq Re \left(\frac{zp^{'}(z)}{p(z)} \right) \\ &= \frac{n}{2} + \sum_{j=1}^{n} \left(-\frac{1}{2} + Re \frac{z}{z - z_j} \right) \\ &\geq \frac{n}{2} + \sum_{j=1}^{n} \left(-\frac{1}{2} + \frac{1}{1 + |z_j|} \right) \\ &= \frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \frac{1 - |z_j|}{1 + |z_j|} \end{aligned} \tag{2.11}$$

or equivalently, for $|z| = 1$

$$\left| p^{'}(z) \right| \geq \left(\frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \frac{1 - |z_j|}{1 + |z_j|} \right) |p(z)|. \tag{2.12}$$

From (2.10) and (2.12), we have for $|z| = 1$

$$|D_\beta p(z)| \geq (|\beta| - 1) \left(\frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \frac{1 - |z_j|}{1 + |z_j|} \right) |p(z)| - n|p(z)|.$$

This completes the proof. □

Remark 2.6. Dividing both sides of (2.9) by $|\beta|$ and letting $|\beta| \to \infty$, we get inequality (1.11) of Theorem 1.2.

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