# APPROXIMATION METHODS FOR SOLVING SPLIT EQUALITY OF VARIATIONAL INEQUALITY AND $f, g$-FIXED POINT PROBLEMS IN REFLEXIVE BANACH SPACES 

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#### Abstract

The purpose of this paper is to introduce and study a method for solving the split equality of variational inequality and $f, g$-fixed point problems in reflexive real Banach spaces, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the fixed point problems are for Bregman relatively $f, g$-nonexpansive mappings. A strong convergence theorem is proved under some mild conditions. Finally, a numerical example is provided to demonstrate the effectiveness of the algorithm.


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## 1. Introduction

Let $E$ be a real Banach space with dual $E^{*}$. Let $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$ be the generalized duality pairing and the induced norm, respectively. If $C$ is a nonempty, closed and convex subset of $E$, then the mapping $A: C \rightarrow E^{*}$ is said to be
(a) $\alpha$-strongly monotone on $C$ if there exists $\alpha>0$ such that

$$
\langle A x-A y, x-y\rangle \geq \alpha\|x-y\|^{2},
$$

for all $x, y \in C$;
(b) monotone on $C$ if

$$
\langle A x-A y, x-y\rangle \geq 0,
$$

for all $x, y \in C$;
(c) $\alpha$ - inverse strongly monotone if there exists $\alpha>0$ such that

$$
\langle A x-A y, x-y\rangle \geq \alpha\|A x-A y\|^{2},
$$

for all $x, y \in C$;
(d) $\alpha$-strongly pseudomonotone on $C$ if there exists $\alpha>0$ such that

$$
\langle A x, y-x\rangle \geq 0 \Longrightarrow\langle A y, y-x\rangle \geq \alpha\|x-y\|^{2}
$$

for all $x, y \in C$;
(e) pseudomonotone on $C$ if

$$
\langle A x, y-x\rangle \geq 0 \Longrightarrow\langle A y, y-x\rangle \geq 0,
$$

for all $x, y \in C$;
(f) $L$ - Lipschitz continuous on $C$ if there exists a constant $L>0$, called the Lipschitz constant, such that

$$
\|A x-A y\| \leq L\|x-y\|
$$

for all $x, y \in C$;
(g) If $L<1$, then $A$ is called a contraction and if $L=1$, then $A$ is said to be nonexpansive;
(h) The mapping $A$ is said to be sequentially weakly continuous if $\left\{A x_{n}\right\}$ converges weakly to $A x$ whenever $\left\{x_{n}\right\}$ is a sequence that converges weakly to $x$.

A point $x \in C$ is called a fixed point of the mapping $G: C \rightarrow E$ if $G x=x$. The set of fixed points of $G$ is denoted by $F(G)$.
Remark 1.1. Every $\alpha$-strongly monotone mapping is monotone and hence pseudomonotone. It is also easy to see that every $\alpha$-strongly monotone mapping is $\alpha$-strongly pseudomonotone.

The variational inequality problem (VIP, in short) is defined as finding a point $x \in C$ such that

$$
\begin{equation*}
\langle A x, y-x\rangle \geq 0 \quad \text { for all } y \in C \tag{1.1}
\end{equation*}
$$

where $C$ is a nonempty, closed and convex subset of $E$ and $A: C \rightarrow E^{*}$ is a mapping. The solution set of the variational inequality problem $\operatorname{VIP}(1.1)$ is denoted by $V I(C, A)$. The concept of variational inequality problem was initially introduced by Hartman and Stampacchia [13] as a natural generalization of boundary value problems. Such problems are applicable in a wide range of applied sciences and mathematics. It helps us to solve new problems that emerge from the fields of applied mathematics, engineering, physics, mechanics, convex programming and the theory of control.

Many authors have studied and proposed different methods for solving VIP(1.1) in different settings (see, for instance, $[2,8,10,11,14,22,29,30,36$, 37] and the references therein).

If $H$ is a real Hilbert space and $A: H \rightarrow H$ is a Lipschitz continuous and strongly monotone, then the projected gradient method introduced by Goldstein [12] is the simplest method to solve (1.1).

In 2020, Thong et al. [28] proposed the following projection type algorithm to solve (1.1) in a Hilbert space setting. Given $l \in(0,1), \mu>0, \beta \in\left(0, \frac{1}{\mu}\right)$ and $x_{1} \in C$, compute:

$$
\left\{\begin{array}{l}
s_{n}=P_{C}\left[x_{n}-\beta A x_{n}\right]  \tag{1.2}\\
w_{n}=x_{n}-\gamma_{n} r_{\beta}\left(x_{n}\right) \\
x_{n+1}=\alpha_{n} f\left(x_{n}\right)+\left(1-\alpha_{n}\right) P_{C_{n}}\left(x_{n}\right)
\end{array}\right.
$$

where $P_{C}$ is the metric projection from $H$ onto $C ; A$ is uniformly continuous pseudomonotone mapping that is sequentially weakly continuous on bounded subsets of $C ; f: C \rightarrow C$ is a contraction mapping with a coefficient $\delta \in[0,1)$; $r_{\beta}\left(x_{n}\right)=x_{n}-s_{n} ;\left\{\alpha_{n}\right\}$ is a sequence of real numbers in $(0,1)$ such that $\lim _{n \rightarrow \infty} \alpha_{n}=0$ and $\sum_{n=1}^{\infty} \alpha_{n}=\infty ; \gamma_{n}=l^{j_{n}}$, where $j_{n}$ is the smallest nonnegative integer $j$ that satisfies

$$
\left\langle A x_{n}-A\left(x_{n}-l^{j} r_{\beta}\left(x_{n}\right)\right), r_{\beta}\left(x_{n}\right)\right\rangle \leq \mu\left\|r_{\beta}\left(x_{n}\right)\right\|^{2},
$$

$C_{n}=\left\{x \in C: h_{n}(x) \leq 0\right\}$ and $h_{n}(x)=\left\langle A w_{n}, x-w_{n}\right\rangle$. They proved that if $V I(C, A) \neq \emptyset$, then the sequence generated by (1.2) converges strongly to some $x \in V I(C, A)$, where $x=P_{V I(C, A)} f(x)$.

In Banach spaces, more general than Hilbert spaces, Jolaoso and Shehu [16] introduced the following single Bregman projection method to solve variational inequality problems (see [33]): Let $C$ be a nonempty, closed and convex subset of $E$. Given $u_{1} \in E, \lambda_{1}>0$ and $\rho \in(0, \alpha)$, compute

$$
\begin{equation*}
u_{n+1}=\nabla f^{*}\left(\nabla f\left(z_{n}\right)-\lambda_{n}\left(A z_{n}-A u_{n}\right)\right), \tag{1.3}
\end{equation*}
$$

$$
\lambda_{n+1}= \begin{cases}\min \left\{\lambda_{n}, \frac{\rho\left\|u_{n}-z_{n}\right\|}{\left\|A u_{n}-A z_{n}\right\|}\right\}, & \text { if } A u_{n} \neq A z_{n} \\ \lambda_{n} & \text { otherwise }\end{cases}
$$

where $z_{n}=P_{C}^{f}\left(\nabla f^{*}\left(\nabla f\left(u_{n}\right)-\lambda_{n} A u_{n}\right)\right), A: E \rightarrow E^{*}$ is a pseudomonotone, sequentially weakly continuous and Lipschitz continuous mapping and $f: E \rightarrow$ $\mathbb{R}$ is a proper, lower semi-continuous, uniformly Fréchet differentiable, $\alpha-$ strongly convex, strongly coercive and Legendre function which is bounded. They proved that the sequence generated by (1.3) converges weakly to some point in $V I(C, A)$ in a reflexive real Banach space $E$ provided that $V I(C, A) \neq$ $\emptyset$. They also proved the strong convergence of the algorithm to a point of $V I(C, A)$ if, in addition, $A$ is strongly pseudomonotone.

Besides these, several authors have proposed and studied different schemes for finding a common point of the solution set of variational inequality and fixed point problems (see, for example, [24, 27, 31, 34]). This method became very important in optimization theory because it is applicable in mathematical models whose constraints can be modeled as both problems.

Takahashi and Toyoda [27] introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an inverse strongly monotone mapping in Hilbert spaces. Given a nonempty, closed and convex subset $D$ of $H$ and $x_{0} \in D$. Let $\left\{x_{n}\right\}$ be the sequence generated by

$$
\begin{equation*}
x_{n+1}=\alpha_{n} x_{n}+\left(1-\alpha_{n}\right) G P_{D}\left(x_{n}-\eta_{n} A x_{n}\right), \tag{1.4}
\end{equation*}
$$

where $n$ is a nonnegative integer, $A: D \rightarrow H$ is $\beta$ - inverse strongly monotone mapping, $G: D \rightarrow H$ is a nonexpansive mapping, $\left\{\alpha_{n}\right\} \subset(0,1)$ and $\left\{\eta_{n}\right\} \subset$ $(0,2 \beta)$. They proved that if $\operatorname{VI}(D, A) \cap F(G) \neq \emptyset$, then the sequence $\left\{x_{n}\right\}$ generated by (1.4) converges weakly to an element $p \in V I(D, A) \cap F(G)$.

In 2021, Wega and Zegeye [34] studied a method of approximating a common element of the set of $f$-fixed points of a Bregman relatively $f$-nonexpansive mapping and the set of solutions of a variational inequality problem for a Lipschitz monotone mapping in a reflexive real Banach space. They introduced the following algorithm to find a point in $V I(K, B) \cap F_{f}(G)$. For a reflexive Banach space $E$ with its dual $E^{*}$, let $g: E \rightarrow(-\infty, \infty]$ be a strongly coercive, bounded, $\lambda$ - strongly convex on bounded subsets of $E$, uniformly Fréchet differentiable Legendre function. Let $K$ be a nonempty, closed and convex subset of $E$. Let $B: K \rightarrow E^{*}$ be a Lipschitz monotone mapping with Lipschitz constant $L$. Let $G: K \rightarrow E^{*}$ be a Bregman relatively $f$-nonexpansive mapping with $\Gamma=V I(K, B) \cap F_{f}(G) \neq \emptyset$. Given $x_{0}, x \in K$, let the sequence
$\left\{x_{n}\right\}$ be generated by

$$
\left\{\begin{array}{l}
s_{n}=P_{K}^{g} \nabla g^{*}\left(\nabla g x_{n}-\eta_{n} B x_{n}\right),  \tag{1.5}\\
t_{n}=\nabla g^{*}\left[\xi_{n} \nabla g x_{n}+\beta_{n} G x_{n}+\tau_{n} \nabla g w_{n}\right], \\
x_{n+1}=P_{K}^{g} \nabla g^{*}\left(\alpha_{n} \nabla g(x)+\left(1-\alpha_{n}\right) \nabla g t_{n}\right),
\end{array}\right.
$$

where $P_{K}^{g}(x)$ is the Bregman projection of $x \in \operatorname{int}(\operatorname{dom} g)$ onto $K, w_{n}=$ $P_{K}^{g} \nabla g^{*}\left(\nabla g x_{n}-\eta_{n} B s_{n}\right), 0<\underline{\eta} \leq \eta_{n} \leq \bar{\eta}<\frac{\lambda}{L},\left\{\alpha_{n}\right\}$ is a sequence in $(0,1)$, $\left\{\xi_{n}\right\},\left\{\beta_{n}\right\},\left\{\tau_{n}\right\} \subset[\delta, 1) \subset(0,1)$ such that $\xi_{n}+\beta_{n}+\tau_{n}=1$. They proved that the sequence $\left\{x_{n}\right\}$ generated by (1.5) converges strongly to some element $\bar{x}$, where $\bar{x}=P_{\Gamma}^{g}(x)$.

One of the generalizations of the variational inequality problem is the split equality variational inequality problem (SEVIP) which is defined as finding a point

$$
\begin{equation*}
(\bar{x}, \bar{u}) \in V I(C, A) \times V I(D, B): T \bar{x}=S \bar{u} \tag{1.6}
\end{equation*}
$$

where $C$ and $D$ are nonempty, closed and convex subsets of real Banach spaces $E_{1}$ and $E_{2}$, respectively, $A: E_{1} \rightarrow E_{1}^{*}$ and $B: E_{2} \rightarrow E_{2}^{*}$ are nonlinear mappings, $T: E_{1} \rightarrow E_{3}$ and $S: E_{2} \rightarrow E_{3}$ are bounded linear mappings with adjoints $T^{*}: E_{3}^{*} \rightarrow E_{1}^{*}$ and $S^{*}: E_{3}^{*} \rightarrow E_{2}^{*}$, respectively, where $E_{3}$ is another Banach space. Some of the special cases of SEVIP are common solutions of variational inequality problem (CSVIP), split variational inequality problem (SVIP), split equality feasibility problem (SEFP) introduced by Moudafi [21], split feasibility problem (SFP) introduced by Censor and Elfving [9] and the split equality null point problem (SENPP).

In 2021, Kwelegano et al. [17] introduced an iterative algorithm which solves SEVIP for uniformly continuous and pseudomonotone mappings that are sequentially weakly continuous in Hilbert spaces and proved a strong convergence of the algorithm under certain conditions.

In 2021, Boikanyo and Zegeye [5] introduced a new algorithm which approximates SEVIP for uniformly continuous pseudomonotone mappings that are sequentially weakly continuous in real Banach spaces and they proved strong convergence results.

Let $C$ be a nonempty, closed and convex subset of a real Banach space $E$ and let $f: E \rightarrow(-\infty,+\infty]$ be a proper, lower semi-continuous, Gâteaux differentiable and convex function. Let $G: C \rightarrow E^{*}$ be any mapping. A point $p \in C$ is called
(1) an $f$-fixed point of $G$ if $G p=\nabla f p$. The set of $f$ - fixed points of $G$ is denoted by $F_{f}(G)$.
(2) an $f$-asymptotic fixed point of $G$ if there exists a sequence $\left\{u_{n}\right\}$ in $C$ such that $u_{n} \rightharpoonup p$ and

$$
\lim _{n \rightarrow \infty}\left\|\nabla f u_{n}-G u_{n}\right\|=0
$$

The set of $f$ - asymptotic fixed points of $G$ is denoted by $\widehat{F_{f}(G)}$.
The mapping $G$ is called Bregman relatively $f$-nonexpansive if the following three properties hold:
(i) $F_{f}(G)$ is nonempty;
(ii) $D_{f}\left(p, \nabla f^{*} G z\right) \leq D_{f}(p, z)$, for all $z \in C, p \in \widehat{F_{f}(G)}$;
(iii) $\widehat{F_{f}(G)}=F_{f}(G)$.

All the results discussed above deal with either of the following: solutions of VIPs; solutions of SEVIPs; finding a common solution of VIPs and fixed point problems. Based on these results, we raise the following important question:

Question 1.2. Can we obtain a method for approximating a solution of split equality of variational inequality and $f, g$-fixed point problems in reflexive real Banach spaces, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the $f, g$-fixed point problems are for Bregman relatively $f, g-$ nonexpansive mappings?

The split equality of variational inequality and $f, g$-fixed point problem is defined as follows: Let $G: C \rightarrow E_{1}^{*}$ and $K: D \rightarrow E_{2}^{*}$ be Bregman relatively $f, g$ - nonexpansive mappings with $f$-fixed points $F_{f}(G)$ and $g$-fixed points $F_{g}(K)$, respectively. The split equality of variational inequality and $f, g-$ fixed point problems is defined as finding a point

$$
\begin{equation*}
(\bar{x}, \bar{u}) \in\left(V I(C, A) \cap F_{f}(G)\right) \times\left(V I(D, B) \cap F_{g}(K)\right): T \bar{x}=S \bar{u}, \tag{1.7}
\end{equation*}
$$

where $E_{1}$ and $E_{2}$ are reflexive real Banach spaces, $A: E_{1} \rightarrow E_{1}^{*}$ and $B$ : $E_{2} \rightarrow E_{2}^{*}$ are any mappings, $f: E_{1} \rightarrow \mathbb{R}$ and $g: E_{2} \rightarrow \mathbb{R}$ are proper, convex, lower semicontinuous, uniformly Fréchet differentiable, strongly convex, strongly coercive Legendre functions which are bounded.

Motivated and inspired by the aforementioned results, in this paper we introduce an algorithm for finding a solution of split equality of variational inequality and $f, g$-fixed point problems, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the fixed point problems are for Bregman relatively $f, g$-nonexpansive mappings in real Banach spaces. We prove a strong convergence theorem for the algorithm proposed. Finally, we provide a numerical example to demonstrate the effectiveness of the algorithm.

## 2. Preliminaries

Under this section, we give definitions and some important results that will be used in the subsequent analysis.

Let $\left\{x_{n}\right\}$ be a sequence in a reflexive real Banach space $E$. The strong and weak convergence of $\left\{x_{n}\right\}$ to a point $x \in E$ are denoted by $x_{n} \rightarrow x$ and $x_{n} \rightharpoonup x$, respectively. Let $U=\{x \in E:\|x\|=1\}$. $E$ is said to be strictly convex if $\frac{\|x+y\|}{2}<1$ for all $x, y \in U$ with $x \neq y$. If the limit

$$
\begin{equation*}
\lim _{t \rightarrow 0} \frac{\|y+t z\|-\|y\|}{t} \tag{2.1}
\end{equation*}
$$

exists for $y, z \in U$, then we say that $E$ is smooth.
Let $f: E \rightarrow \mathbb{R}$ be a convex function. The domain of $f$, denoted by $\operatorname{dom} f$, is defined as $\operatorname{dom} f=\{x \in E: f(x)<+\infty\}$. The function $f$ is said to be proper if $\operatorname{dom} f \neq \emptyset$. If a function is proper, convex and lower semi-continuous, then it is continuous (see, [4]). The Fenchel conjugate of $f$ is the function $f^{*}$ : $E^{*} \rightarrow \mathbb{R}$, defined by

$$
f^{*}\left(x^{*}\right)=\sup \left\{\left\langle x^{*}, x\right\rangle-f(x): x \in E\right\}
$$

for any $x^{*} \in E^{*}$. The directional derivative of $f$ at $x \in \operatorname{int}(\operatorname{dom} f)$ in the direction of $y$ is defined as

$$
\begin{equation*}
f^{o}(x, y)=\lim _{t \downarrow 0} \frac{f(x+t y)-f(x)}{t} \tag{2.2}
\end{equation*}
$$

provided that this limit exists. We say that $f$ is Gâteaux differentiable at $x$ if the limit in (2.2) exists for every $y \in E$. In this case, we define the gradient of $f$ at $x$ to be the linear function $\langle\nabla f(x), y\rangle=f^{o}(x, y)$ for all $y \in E$. The function $f$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \operatorname{int}(\operatorname{dom} f)$. If the limit in (2.2) is attained uniformly for any $y \in U$, then we say that $f$ is uniformly Fréchet differentiable at $x$.

A function $f: E \rightarrow \mathbb{R}$ is said to be a Legendre function if and only if it satisfies the following conditions:
(A) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset, f$ is Gâteaux differentiable and $\operatorname{dom} \nabla f=\operatorname{int}(\operatorname{dom} f)$;
(B) $\operatorname{int}\left(\operatorname{dom} f^{*}\right) \neq \emptyset, f^{*}$ is Gâteaux differentiable and $\operatorname{dom} \nabla f^{*}=\operatorname{int}(\operatorname{dom}$ $\left.f^{*}\right)$.
If $E$ is a smooth and strictly convex Banach space, then the function $f(x)=$ $\frac{1}{p}\|x\|^{p}(1<p<\infty)$ is a proper, lower semi-continuous Legendre function with Fenchel conjugate $f^{*}\left(x^{*}\right)=\frac{1}{q}\left\|x^{*}\right\|^{q}(1<q<\infty)$, (see, for instance, [3]), where
$\frac{1}{p}+\frac{1}{q}=1$. In this case, the gradient of $f$ is equal to the generalized duality mapping, $J_{p}$, of $E$. That is, $\nabla f=J_{p}$, where $J_{p}: E \rightarrow 2^{E^{*}}$ is defined as

$$
J_{p}(x)=\left\{x^{*} \in E^{*}:\left\langle x^{*}, x\right\rangle=\|x\|^{p},\left\|x^{*}\right\|=\|x\|^{p-1}\right\}
$$

If $p=2$, then we write $J_{p}=J$ and we call it the normalized duality mapping and if, in addition, $E=H$, where $H$ is a real Hilbert space, then $J=I$, where $I$ is the identity mapping on $H$. If $f: E \rightarrow(-\infty,+\infty]$ is a Legendre function and $E$ is a reflexive Banach space, then $\nabla f^{*}=(\nabla f)^{-1}$ (see, [6]). We also have that $f$ is a Legendre function if and only if $f^{*}$ is a Legendre function (see, [3]).

Lemma 2.1. ([26]) If $E$ is a real Banach space and $J_{E}$ is the normalized duality mapping on $E$, then

$$
\|x+y\|^{2} \leq\|x\|^{2}+2\left\langle j_{E}(x+y), y\right\rangle
$$

for all $x, y \in E$ and all $j_{E}(x+y) \in J_{E}(x+y)$.
Definition 2.2. The function $f: E \rightarrow \mathbb{R} \cup\{+\infty\}$ is said to be strongly coercive if $\frac{f(x)}{\|x\|} \rightarrow+\infty$ as $\|x\| \rightarrow+\infty$.

Definition 2.3. Let $E$ be a Banach space and $f: E \rightarrow \mathbb{R} \cup\{+\infty\}$ be a Gâteaux differentiable convex function. The function $D_{f}: \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \rightarrow$ $[0,+\infty)$ defined by

$$
D_{f}(y, x)=f(y)-f(x)-\langle\nabla f(x), y-x\rangle
$$

is called the Bregman distance with respect to $f$.
The Bregman distance has the following important properties:
For any $w, x, y, z \in E$,
(i) Three point identity:

$$
\begin{equation*}
D_{f}(w, x)+D_{f}(x, y)-D_{f}(w, y)=\langle\nabla f(x)-\nabla f(y), x-w\rangle . \tag{2.3}
\end{equation*}
$$

(ii) Four point identity:

$$
\begin{align*}
& D_{f}(x, z)+D_{f}(w, y)-D_{f}(x, y)-D_{f}(w, z) \\
& \quad=\langle\nabla f(y)-\nabla f(z), x-w\rangle . \tag{2.4}
\end{align*}
$$

Definition 2.4. A Gâteaux differentiable function $f: E \rightarrow \mathbb{R} \cup\{+\infty\}$ defined on a reflexive real Banach space $E$ is said to be strongly convex if there exists a constant $\beta>0$ such that

$$
\langle\nabla f(x)-\nabla f(y), x-y\rangle \geq \beta\|x-y\|^{2}
$$

for all $x, y \in \operatorname{dom} f$, or equivalently

$$
f(y) \geq f(x)+\langle\nabla f(x), y-x\rangle+\frac{\beta}{2}\|x-y\|^{2} .
$$

If $E$ is a smooth and strictly convex Banach space, then $f(x)=\frac{1}{2}\|x\|^{2}$ is a strongly coercive, bounded, uniformly Fréchet differentiable and strongly convex function with strong convexity constant $\beta \in(0,1]$ and Fenchel conjugate $f^{*}\left(x^{*}\right)=\frac{1}{2}| | x^{*} \|^{2}$.

It can be easily shown that if $f$ is a strongly convex function with constant $\beta>0$, then for all $y \in \operatorname{domf}$ and $x \in \operatorname{int}(\operatorname{domf})$, we have

$$
\begin{equation*}
D_{f}(y, x) \geq \frac{\beta}{2}\|x-y\|^{2} . \tag{2.5}
\end{equation*}
$$

Definition 2.5. Let $C \subseteq \operatorname{int}(\operatorname{dom} f)$ be a nonempty, closed and convex subset of real Banach space $E$, where $f: E \rightarrow \mathbb{R} \cup\{+\infty\}$ is a convex and Gâteaux differentiable function. The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto $C$ is the unique vector $P_{C}^{f}(x)$ of $C$ with the property

$$
D_{f}\left(P_{C}^{f}(x), x\right)=\inf \left\{D_{f}(y, x): y \in C\right\} .
$$

The Bregman projection also satisfies the following properties:

$$
\begin{equation*}
z=P_{C}^{f}(x) \text { if and only if }\langle\nabla f(x)-\nabla f(z), y-z\rangle \leq 0, \text { for all } y \in C, \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{f}\left(y, P_{C}^{f}(x)\right)+D_{f}\left(P_{C}^{f}(x), x\right) \leq D_{f}(y, x), \text { for all } x \in E, y \in C \tag{2.7}
\end{equation*}
$$

Lemma 2.6. ([1]) Let $E_{1}$ and $E_{2}$ be reflexive Banach spaces. Then, $E=$ $E_{1} \times E_{2}$ is also a reflexive Banach space with dual $E^{*}=E_{1}^{*} \times E_{2}^{*}$ and duality pairing

$$
\left\langle\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right\rangle=\left\langle x_{1}, x_{2}\right\rangle+\left\langle y_{1}, y_{2}\right\rangle
$$

for all $\left(x_{1}, y_{1}\right) \in E,\left(x_{2}, y_{2}\right) \in E^{*}$ and $\left(x_{n}, y_{n}\right) \rightharpoonup(x, y)$ implies $x_{n} \rightharpoonup x$ and $y_{n} \rightharpoonup y$.

If $C$ is a nonempty, closed and convex subset of $E, f: E_{1} \rightarrow \mathbb{R} \cup\{+\infty\}$, $g: E_{2} \rightarrow \mathbb{R} \cup\{+\infty\},(x, y) \in E,\left(x^{*}, y^{*}\right)=P_{C}^{h}(x, y)$, where $h=(f, g)$ and $\nabla h=(\nabla f, \nabla g)$, then

$$
\begin{equation*}
\left\langle(u, v)-\left(x^{*}, y^{*}\right),(\nabla f(x), \nabla g(y))-\left(\nabla f\left(x^{*}\right), \nabla g\left(y^{*}\right)\right)\right\rangle \leq 0 \tag{2.8}
\end{equation*}
$$

for all $(u, v) \in C$.

Lemma 2.7. ([25]) If $E$ is a reflexive real Banach space and $f: E \rightarrow(-\infty,+\infty]$ is a proper, lower semi-continuous, convex and Gâteaux differentiable function, then $f^{*}: E^{*} \rightarrow(-\infty,+\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $x \in E$, we have

$$
D_{f}\left(x, \nabla f^{*}\left(\sum_{i=1}^{N} s_{i} \nabla f\left(y_{i}\right)\right)\right) \leq \sum_{i=1}^{N} s_{i} D_{f}\left(x, y_{i}\right),
$$

where $\left\{y_{i}\right\}_{i=1}^{N} \subseteq E$ and $\left\{s_{i}\right\}_{i=1}^{N} \subseteq(0,1)$ with $\sum_{i=1}^{N} s_{i}=1$.
We say that a function $f$ is uniformly convex with modulus $\phi$ if for all $x, y \in \operatorname{dom} f$ and $\gamma \in[0,1]$, we have

$$
f(\gamma x+(1-\gamma) y) \leq \gamma f(x)+(1-\gamma) f(y)-\gamma(1-\gamma) \phi(\|x-y\|),
$$

where $\phi$ is an increasing function and $\phi(x)=0$ only for $x=0$.
The subdifferential $\partial f$ of $f$ at $x$ is defined by

$$
\partial f(x)=\left\{x^{*} \in E^{*}:\left\langle x^{*}, y-x\right\rangle \leq f(y)-f(x), \forall y \in E\right\}(\text { see },[15])
$$

Lemma 2.8. ([35]) Let $f$ be a convex and lower semi-continuous function on a Banach space E. The following assertions are equivalent:
(i) $f$ is uniformly convex;
(ii) there exists modulus $\phi$, for all $\left(x, x^{*}\right),\left(y, y^{*}\right) \in G p h(\partial f)$ such that

$$
f(y) \geq f(x)+\left\langle x^{*}, y-x\right\rangle+\phi(\|x-y\|) ;
$$

(iii) dom $f^{*}=E^{*}, f^{*}$ is Fréchet differentiable and $\nabla f^{*}$ is uniformly continuous.

Note that a strongly convex function is uniformly convex with $\phi(x)=$ $\frac{\beta}{2}\|x\|^{2}$ and hence the class of uniformly convex functions contains the class of strongly convex functions.

Lemma 2.9. ([23]) Let $E$ be a Banach space and $f: E \rightarrow(-\infty,+\infty]$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of $E$. Let $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$ be bounded sequences in $E$. Then, $\lim _{n \rightarrow \infty} D_{f}\left(x_{n}, u_{n}\right)$ $=0$ if and only if $\lim _{n \rightarrow \infty}\left(x_{n}-u_{n}\right)=0$.

Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable Legendre function. The nonnegative real-valued function $V_{f}: E \times E^{*} \rightarrow[0,+\infty)$ defined by

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)=f(x)-\left\langle x^{*}, x\right\rangle+f^{*}\left(x^{*}\right) \text { for all } x \in E, \quad x^{*} \in E^{*}, \tag{2.9}
\end{equation*}
$$

satisfies the properties

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)=D_{f}\left(x, \nabla f^{*}\left(x^{*}\right)\right), \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{f}\left(x, x^{*}\right)+\left\langle y^{*}, \nabla f^{*}\left(x^{*}\right)-x\right\rangle \leq V_{f}\left(x, x^{*}+y^{*}\right) \tag{2.11}
\end{equation*}
$$

for all $x \in E, x^{*}, y^{*} \in E^{*}$
Lemma 2.10. ([20]) Let $C$ be a nonempty, closed and convex subset of a reflexive real Banach space $E$. If $A: C \rightarrow E^{*}$ is a continuous pseudomonotone mapping, then $\operatorname{VI}(C, A)$ is closed and convex. Moreover, $\langle A p, q-p\rangle \geq 0$ for all $q \in C$ if and only if $\langle A q, q-p\rangle \geq 0$ for all $q \in C$.

Lemma 2.11. ([5]) If $\left\{c_{n}\right\}$ is a sequence of nonnegative real numbers such that

$$
c_{n+1} \leq\left(1-\alpha_{n}\right) c_{n}+\alpha_{n} d_{n},
$$

where $\left\{\alpha_{n}\right\} \subset(0,1)$ with $\sum_{n=1}^{\infty} \alpha_{n}=\infty$, and $\left\{d_{n}\right\}$ is a sequence of real numbers with $\limsup \sup _{n \rightarrow \infty} d_{n} \leq 0$, then $\lim _{n \rightarrow \infty} c_{n}=0$.

Lemma 2.12. ([18]) Let $\left\{a_{n}\right\}$ be a sequence of nonnegative real numbers. If $\left\{a_{n_{i}}\right\}$ is a subsequence of $\left\{a_{n}\right\}$ such that $a_{n_{i}}<a_{n_{i}+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\left\{m_{k}\right\}$ of $\mathbb{N}$ such that $\lim _{k \rightarrow \infty} m_{k}=\infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$ :

$$
a_{m_{k}} \leq a_{m_{k}+1} \quad \text { and } \quad a_{k} \leq a_{m_{k}+1} .
$$

In fact, $m_{k}=\max \left\{n \leq k: a_{n}<a_{n+1}\right\}$.
The modulus of total convexity of a G $\hat{a}$ teaux differentiable function $f$ is the function $v_{f}: \operatorname{int}(\operatorname{domf}) \times[0, \infty) \rightarrow[0, \infty)$ defined by

$$
v_{f}(x, t)=\inf \left\{D_{f}(y, x): y \in \operatorname{dom} f,\|y-x\|=t\right\} .
$$

In this case we say that $f$ is totally convex at a point $x \in \operatorname{int}(\operatorname{dom} f)$ if $v_{f}(x, t)>0$ whenever $t>0$. The function $f$ is said to be totally convex if it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$.

On bounded subsets of $E$, the concepts of uniform convexity and total convexity are the same (see, [7]).

Lemma 2.13. ([19]) Let $E$ be a reflexive real Banach space and $f: E \rightarrow \mathbb{R}$ be a totally convex function. If $\left\{D_{f}\left(x_{n}, x_{0}\right)\right\}$ is bounded for any $x_{0} \in E$, then $\left\{x_{n}\right\}$ is bounded.

Lemma 2.14. ([32]) Let $f$ be a continuous, convex and strongly coercive real valued function defined on a reflexive real Banach space E. Then the following are equivalent:
(i) $f$ is uniformly smooth and bounded on bounded subsets of $E$;
(ii) $f^{*}: E^{*} \rightarrow \mathbb{R}$ is Fréchet differentiable and $\nabla f^{*}$ is uniformly norm-tonorm continuous on bounded subsets of $E^{*}$;
(iii) $f^{*}$ is strongly coercive, uniformly convex on bounded subsets of $E^{*}$ and $\operatorname{domf} f^{*}=E^{*}$.

Lemma 2.15. Let $C$ be a subset of a reflexive real Banach space $E$ and $A$ : $C \rightarrow E^{*}$ be any mapping. Let $f: E \rightarrow \mathbb{R}$ be a strongly convex function with constant $\eta$ and has Lipschitz continuous gradient with constant $\gamma$. If $\gamma \leq \eta$, then for any $x \in E$ and $\alpha \geq \beta>0$, the following inequality holds:

$$
\frac{\left\|x-P_{C}^{f} \nabla f^{*}(\nabla f(x)-\alpha A x)\right\|}{\alpha} \leq \frac{\left\|x-P_{C}^{f} \nabla f^{*}(\nabla f(x)-\beta A x)\right\|}{\beta} .
$$

Proof. Let $x_{\alpha}=P_{C}^{f} \nabla f^{*}(\nabla f(x)-\alpha A x)$ and $x_{\beta}=P_{C}^{f} \nabla f^{*}(\nabla f(x)-\beta A x)$. Then it follows from (2.6) that

$$
\left\langle\nabla f(x)-\alpha A x-\nabla f\left(x_{\alpha}\right), y-x_{\alpha}\right\rangle \leq 0
$$

and

$$
\left\langle\nabla f(x)-\beta A x-\nabla f\left(x_{\beta}\right), y-x_{\beta}\right\rangle \leq 0
$$

for all $y \in C$, which implies that

$$
\left\langle\frac{\nabla f\left(x_{\alpha}\right)-\nabla f(x)}{\alpha}+A x, x_{\beta}-x_{\alpha}\right\rangle \geq 0
$$

and

$$
\left\langle\frac{\nabla f\left(x_{\beta}\right)-\nabla f(x)}{\beta}+A x, x_{\alpha}-x_{\beta}\right\rangle \geq 0 .
$$

Adding both inequalities and using the Cauchy-Schwarz inequality, strong convexity of $f$ and Lipschitz continuity of $\nabla f$, we get

$$
\begin{aligned}
0 \leq & \left\langle\frac{\nabla f(x)-\nabla f\left(x_{\alpha}\right)}{\alpha}-\frac{\nabla f(x)-\nabla f\left(x_{\beta}\right)}{\beta}, x_{\alpha}-x_{\beta}\right\rangle \\
= & \left\langle\frac{\nabla f(x)-\nabla f\left(x_{\alpha}\right)}{\alpha}-\frac{\nabla f(x)-\nabla f\left(x_{\beta}\right)}{\beta},\left(x-x_{\beta}\right)-\left(x-x_{\alpha}\right)\right\rangle \\
= & -\left\langle\frac{\nabla f(x)-\nabla f\left(x_{\alpha}\right)}{\alpha}, x-x_{\alpha}\right\rangle+\left\langle\frac{\nabla f(x)-\nabla f\left(x_{\alpha}\right)}{\alpha}, x-x_{\beta}\right\rangle \\
& -\left\langle\frac{\nabla f(x)-\nabla f\left(x_{\beta}\right)}{\beta}, x-x_{\beta}\right\rangle+\left\langle\frac{\nabla f(x)-\nabla f\left(x_{\beta}\right)}{\beta}, x-x_{\alpha}\right\rangle \\
\leq & -\frac{1}{\alpha}\left\langle\nabla f(x)-\nabla f\left(x_{\alpha}\right), x-x_{\alpha}\right\rangle-\frac{1}{\beta}\left\langle\nabla f(x)-\nabla f\left(x_{\beta}\right), x-x_{\beta}\right\rangle \\
& +\frac{1}{\alpha}\left\|\nabla f(x)-\nabla f\left(x_{\alpha}\right)\right\|\left\|x-x_{\beta}\right\|+\frac{1}{\beta}\left\|\nabla f(x)-\nabla f\left(x_{\beta}\right)\right\|\left\|x-x_{\alpha}\right\|
\end{aligned}
$$

$$
\begin{align*}
& \leq-\frac{\eta}{\alpha}\left\|x-x_{\alpha}\right\|^{2}-\frac{\eta}{\beta}\left\|x-x_{\beta}\right\|^{2}+\frac{\gamma}{\alpha}\left\|x-x_{\alpha}\right\|\left\|x-x_{\beta}\right\| \\
& +\frac{\gamma}{\beta}\left\|x-x_{\beta}\right\|\left\|x-x_{\alpha}\right\| \\
& \leq \frac{\eta}{\alpha}\left\|x-x_{\alpha}\right\|^{2}-\frac{\eta}{\beta}\left\|x-x_{\beta}\right\|^{2}+\frac{\eta}{\alpha}\left\|x-x_{\alpha}\right\|\left\|x-x_{\beta}\right\| \\
& +\frac{\eta}{\beta}\left\|x-x_{\beta}\right\|\left\|x-x_{\alpha}\right\| . \tag{2.12}
\end{align*}
$$

Thus, from (2.12), we obtain

$$
\begin{equation*}
0 \geq\left(\left\|x-x_{\alpha}\right\|-\left\|x-x_{\beta}\right\|\right)\left(\frac{\left\|x-x_{\alpha}\right\|}{\alpha}-\frac{\left\|x-x_{\beta}\right\|}{\beta}\right) . \tag{2.13}
\end{equation*}
$$

Now, suppose on the contrary that we have

$$
\begin{equation*}
\frac{\left\|x-x_{\alpha}\right\|}{\alpha}>\frac{\left\|x-x_{\beta}\right\|}{\beta}, \tag{2.14}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left\|x-x_{\alpha}\right\|>\left\|x-x_{\beta}\right\| . \tag{2.15}
\end{equation*}
$$

Combining (2.14) and (2.15), we obtain

$$
0<\left(\left\|x-x_{\alpha}\right\|-\left\|x-x_{\beta}\right\|\right)\left(\frac{\left\|x-x_{\alpha}\right\|}{\alpha}-\frac{\left\|x-x_{\beta}\right\|}{\beta}\right),
$$

which is a contradiction to (2.13) hence the proof is completed.
Examples of functions that satisfy conditions of the hypothesis in Lemma 2.15 are functions of the type $f(x)=k\|x\|^{2}$, for $k>0$. One can show that $f$ is strongly convex with strong convexity constant $2 k$ and $\nabla f$ is Lipschitz continuous with Lipschitz constant $2 k$.

## 3. Main results

The following assumptions will be used in the sequel.

## Condition 3.1.

(A1) Let $C$ and $D$ be nonempty, closed and convex subsets of the smooth, strictly convex and reflexive real Banach spaces $E_{1}$ and $E_{2}$, respectively.
(A2) Let $f: E_{1} \rightarrow \mathbb{R}$ and $g: E_{2} \rightarrow \mathbb{R}$ be proper, lower semi-continuous, strongly coercive, uniformly Fréchet differentiable, strongly convex Legendre functions which are bounded on bounded subsets of $E_{1}$ and $E_{2}$, respectively. Let $f$ and $g$ have Lipschitz continuous gradients with the strong convexity constant of $f$ (respectively, $g$ ) greater than or equal to the Lipschitz constant of $\nabla f$ (respectively, $\nabla g$ ).

## Condition 3.2.

$(B 1)$ Let $A: C \rightarrow E_{1}^{*}$ and $B: D \rightarrow E_{2}^{*}$ be uniformly continuous, pseudomonotone and sequentially weakly continuous mappings;
(B2) Let $G: E_{1} \rightarrow E_{1}^{*}$ and $K: E_{2} \rightarrow E_{2}^{*}$ be Bregman relatively $f$-nonexpansive and Bregman relatively $g$-nonexpansive mappings, respectively;
(B3) Let $T: E_{1} \rightarrow E_{3}$ and $S: E_{2} \rightarrow E_{3}$ be bounded linear mappings with adjoints $T^{*}: E_{3}^{*} \rightarrow E_{1}^{*}$ and $S^{*}: E_{3}^{*} \rightarrow E_{2}^{*}$, respectively, where $E_{3}$ is another smooth, strictly convex real reflexive Banach space;
(B4) Let the set of solutions of (1.7), denoted by $\Upsilon$, be nonempty, that is, $\Upsilon=\left\{(x, u) \in\left(V I(C, A) \cap F_{f}(G)\right) \times\left(V I(D, B) \cap F_{g}(K)\right): T x=S u\right\} \neq \emptyset$.

## Condition 3.3.

$(C 1)$ Let $\beta=\min \left\{\beta_{1}, \beta_{2}\right\}$, where $\beta_{1}$ and $\beta_{2}$ are the strong convexity constants of $f$ and $g$, respectively;
$(C 2)$ Let $\left\{\alpha_{n}\right\} \subseteq(0,1)$ be such that $\lim _{n \rightarrow \infty} \alpha_{n}=0$ and $\sum_{n=1}^{\infty} \alpha_{n}=\infty$.

## Algorithm A.

Initialization: Choose $\left(x_{1}, u_{1}\right) \in E_{1} \times E_{2}, \mu \in(0, \beta), l, \gamma, \tau \in(0,1)$. For $x \in C, u \in D$ define the algorithm as follows:
Step 1: Given the current iterates $x_{n}$ and $u_{n}$, compute

$$
\left\{\begin{array}{l}
z_{n}=P_{C}^{f}\left[\nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)\right],  \tag{3.1}\\
w_{n}=P_{D}^{g}\left[\nabla g^{*}\left(\nabla g\left(u_{n}\right)-\gamma_{n} S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right)\right],
\end{array}\right.
$$

where $0<\rho \leq \gamma_{n} \leq \rho_{n}$ for $n \in\left\{m \in \mathbb{N}: T x_{m}-S u_{m} \neq 0\right\}$, otherwise $\gamma_{n}=\rho$, for some $\rho>0$, and

$$
\rho_{n}=\min \left\{\rho+1, \frac{\beta\left\|T x_{n}-S u_{n}\right\|^{2}}{2\left[\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\|^{2}+\left\|S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right\|^{2}\right]}\right\} .
$$

Step 2: Compute

$$
\left\{\begin{array}{l}
y_{n}=P_{C}^{f}\left(\nabla f^{*}\left(\nabla f\left(z_{n}\right)-\lambda_{n} A z_{n}\right)\right) \\
v_{n}=P_{D}^{g}\left(\nabla g^{*}\left(\nabla g\left(w_{n}\right)-\eta_{n} B w_{n}\right)\right)
\end{array}\right.
$$

where $\lambda_{n}=\gamma^{l_{m}}$, for $j_{m}$ is the smallest nonnegative integer $j$ satisfying

$$
\begin{equation*}
\gamma l^{j}\left\|A y_{n}-A z_{n}\right\| \leq \mu\left\|y_{n}-z_{n}\right\|, \tag{3.2}
\end{equation*}
$$

and $\eta_{n}=\gamma l^{k_{m}}$, for $k_{m}$ is the smallest nonnegative integer $k$ satisfying

$$
\begin{equation*}
\gamma l^{k}\left\|B v_{n}-B w_{n}\right\| \leq \mu\left\|v_{n}-w_{n}\right\| \tag{3.3}
\end{equation*}
$$

Step 3: Compute

$$
a_{n}=\nabla f^{*}\left(\nabla f\left(y_{n}\right)-\lambda_{n}\left(A y_{n}-A z_{n}\right)\right),
$$

$$
\begin{gather*}
b_{n}=\nabla g^{*}\left(\nabla g\left(v_{n}\right)-\eta_{n}\left(B v_{n}-B w_{n}\right)\right), \\
\left\{\begin{array}{l}
x_{n+1}=\nabla f^{*}\left(\alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right)\left[\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right]\right), \\
u_{n+1}=\nabla g^{*}\left(\alpha_{n} \nabla g(u)+\left(1-\alpha_{n}\right)\left[\tau \nabla g\left(b_{n}\right)+(1-\tau) K\left(b_{n}\right)\right]\right) .
\end{array}\right. \tag{3.4}
\end{gather*}
$$

Set $n:=n+1$ and go to Step 1.
Hereunder, we present some results that are fundamental to the convergence analysis of the sequences generated by Algorithm A. We begin by proving that the proposed algorithm is well-defined.

Lemma 3.1. Assume that Conditions (A1) - (A2), (B1) - (B4) and (C1) $(C 2)$ hold. Then the Armijo line-search rules (3.2) and (3.3) are well-defined.

Proof. If $z_{n} \in V I(C, A)$, then $z_{n}=P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\lambda_{n} A z_{n}\right)$. In this case, we have $z_{n}=y_{n}$ and hence (3.2) holds for $j=0$. Now, we consider the case when $z_{n} \notin V I(C, A)$ and assume on the contrary that for all $j \geq 0$ we have

$$
\gamma l^{j}\left\|A y_{n}-A z_{n}\right\|>\mu\left\|y_{n}-z_{n}\right\| .
$$

That is,

$$
\begin{align*}
& \left\|A P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)-A z_{n}\right\| \\
& \quad>\frac{\mu}{\gamma l^{j}}\left\|P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)-z_{n}\right\| . \tag{3.5}
\end{align*}
$$

Since $P_{C}^{f}$ and $\nabla f^{*}$ are continuous, we have that

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left\|z_{n}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)\right\|=0 \tag{3.6}
\end{equation*}
$$

By the uniform continuity of the mapping $A$ on bounded subsets of $C$, we obtain

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left\|A P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)-A z_{n}\right\|=0 \tag{3.7}
\end{equation*}
$$

From (3.5) and (3.7), we have

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \frac{\left\|z_{n}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)\right\|}{\gamma l^{j}}=0 . \tag{3.8}
\end{equation*}
$$

Since $\nabla f$ is Lipschitz continuous, there exists a real number $R>0$ such that

$$
\begin{aligned}
0 & \leq \lim _{j \rightarrow \infty} \frac{\left\|\nabla f\left(z_{n}\right)-\nabla f\left(P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)\right)\right\|}{\gamma l^{j}} \\
& \leq R \lim _{j \rightarrow \infty} \frac{\left\|z_{n}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)\right\|}{\gamma l^{j}},
\end{aligned}
$$

which implies from (3.8) that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \frac{\left\|\nabla f\left(z_{n}\right)-\nabla f P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)\right\|}{\gamma l^{j}}=0 . \tag{3.9}
\end{equation*}
$$

Let $k_{j}=P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n}\right)-\gamma l^{j} A z_{n}\right)$. Then, by (2.6), we get

$$
\left\langle\nabla f\left(k_{j}\right)-\nabla f\left(z_{n}\right)+\gamma l^{j} A z_{n}, y-k_{j}\right\rangle \geq 0, \quad \forall y \in C
$$

which implies that

$$
\begin{equation*}
\left\langle\frac{\nabla f\left(k_{j}\right)-\nabla f\left(z_{n}\right)}{\gamma l^{j}}, y-k_{j}\right\rangle+\left\langle A z_{n}, z_{n}-k_{j}\right\rangle+\left\langle A z_{n}, y-z_{n}\right\rangle \geq 0, \forall y \in C \tag{3.10}
\end{equation*}
$$

Taking the limit as $j \rightarrow \infty$ in (3.10) and using (3.6) and (3.9), we obtain

$$
\begin{equation*}
\left\langle A z_{n}, y-z_{n}\right\rangle \geq 0 \text { for all } y \in C \tag{3.11}
\end{equation*}
$$

and this implies that $z_{n} \in V I(C, A)$, which is a contradiction. Hence, (3.2) holds. Similarly, one can show that (3.3) holds and hence the proof is complete.

Theorem 3.2. Assume that Conditions (A1) - (A2), (B1) - (B4) and (C1)$(C 2)$ hold. Then the sequences $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$ generated by Algorithm $A$ are bounded.

Proof. Denote

$$
q_{n}=\nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)
$$

and

$$
t_{n}=\nabla g^{*}\left(\nabla g\left(u_{n}\right)-\gamma_{n} S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right)
$$

Let $(\bar{x}, \bar{u}) \in \Upsilon$. Then by (3.4), Lemma 2.7 and Bregman relatively $f$-nonexpansiveness of $G$, we have

$$
\begin{align*}
D_{f}\left(\bar{x}, x_{n+1}\right)= & D_{f}\left(\bar{x}, \nabla f^{*}\left(\alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right)\left[\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right]\right)\right) \\
\leq & \alpha_{n} D_{f}(\bar{x}, x)+\left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, \nabla f^{*}\left[\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right]\right) \\
\leq & \alpha_{n} D_{f}(\bar{x}, x)+\left(1-\alpha_{n}\right) \tau D_{f}\left(\bar{x}, a_{n}\right) \\
& +\left(1-\alpha_{n}\right)(1-\tau) D_{f}\left(\bar{x}, \nabla f^{*}\left(G\left(a_{n}\right)\right)\right) \\
\leq & \alpha_{n} D_{f}(\bar{x}, x)+\left(1-\alpha_{n}\right) \tau D_{f}\left(\bar{x}, a_{n}\right) \\
& +\left(1-\alpha_{n}\right)(1-\tau) D_{f}\left(\bar{x}, a_{n}\right) \\
= & \alpha_{n} D_{f}(\bar{x}, x)+\left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, a_{n}\right) . \tag{3.12}
\end{align*}
$$

From the definition of $a_{n}$, we have

$$
\begin{align*}
D_{f}\left(\bar{x}, a_{n}\right)= & D_{f}\left(\bar{x}, \nabla f^{*}\left(\nabla f\left(y_{n}\right)-\lambda_{n}\left(A y_{n}-A z_{n}\right)\right)\right) \\
= & f(\bar{x})-\left\langle\nabla f\left(y_{n}\right)-\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle-f\left(a_{n}\right) \\
= & f(\bar{x})+\left\langle\nabla f\left(y_{n}\right), a_{n}-\bar{x}\right\rangle+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle-f\left(a_{n}\right) \\
= & f(\bar{x})-\left\langle\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle-f\left(y_{n}\right)+\left\langle\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle+f\left(y_{n}\right) \\
& +\left\langle\nabla f\left(y_{n}\right), a_{n}-\bar{x}\right\rangle+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle-f\left(a_{n}\right) \\
= & D_{f}\left(\bar{x}, y_{n}\right)+\left\langle\nabla f\left(y_{n}\right), a_{n}-y_{n}\right\rangle \\
& +f\left(y_{n}\right)-f\left(a_{n}\right)+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle \\
= & D_{f}\left(\bar{x}, y_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle . \tag{3.13}
\end{align*}
$$

Using (2.4), we get

$$
D_{f}\left(\bar{x}, y_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)=D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, z_{n}\right)+\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-a_{n}\right\rangle .
$$

Thus, (3.13) becomes

$$
\begin{align*}
D_{f}\left(\bar{x}, a_{n}\right)= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, z_{n}\right)+\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-a_{n}\right\rangle \\
& +\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle . \tag{3.14}
\end{align*}
$$

Furthermore, from (2.3), we obtain

$$
\begin{equation*}
D_{f}\left(a_{n}, z_{n}\right)=D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)-\left\langle\nabla f\left(y_{n}\right)-\nabla f\left(z_{n}\right), y_{n}-a_{n}\right\rangle . \tag{3.15}
\end{equation*}
$$

Therefore, from (3.14) and (3.15) we obtain

$$
\begin{aligned}
D_{f}\left(\bar{x}, a_{n}\right)= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-a_{n}\right\rangle \\
& +\left\langle\nabla f\left(y_{n}\right)-\nabla f\left(z_{n}\right), y_{n}-a_{n}\right\rangle+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle \\
= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle \\
& +\left\langle\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-a_{n}\right\rangle\right. \\
= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle \\
& +\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-y_{n}+y_{n}-a_{n}\right\rangle \\
= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\left\langle\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle \\
& +\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), \bar{x}-y_{n}\right\rangle+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), y_{n}-a_{n}\right\rangle
\end{aligned}
$$

$$
\begin{align*}
= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right)+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right)\right. \\
& \left.+\nabla f\left(z_{n}\right)-\nabla f\left(y_{n}\right), \bar{x}-y_{n}\right\rangle+\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), y_{n}-a_{n}\right\rangle \\
= & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right)  \tag{3.16}\\
& -\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right)-\left(\nabla f\left(y_{n}\right)-\nabla f\left(z_{n}\right)\right), y_{n}-\bar{x}\right\rangle \\
& +\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), y_{n}-a_{n}\right\rangle .
\end{align*}
$$

Since $y_{n}=P_{C}^{f}\left[\nabla f^{*}\left(\nabla f\left(z_{n}\right)-\lambda_{n} A z_{n}\right)\right]$, by (2.6) we get

$$
\begin{equation*}
\left\langle\nabla f\left(z_{n}\right)-\lambda_{n} A z_{n}-\nabla f\left(y_{n}\right), y_{n}-\bar{x}\right\rangle \geq 0 . \tag{3.17}
\end{equation*}
$$

Since $\bar{x} \in V I(C, A)$ and $y_{n} \in C$, we have $\left\langle A \bar{x}, y_{n}-\bar{x}\right\rangle \geq 0$. Moreover, the fact that $A$ is pseudomonotone implies that $\left\langle A y_{n}, y_{n}-\bar{x}\right\rangle \geq 0$, and thus

$$
\begin{equation*}
\left\langle\lambda_{n} A y_{n}, y_{n}-\bar{x}\right\rangle \geq 0 . \tag{3.18}
\end{equation*}
$$

Combining (3.17) and (3.18), we get

$$
\begin{equation*}
\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right)-\left(\nabla f\left(y_{n}\right)-\nabla f\left(z_{n}\right)\right), y_{n}-\bar{x}\right\rangle \geq 0 . \tag{3.19}
\end{equation*}
$$

Thus, from (3.16) and (3.19), we obtain

$$
\begin{align*}
D_{f}\left(\bar{x}, a_{n}\right) \leq & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\left\langle\lambda_{n}\left(A y_{n}-A z_{n}\right), y_{n}-a_{n}\right\rangle . \tag{3.20}
\end{align*}
$$

Furthermore, from (3.20), Cauchy Schwarz inequality, (3.2) and (2.5), we get

$$
\begin{align*}
D_{f}\left(\bar{x}, a_{n}\right) \leq & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\lambda_{n}\left\|y_{n}-a_{n}\right\|\left\|A y_{n}-A z_{n}\right\| \\
\leq & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\mu\left\|y_{n}-a_{n}\right\|\left\|y_{n}-z_{n}\right\| \\
\leq & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\frac{\mu}{2}\left(\left\|y_{n}-a_{n}\right\|^{2}+\left\|y_{n}-z_{n}\right\|^{2}\right)  \tag{3.21}\\
\leq & D_{f}\left(\bar{x}, z_{n}\right)-D_{f}\left(a_{n}, y_{n}\right)-D_{f}\left(y_{n}, z_{n}\right) \\
& +\frac{\mu}{\beta}\left(D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)\right) \\
= & D_{f}\left(\bar{x}, z_{n}\right)-\left(1-\frac{\mu}{\beta}\right)\left(D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)\right) .
\end{align*}
$$

Using (2.7), (2.10) and (2.11), we get

$$
\begin{align*}
D_{f}\left(\bar{x}, z_{n}\right) \leq & D_{f}\left(\bar{x}, \nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)\right)-D_{f}\left(z_{n}, q_{n}\right) \\
\leq & D_{f}\left(\bar{x}, \nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)\right) \\
= & V_{f}\left(\bar{x}, \nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right) \\
\leq & V_{f}\left(\bar{x}, \nabla f\left(x_{n}\right)\right) \\
- & \left\langle\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right),\right. \\
& \left.\nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)-\bar{x}\right\rangle \\
= & D_{f}\left(\bar{x}, x_{n}\right)-\gamma_{n}\left\langle T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right), q_{n}-\bar{x}\right\rangle \\
= & D_{f}\left(\bar{x}, x_{n}\right)-\gamma_{n}\left\langle J_{E_{3}}\left(T x_{n}-S u_{n}\right), T q_{n}-T \bar{x}\right\rangle . \tag{3.22}
\end{align*}
$$

Substituting (3.22) into (3.21), we obtain

$$
\begin{align*}
D_{f}\left(\bar{x}, a_{n}\right) \leq & D_{f}\left(\bar{x}, x_{n}\right)-\left(1-\frac{\mu}{\beta}\right)\left(D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)\right)  \tag{3.23}\\
& -\gamma_{n}\left\langle J_{E_{3}}\left(T x_{n}-S u_{n}\right), T q_{n}-T \bar{x}\right\rangle .
\end{align*}
$$

Thus, from (3.23) and (3.12), we get

$$
\begin{align*}
D_{f}\left(\bar{x}, x_{n+1}\right) \leq & \alpha_{n} D_{f}(\bar{x}, x)+\left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, x_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right)\left(D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)\right)  \tag{3.24}\\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(T x_{n}-S u_{n}\right), T q_{n}-T \bar{x}\right\rangle .
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
D_{g}\left(\bar{u}, u_{n+1}\right) \leq & \alpha_{n} D_{g}(\bar{u}, u)+\left(1-\alpha_{n}\right) D_{g}\left(\bar{u}, u_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right)\left(D_{g}\left(b_{n}, v_{n}\right)+D_{g}\left(v_{n}, w_{n}\right)\right)  \tag{3.25}\\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(S u_{n}-T x_{n}\right), S t_{n}-S \bar{u}\right\rangle .
\end{align*}
$$

Now, denote

$$
\Omega_{n}=D_{f}\left(\bar{x}, x_{n}\right)+D_{g}\left(\bar{u}, u_{n}\right)
$$

and

$$
\Sigma=D_{f}(\bar{x}, x)+D_{g}(\bar{u}, u) .
$$

Since $\mu \in(0, \beta)$ and $\beta>0$, we have that $1>\frac{\mu}{\beta}>0$. Thus, $1-\frac{\mu}{\beta}>0$. Then, combining (3.24) and (3.25) and using the fact that $T \bar{x}=S \bar{u}$, we get

$$
\begin{equation*}
\Omega_{n+1} \leq \alpha_{n} \Sigma+\left(1-\alpha_{n}\right) \Omega_{n}-\left(1-\alpha_{n}\right) \gamma_{n}\left\langle T q_{n}-S t_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle . \tag{3.26}
\end{equation*}
$$

But, we have by the Cauchy Schwarz inequality that

$$
\begin{align*}
-\left\langle T q_{n}-S t_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle= & -\left\langle T x_{n}-S u_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle \\
& -\left\langle T q_{n}-T x_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle \\
& -\left\langle S u_{n}-S t_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle \\
= & -\left\|T x_{n}-S u_{n}\right\|^{2} \\
& -\left\langle q_{n}-x_{n}, T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle  \tag{3.27}\\
& -\left\langle u_{n}-t_{n}, S^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle \\
\leq & -\left\|T x_{n}-S u_{n}\right\|^{2} \\
& +\left\|q_{n}-x_{n}\right\|\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\| \\
& +\left\|u_{n}-t_{n}\right\|\left\|S^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\| .
\end{align*}
$$

From the strong convexity of $f$ and the definition of $q_{n}$, we have

$$
\begin{align*}
\left\|q_{n}-x_{n}\right\| & =\left\|\nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)-\nabla f^{*}\left(\nabla f\left(x_{n}\right)\right)\right\| \\
& \leq \frac{1}{\beta_{1}}\left\|\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\| \\
& \leq \frac{\gamma_{n}}{\beta}\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\| . \tag{3.28}
\end{align*}
$$

Similarly, the strong convexity of $g$ and the definition of $t_{n}$ gives

$$
\begin{equation*}
\left\|t_{n}-u_{n}\right\| \leq \frac{\gamma_{n}}{\beta}\left\|S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right\| . \tag{3.29}
\end{equation*}
$$

Substituting (3.29) and (3.28) into (3.27) and applying the property of $\gamma_{n}$, we get

$$
\begin{align*}
-\gamma_{n}\left\langle T q_{n}-S t_{n}, J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\rangle \leq & -\gamma_{n}\left\|T x_{n}-S u_{n}\right\|^{2} \\
& +\frac{\gamma_{n}^{2}}{\beta}\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\|^{2} \\
& +\frac{\gamma_{n}^{2}}{\beta}\left\|S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right\|^{2} \\
\leq & -\frac{\rho}{2}\left\|T x_{n}-S u_{n}\right\|^{2}-\frac{\gamma_{n}}{2}\left\|T x_{n}-S u_{n}\right\|^{2} \\
& +\frac{\gamma_{n}^{2}}{\beta}\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\|^{2} \\
& +\frac{\gamma_{n}^{2}}{\beta}\left\|S^{*} J_{E_{3}}\left(S u_{n}-T x_{n}\right)\right\|^{2} \\
\leq & -\frac{\rho}{2}\left\|T x_{n}-S u_{n}\right\|^{2} \tag{3.30}
\end{align*}
$$

and substituting (3.30) into (3.26), we obtain

$$
\begin{aligned}
\Omega_{n+1} & \leq \alpha_{n} \Sigma+\left(1-\alpha_{n}\right) \Omega_{n}-\left(1-\alpha_{n}\right) \frac{\rho}{2}\left\|T x_{n}-S u_{n}\right\|^{2} \\
& \leq \alpha_{n} \Sigma+\left(1-\alpha_{n}\right) \Omega_{n}
\end{aligned}
$$

which implies by the mathematical induction that $\Omega_{n} \leq \max \left\{\Omega_{1}, \Sigma\right\}$. Hence we have that $\left\{D_{f}\left(\bar{x}, x_{n}\right)+D_{g}\left(\bar{u}, u_{n}\right)\right\}$ is bounded which implies that the sequences $\left\{D_{f}\left(\bar{x}, x_{n}\right)\right\}$ and $\left\{D_{g}\left(\bar{u}, u_{n}\right)\right\}$ are bounded. By Lemma 2.13, we have that $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$ are bounded.

Lemma 3.3. Assume that Conditions $(A 1)-(A 2),(B 1)-(B 4)$ and $(C 1)-$ (C2) hold. Let $\left\{z_{n}\right\},\left\{y_{n}\right\},\left\{w_{n}\right\}$ and $\left\{v_{n}\right\}$ be as defined in Algorithm A. Then we have the following statements:
(1) If there exist subsequences $\left\{z_{n_{k}}\right\}$ and $\left\{y_{n_{k}}\right\}$ of $\left\{z_{n}\right\}$ and $\left\{y_{n}\right\}$, respectively, such that $z_{n_{k}} \rightharpoonup p \in C$ and $\left\|z_{n_{k}}-y_{n_{k}}\right\| \rightarrow 0$ as $k \rightarrow \infty$, then
(i) $0 \leq \liminf _{k \rightarrow \infty}\left\langle A z_{n_{k}}, z-z_{n_{k}}\right\rangle$ for all $z \in C$; (ii) $p \in V I(C, A)$.
(2) If there exist subsequences $\left\{w_{n_{k}}\right\}$ and $\left\{v_{n_{k}}\right\}$ of $\left\{w_{n}\right\}$ and $\left\{v_{n}\right\}$, respectively, such that $w_{n_{k}} \rightharpoonup q \in D$ and $\left\|w_{n_{k}}-v_{n_{k}}\right\| \rightarrow 0$ as $k \rightarrow \infty$, then
(i) $0 \leq \liminf _{k \rightarrow \infty}\left\langle B w_{n_{k}}, w-w_{n_{k}}\right\rangle$ for all $w \in D$;
(ii) $q \in V I(D, B)$.

Proof. (1) Let the hypotheses be satisfied.
(i) Put $s_{n_{k}}=P_{C}^{f} \nabla f^{*}\left(\nabla f z_{n_{k}}-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)$. By Lemma 2.15 and (3.6) we have

$$
\begin{equation*}
\left\|z_{n_{k}}-s_{n_{k}}\right\| \leq \frac{1}{l}\left\|z_{n_{k}}-y_{n_{k}}\right\| \rightarrow 0, \text { as } k \rightarrow \infty . \tag{3.31}
\end{equation*}
$$

Therefore, $s_{n_{k}} \rightharpoonup p \in C$. Thus, we have that $\left\{s_{n_{k}}\right\}$ is bounded. Since $A$ is uniformly continuous on bounded subsets of $E_{1}$, we have

$$
\begin{equation*}
\left\|A z_{n_{k}}-A s_{n_{k}}\right\| \rightarrow 0, \text { as } k \rightarrow \infty \tag{3.32}
\end{equation*}
$$

By the Armijo line-search rule (3.2), we have

$$
\begin{aligned}
& \lambda_{n_{k}} l^{-1}\left\|A P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)-A z_{n_{k}}\right\| \\
& \quad>\mu\left\|z_{n_{k}}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right\|,
\end{aligned}
$$

which implies that

$$
\begin{align*}
& \frac{1}{\mu}\left\|A P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)-A z_{n_{k}}\right\| \\
& \quad>\frac{\left\|z_{n_{k}}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right\|}{\lambda_{n_{k}} l^{-1}} \tag{3.33}
\end{align*}
$$

From (3.32) and (3.33), we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\left\|z_{n_{k}}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right\|}{\lambda_{n_{k}} l^{-1}}=0 . \tag{3.34}
\end{equation*}
$$

Since $\nabla f$ is Lipschitz continuous, we have

$$
\begin{align*}
0 & \leq \lim _{k \rightarrow \infty} \frac{\left\|\nabla f z_{n_{k}}-\nabla f\left(P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right)\right\|}{\lambda_{n_{k}} l^{-1}}  \tag{3.35}\\
& \leq L \lim _{k \rightarrow \infty} \frac{\left\|z_{n_{k}}-P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right\|}{\lambda_{n_{k}} l^{-1}}
\end{align*}
$$

for some $L>0$. Thus, we obtain from (3.34) and (3.35) that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\left\|\nabla f z_{n_{k}}-\nabla f\left(P_{C}^{f} \nabla f^{*}\left(\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}\right)\right)\right\|}{\lambda_{n_{k}} l^{-1}}=0 . \tag{3.36}
\end{equation*}
$$

From the definition of $s_{n_{k}}$ and (2.6), we obtain

$$
\left\langle\nabla f\left(z_{n_{k}}\right)-\lambda_{n_{k}} l^{-1} A z_{n_{k}}-\nabla f\left(s_{n_{k}}\right), z-s_{n_{k}}\right\rangle \leq 0 \text { for all } z \in C .
$$

This implies that

$$
\begin{align*}
\left\langle\frac{\nabla f\left(z_{n_{k}}\right)-\nabla f\left(s_{n_{k}}\right)}{\lambda_{n_{k}} l^{-1}}, z-s_{n_{k}}\right\rangle & +\left\langle A z_{n_{k}}, s_{n_{k}}-z_{n_{k}}\right\rangle  \tag{3.37}\\
& \leq\left\langle A z_{n_{k}}, z-z_{n_{k}}\right\rangle \text { for all } z \in C .
\end{align*}
$$

Taking the limit on both sides of (3.37) as $k \rightarrow \infty$ and using (3.36), (3.31), uniform continuity of $A$ and the boundedness of the sequences $\left\{z_{n_{k}}\right\}$ and $\left\{s_{n_{k}}\right\}$, we obtain

$$
\begin{equation*}
\liminf _{k \rightarrow \infty}\left\langle A z_{n_{k}}, z-z_{n_{k}}\right\rangle \geq 0 \text { for all } z \in C . \tag{3.38}
\end{equation*}
$$

(ii) Let $\left\{\varepsilon_{k}\right\}$ be a sequence of decreasing nonnegative numbers such that $\varepsilon_{k} \rightarrow 0$ as $k \rightarrow \infty$. For each $\varepsilon_{k}$, we choose $N_{k}$ to be the smallest positive integer such that

$$
\begin{equation*}
\left\langle A z_{n_{k}}, z-z_{n_{k}}\right\rangle+\varepsilon_{k} \geq 0 \text { for all } k \geq N_{k} \tag{3.39}
\end{equation*}
$$

where the existence of $N_{k}$ follows from (3.38). Since $\left\{\varepsilon_{k}\right\}$ is decreasing, $\left\{N_{k}\right\}$ is increasing. If there exists $N>0$ such that $A z_{N_{k}}=0$ for all $k \geq N$, then

$$
\left\langle A z_{N_{k}}, z-z_{N_{k}}\right\rangle \geq 0
$$

for all $k \geq N$ and $z \in C$. Since $A$ is pseudomonotone, we have that

$$
\begin{equation*}
\left\langle A z, z-z_{N_{k}}\right\rangle \geq 0 \quad \text { for all } k \geq N \text { and } z \in C . \tag{3.40}
\end{equation*}
$$

Taking the limit on both sides of (3.40) as $k \rightarrow \infty$, we obtain

$$
\langle A z, z-p\rangle \geq 0 \text { for all } k \geq N \text { and } z \in C .
$$

By Lemma 2.10, we conclude that $p \in V I(C, A)$. If there exists a subsequence $\left\{N_{k_{i}}\right\}$ of $\left\{N_{k}\right\}$, again denoted by $\left\{N_{k}\right\}$, such that $A_{z_{N_{k}}} \neq 0$ for all $k \in \mathbb{N}$, then setting $t_{N_{k}}=\frac{J_{E_{1}}^{-1} A z_{N_{k}}}{\left\|A z_{N_{k}}\right\|^{2}}$, we get $\left\langle A z_{N_{k}}, t_{N_{k}}\right\rangle=1$ for each $k$. Therefore, from

$$
\begin{equation*}
\left\langle A z_{N_{k}}, z+\varepsilon_{k} t_{N_{k}}-z_{N_{k}}\right\rangle \geq 0 . \tag{3.39}
\end{equation*}
$$

Since $A$ is pseudomonotone, we have that

$$
\begin{equation*}
\left\langle A\left(z+\varepsilon_{k} t_{N_{k}}\right), z+\varepsilon_{k} t_{N_{k}}-z_{N_{k}}\right\rangle \geq 0 . \tag{3.41}
\end{equation*}
$$

Since $\left\{z_{n_{k}}\right\}$ converges weakly to $p$ as $k \rightarrow \infty$ and $A$ is sequentially weakly continuous, we have that $\left\{A z_{n_{k}}\right\}$ converges weakly to $A p$. Suppose $A p \neq$ 0 (otherwise, $p \in V I(C, A)$ ). Then by the sequentially weakly lower semicontinuity of the norm, we get

$$
0<\|A p\| \leq \liminf _{k \rightarrow \infty}\left\|A z_{n_{k}}\right\| .
$$

Since $\left\{z_{N_{k}}\right\} \subset\left\{z_{n_{k}}\right\}$ and $\varepsilon_{k} \rightarrow 0$ as $k \rightarrow \infty$, we get that

$$
\begin{aligned}
0 & \leq \limsup _{k \rightarrow \infty}\left\|\varepsilon_{k} t_{N_{k}}\right\|=\limsup _{k \rightarrow \infty}\left(\frac{\varepsilon_{k}}{\left\|A z_{n_{k}}\right\|}\right) \\
& \leq \frac{\limsup _{k \rightarrow \infty} \varepsilon_{k}}{\liminf }{ }_{k \rightarrow \infty}\left\|A z_{n_{k}}\right\|
\end{aligned} \frac{0}{\|A p\|}=0 .
$$

Hence, $\lim \sup _{k \rightarrow \infty}\left\|\varepsilon_{k} t_{N_{k}}\right\|=0$. So, taking the limit on both sides of (3.41) as $k \rightarrow \infty$, we get

$$
\langle A z, z-p\rangle \geq 0 \text { for all } z \in C \text {. }
$$

Therefore, by Lemma 2.10, we have $p \in V I(C, A)$.
(2) Part (2) of the lemma can be proved in a similar way.

Theorem 3.4. Suppose that Conditions ( $A 1)-(A 2),(B 1)-(B 4)$ and $(C 1)-$ $(C 2)$ hold. Then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$ converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon$, where $(\bar{x}, \bar{u})=P_{\Upsilon}^{h}(x, u)$, for $h=(f, g)$.
Proof. Let $(\bar{x}, \bar{u})=P_{\Upsilon}^{h}(x, u)$. Denote $C_{n}=\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)$ and

$$
\begin{aligned}
\Delta_{n} & =\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n}-\bar{x}\right\rangle+\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n}-\bar{u}\right\rangle \\
& =\left\langle(\nabla f(x), \nabla g(u))-(\nabla f(\bar{x}), \nabla g(\bar{u})),\left(x_{n}, u_{n}\right)-(\bar{x}, \bar{u})\right\rangle .
\end{aligned}
$$

Then, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\Delta_{n+1}= & \left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n}-\bar{x}\right\rangle+\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n}-\bar{u}\right\rangle \\
& +\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-x_{n}\right\rangle \\
& +\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n+1}-u_{n}\right\rangle \\
\leq & \Delta_{n}+\Lambda\left[\left\|x_{n+1}-x_{n}\right\|+\left\|u_{n+1}-u_{n}\right\|\right],
\end{aligned}
$$

for some constant $\Lambda>0$. From (3.4), (2.10) and (2.11) we obtain

$$
\begin{aligned}
& D_{f}\left(\bar{x}, x_{n+1}\right) \\
&= D_{f}\left(\bar{x}, \nabla f^{*}\left(\alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right) C_{n}\right)\right) \\
&= V_{f}\left(\bar{x}, \alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right) \nabla f\left(\nabla f^{*}\left(C_{n}\right)\right)\right) \\
& \leq V_{f}\left(\bar{x}, \alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right) \nabla f\left(\nabla f^{*}\left(C_{n}\right)\right)-\alpha_{n}(\nabla f(x)-\nabla f(\bar{x}))\right) \\
& \quad-\left\langle-\alpha_{n}(\nabla f(x)-\nabla f(\bar{x})),\right. \\
&\left.\nabla f^{*}\left(\alpha_{n} \nabla f(x)+\left(1-\alpha_{n}\right) \nabla f\left(\nabla f^{*}\left(C_{n}\right)\right)\right)-\bar{x}\right\rangle \\
&= V_{f}\left(\bar{x}, \alpha_{n} \nabla f(\bar{x})+\left(1-\alpha_{n}\right) \nabla f\left(\nabla f^{*}\left(C_{n}\right)\right)\right) \\
&+\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle \\
&= D_{f}\left(\bar{x}, \nabla f^{*}\left(\alpha_{n} \nabla f(\bar{x})+\left(1-\alpha_{n}\right) \nabla f\left(\nabla f^{*}\left(C_{n}\right)\right)\right)\right) \\
&+\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle \\
& \leq \alpha_{n} D_{f}(\bar{x}, \bar{x})+\left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, \nabla f^{*}\left(C_{n}\right)\right) \\
&+\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle,
\end{aligned}
$$

which implies by Lemma 2.7 that

$$
\begin{align*}
D_{f}\left(\bar{x}, x_{n+1}\right) \leq & \left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, \nabla f^{*}\left(C_{n}\right)\right) \\
& +\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle . \tag{3.42}
\end{align*}
$$

Furthermore, from (2.9), (2.10), part (iii) of Lemma 2.14 and Bregman relatively $f$-nonexpansiveness of $G$, we have

$$
\begin{aligned}
D_{f}\left(\bar{x}, \nabla f^{*}\left(C_{n}\right)\right)= & D_{f}\left(\bar{x}, \nabla f^{*}\left(\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right)\right) \\
= & V_{f}\left(\bar{x}, \tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right) \\
= & f(\bar{x})+f^{*}\left(\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right)\right) \\
& -\left\langle\tau \nabla f\left(a_{n}\right)+(1-\tau) G\left(a_{n}\right), \bar{x}\right\rangle \\
\leq & f(\bar{x})+\tau f^{*}\left(\nabla f\left(a_{n}\right)\right)+(1-\tau) f^{*}\left(G\left(a_{n}\right)\right) \\
& -\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right) \\
& -\tau\left\langle\nabla f\left(a_{n}\right), \bar{x}\right\rangle-(1-\tau)\left\langle G\left(a_{n}\right), \bar{x}\right\rangle \\
= & \tau V_{f}\left(\bar{x}, \nabla f\left(a_{n}\right)\right)+(1-\tau) V_{f}\left(\bar{x}, G\left(a_{n}\right)\right) \\
& -\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right)
\end{aligned}
$$

$$
\begin{align*}
= & \tau D_{f}\left(\bar{x}, a_{n}\right)+(1-\tau) D_{f}\left(\bar{x}, \nabla f^{*} G\left(a_{n}\right)\right) \\
& -\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right) \\
\leq & \tau D_{f}\left(\bar{x}, a_{n}\right)+(1-\tau) D_{f}\left(\bar{x}, a_{n}\right)  \tag{3.43}\\
& -\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right) \\
= & D_{f}\left(\bar{x}, a_{n}\right)-\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right),
\end{align*}
$$

where $\phi_{1}$ is the modulus of uniform convexity of $f^{*}$.
Substituting (3.43) into (3.42), we get

$$
\begin{align*}
D_{f}\left(\bar{x}, x_{n+1}\right) \leq & \left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, a_{n}\right) \\
& -\left(1-\alpha_{n}\right) \tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right)  \tag{3.44}\\
& +\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle .
\end{align*}
$$

Again using (3.23), we obtain

$$
\begin{align*}
D_{f}\left(\bar{x}, x_{n+1}\right) \leq & \left(1-\alpha_{n}\right) D_{f}\left(\bar{x}, x_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right)\left(D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)\right) \\
& -\left(1-\alpha_{n}\right) \tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right)  \tag{3.45}\\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(T x_{n}-S u_{n}\right), T q_{n}-T \bar{x}\right\rangle \\
& +\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle .
\end{align*}
$$

Similarly, we get

$$
\begin{align*}
D_{g}\left(\bar{u}, u_{n+1}\right) \leq & \left(1-\alpha_{n}\right) D_{g}\left(\bar{u}, u_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right)\left(D_{g}\left(b_{n}, v_{n}\right)+D_{g}\left(v_{n}, w_{n}\right)\right) \\
& -\left(1-\alpha_{n}\right) \tau(1-\tau) \phi_{2}\left(\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\|\right)  \tag{3.46}\\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(S u_{n}-T x_{n}\right), S t_{n}-S \bar{u}\right\rangle \\
& +\alpha_{n}\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n+1}-\bar{u}\right\rangle,
\end{align*}
$$

where $\phi_{2}$ is the modulus of uniform convexity of $g^{*}$.
Let $\Theta_{n}=D_{f}\left(\bar{x}, x_{n}\right)+D_{g}\left(\bar{u}, u_{n}\right)$. Then, combining (3.45) and (3.46) and using the relation in (3.30), we obtain

$$
\begin{align*}
\Theta_{n+1} \leq & \left(1-\alpha_{n}\right) \Theta_{n}-\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{f}\left(a_{n}, y_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{f}\left(y_{n}, z_{n}\right)-\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{g}\left(b_{n}, v_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{g}\left(v_{n}, w_{n}\right) \\
& -\left(1-\alpha_{n}\right) \tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right) \\
& -\left(1-\alpha_{n}\right) \tau(1-\tau) \phi_{2}\left(\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\|\right) \\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(T x_{n}-S u_{n}\right), T q_{n}-T \bar{x}\right\rangle \\
& -\left(1-\alpha_{n}\right) \gamma_{n}\left\langle J_{E_{3}}\left(S u_{n}-T x_{n}\right), S t_{n}-S \bar{u}\right\rangle \\
& +\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle+\alpha_{n}\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n+1}-\bar{u}\right\rangle \\
\leq & \left(1-\alpha_{n}\right) \Theta_{n}-\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{f}\left(a_{n}, y_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{f}\left(y_{n}, z_{n}\right)-\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{g}\left(b_{n}, v_{n}\right) \\
& -\left(1-\alpha_{n}\right)\left(1-\frac{\mu}{\beta}\right) D_{g}\left(v_{n}, w_{n}\right) \\
& -\tau(1-\tau)\left(1-\alpha_{n}\right) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right) \\
& -\tau(1-\tau)\left(1-\alpha_{n}\right) \phi_{2}\left(\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\|\right) \\
& -\left(1-\alpha_{n}\right) \frac{\rho}{2}\left\|T x_{n}-S u_{n}\right\|^{2}+\alpha_{n}\left\langle\nabla f(x)-\nabla f(\bar{x}), x_{n+1}-\bar{x}\right\rangle \\
& +\alpha_{n}\left\langle\nabla g(u)-\nabla g(\bar{u}), u_{n+1}-\bar{u}\right\rangle . \tag{3.47}
\end{align*}
$$

This implies that

$$
\begin{align*}
(1- & \left.\frac{\mu}{\beta}\right)\left[D_{f}\left(a_{n}, y_{n}\right)+D_{f}\left(y_{n}, z_{n}\right)+D_{g}\left(b_{n}, v_{n}\right)+D_{g}\left(v_{n}, w_{n}\right)\right] \\
& +\frac{\rho}{2}\left\|T x_{n}-S u_{n}\right\|^{2}+\tau(1-\tau) \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right)  \tag{3.48}\\
& +\tau(1-\tau) \phi_{2}\left(\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\|\right) \\
\leq & \Theta_{n}-\Theta_{n+1}+\alpha_{n} M
\end{align*}
$$

for some $M>0$, where the existence of such $M$ is guaranteed by the boundedness of $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$.

Now, we show that the sequence $\left\{\Theta_{n}\right\}$ of real numbers, converges strongly to zero by considering two cases:

Case I. If there exists a natural number $n_{0}$ such that $\Theta_{n+1} \leq \Theta_{n}$ for all $n \geq n_{0}$, then $\left\{\Theta_{n}\right\}$ converges. Taking the limit as $n \rightarrow \infty$ in (3.48), we get

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|T x_{n}-S u_{n}\right\|^{2}=0 \tag{3.49}
\end{equation*}
$$

and

$$
\lim _{n \rightarrow \infty} \phi_{1}\left(\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|\right)=0=\lim _{n \rightarrow \infty} \phi_{2}\left(\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\|\right),
$$

which implies that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|\nabla f\left(a_{n}\right)-G\left(a_{n}\right)\right\|=0=\lim _{n \rightarrow \infty}\left\|\nabla g\left(b_{n}\right)-K\left(b_{n}\right)\right\| . \tag{3.50}
\end{equation*}
$$

From the definition of $z_{n},(2.10),(2.11)$, property of the Bregman projection, and Cauchy-Schwarz inequality, we have

$$
\begin{align*}
D_{f} & \left(x_{n}, z_{n}\right) \\
\leq & D_{f}\left(x_{n}, \nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)\right) \\
= & V_{f}\left(x_{n}, \nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right) \\
\leq & V_{f}\left(x_{n}, \nabla f\left(x_{n}\right)\right) \\
& -\left\langle\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right), \nabla f^{*}\left(\nabla f\left(x_{n}\right)-\gamma_{n} T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right)-x_{n}\right\rangle \\
= & D_{f}\left(x_{n}, x_{n}\right)-\gamma_{n}\left\langle T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right), q_{n}-x_{n}\right\rangle \\
\leq & \gamma_{n}\left\|T^{*} J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\|\left\|q_{n}-x_{n}\right\| . \tag{3.51}
\end{align*}
$$

Substituting (3.28) into (3.51), then taking the limit on both sides and using (3.49), we obtain

$$
\begin{aligned}
0 & \leq \lim _{n \rightarrow \infty} D_{f}\left(x_{n}, z_{n}\right) \leq \lim _{n \rightarrow \infty}\left(\frac{\gamma_{n}^{2}}{\beta}\|T\|^{2}\left\|J_{E_{3}}\left(T x_{n}-S u_{n}\right)\right\|^{2}\right) \\
& =\lim _{n \rightarrow \infty}\left(\frac{\gamma_{n}^{2}}{\beta}\|T\|^{2}\left\|T x_{n}-S u_{n}\right\|^{2}\right)=0 .
\end{aligned}
$$

This implies that $\lim _{n \rightarrow \infty} D_{f}\left(x_{n}, z_{n}\right)=0$ and hence by Lemma 2.9 we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-z_{n}\right\|=0 \tag{3.52}
\end{equation*}
$$

Similarly, one can show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n}-w_{n}\right\|=0 \tag{3.53}
\end{equation*}
$$

Moreover, by taking the limit as $n \rightarrow \infty$ on both sides of (3.48), we obtain

$$
\begin{aligned}
\lim _{n \rightarrow \infty} D_{f}\left(a_{n}, y_{n}\right) & =0, \quad \lim _{n \rightarrow \infty} D_{f}\left(y_{n}, z_{n}\right)=0 \\
\lim _{n \rightarrow \infty} D_{g}\left(b_{n}, v_{n}\right) & =0, \quad \lim _{n \rightarrow \infty} D_{g}\left(v_{n}, w_{n}\right)=0,
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|a_{n}-y_{n}\right\|=0, \quad \lim _{n \rightarrow \infty}\left\|y_{n}-z_{n}\right\|=0 \tag{3.54}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|b_{n}-v_{n}\right\|=0, \quad \lim _{n \rightarrow \infty}\left\|v_{n}-w_{n}\right\|=0 \tag{3.55}
\end{equation*}
$$

From (3.52) and (3.54), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-y_{n}\right\| \leq \lim _{n \rightarrow \infty}\left\|x_{n}-z_{n}\right\|+\lim _{n \rightarrow \infty}\left\|z_{n}-y_{n}\right\|=0 \tag{3.56}
\end{equation*}
$$

and from (3.53) and (3.55), we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n}-v_{n}\right\| \leq \lim _{n \rightarrow \infty}\left\|u_{n}-w_{n}\right\|+\lim _{n \rightarrow \infty}\left\|w_{n}-v_{n}\right\|=0 . \tag{3.57}
\end{equation*}
$$

From (3.54) and (3.56), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|a_{n}-x_{n}\right\| \leq \lim _{n \rightarrow \infty}\left\|a_{n}-y_{n}\right\|+\lim _{n \rightarrow \infty}\left\|y_{n}-x_{n}\right\|=0 . \tag{3.58}
\end{equation*}
$$

From (3.4) and (3.50), we have

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left\|\nabla f\left(x_{n+1}\right)-\nabla f\left(a_{n}\right)\right\| \leq & \lim _{n \rightarrow \infty}\left(\alpha_{n}\left\|\nabla f(x)-\tau \nabla f\left(a_{n}\right)\right\|\right) \\
& +(1-\tau) \lim _{n \rightarrow \infty}\left(\left(1-\alpha_{n}\right)\left\|G\left(a_{n}\right)-\nabla f\left(a_{n}\right)\right\|\right) \\
= & 0 . \tag{3.59}
\end{align*}
$$

From (3.59) and part (iii) of Lemma 2.8, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n+1}-a_{n}\right\|=0 \tag{3.60}
\end{equation*}
$$

Therefore, from (3.60) and (3.58), we obtain

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\| & \leq \lim _{n \rightarrow \infty}\left\|x_{n+1}-a_{n}\right\|+\lim _{n \rightarrow \infty}\left\|a_{n}-x_{n}\right\| \\
& =0 . \tag{3.61}
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n+1}-u_{n}\right\|=0 \tag{3.62}
\end{equation*}
$$

Since $\left\{\left(x_{n}, u_{n}\right)\right\}$ is bounded and $E_{1} \times E_{2}$ is reflexive (Lemma 2.6), there exists a subsequence $\left\{\left(x_{n_{k}}, u_{n_{k}}\right)\right\}$ of $\left\{\left(x_{n}, u_{n}\right)\right\}$ which converges weakly to some $\left(x^{*}, u^{*}\right) \in E_{1} \times E_{2}$ and

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \Delta_{n}=\lim _{k \rightarrow \infty} \Delta_{n_{k}} . \tag{3.63}
\end{equation*}
$$

Consequently, we have $x_{n_{k}} \rightharpoonup x^{*}$ and $u_{n_{k}} \rightharpoonup u^{*}$. From (3.52) and (3.53), we have $z_{n_{k}} \rightharpoonup x^{*}$ and $w_{n_{k}} \rightharpoonup u^{*}$, respectively. So, from (3.54), the fact that $z_{n_{k}} \rightharpoonup x^{*}$ and Lemma 3.3, we obtain $x^{*} \in V I(C, A)$. Similarly, one can show that $u^{*} \in V I(D, B)$. From (3.58), we have $a_{n} \rightharpoonup x^{*}$. Thus, with the help of (3.50) and the definition of $f$-asymptotic fixed points, we conclude that
$x^{*} \in \widehat{F_{f}(G)}$. From the Bregman relatively $f$-nonexpansivity of $G$, we have $x^{*} \in F_{f}(G)$. So, $x^{*} \in V I(C, A) \cap F_{f}(G)$.

Similarly, we can show that $u^{*} \in V I(D, B) \cap F_{g}(K)$.
Moreover, by Lemma 2.1 we have

$$
\begin{align*}
\left\|T x^{*}-S u^{*}\right\|^{2}= & \left\|T x_{n_{k}}-S u_{n_{k}}+T x^{*}-T x_{n_{k}}+S u_{n_{k}}-S u^{*}\right\|^{2} \\
\leq & \left\|T x_{n_{k}}-S u_{n_{k}}\right\|^{2}  \tag{3.64}\\
& +2\left\langle J_{E_{3}}\left(T x^{*}-S u^{*}\right), T x^{*}-T x_{n_{k}}+S u_{n_{k}}-S u^{*}\right\rangle .
\end{align*}
$$

Since $T$ and $S$ are sequentially weakly continuous, we have that $x_{n_{k}} \rightharpoonup x^{*}$ implies $T x_{n_{k}} \rightharpoonup T x^{*}$, and $u_{n_{k}} \rightharpoonup u^{*}$ implies $S u_{n_{k}} \rightharpoonup S u^{*}$. Thus, we obtain using (3.49) that $T x^{*}=S u^{*}$. Therefore, $\left(x^{*}, u^{*}\right) \in \Upsilon$.

Furthermore, from the definition of $\Delta_{n+1},(3.61),(3.62),(3.63)$ and (2.8) we have

$$
\begin{align*}
\limsup _{n \rightarrow \infty} \Delta_{n+1} & \leq \limsup _{n \rightarrow \infty} \Delta_{n}+\Lambda \limsup _{n \rightarrow \infty}\left[\left\|x_{n+1}-x_{n}\right\|+\left\|u_{n+1}-u_{n}\right\|\right] \\
& =\lim _{k \rightarrow \infty} \Delta_{n_{k}}+\Lambda \lim _{k \rightarrow \infty}\left[\left\|x_{n_{k}+1}-x_{n_{k}}\right\|+\left\|u_{n_{k}+1}-u_{n_{k}}\right\|\right] \\
& =\lim _{k \rightarrow \infty}\left\langle(\nabla f(x), \nabla g(u))-(\nabla f(\bar{x}), \nabla g(\bar{u})),\left(x_{n_{k}}, u_{n_{k}}\right)-(\bar{x}, \bar{u})\right\rangle \\
& =\left\langle(\nabla f(x), \nabla g(u))-(\nabla f(\bar{x}), \nabla g(\bar{u})),\left(x^{*}, u^{*}\right)-(\bar{x}, \bar{u})\right\rangle \\
& \leq 0 . \tag{3.65}
\end{align*}
$$

From (3.47) we have

$$
\begin{equation*}
\Theta_{n+1} \leq\left(1-\alpha_{n}\right) \Theta_{n}+\alpha_{n} \Delta_{n+1} . \tag{3.66}
\end{equation*}
$$

So, (3.65), (3.66), Lemma 2.11 and the condition on $\alpha_{n}$ give that

$$
\lim _{n \rightarrow \infty} \Theta_{n}=0
$$

which implies that

$$
\lim _{n \rightarrow \infty} D_{f}\left(\bar{x}, x_{n}\right)=0
$$

and

$$
\lim _{n \rightarrow \infty} D_{g}\left(\bar{u}, u_{n}\right)=0
$$

Thus, by Lemma 2.9 we obtain $\lim _{n \rightarrow \infty} x_{n}=\bar{x}$ and $\lim _{n \rightarrow \infty} u_{n}=\bar{u}$.
Case II. If there exists a subsequence $\left\{\Theta_{n_{i}}\right\}$ of $\left\{\Theta_{n}\right\}$ with $\Theta_{n_{i}}<\Theta_{n_{i}+1}$ for all $i \geq 0$, then by Lemma 2.12, we can find a nondecreasing sequence $\left\{m_{k}\right\}$ of positive integers such that $\lim _{k \rightarrow \infty} m_{k}=\infty$ and

$$
\begin{equation*}
\Theta_{m_{k}} \leq \Theta_{m_{k}+1} \quad \text { and } \quad \Theta_{k} \leq \Theta_{m_{k}+1} \tag{3.67}
\end{equation*}
$$

for all positive integers $k$. Thus, (3.48) becomes

$$
\begin{align*}
(1- & \left.\frac{\mu}{\beta}\right)\left[D_{f}\left(a_{m_{k}}, y_{m_{k}}\right)+D_{f}\left(y_{m_{k}}, z_{m_{k}}\right)+D_{g}\left(b_{m_{k}}, v_{m_{k}}\right)+D_{g}\left(v_{m_{k}}, w_{m_{k}}\right)\right] \\
& +\tau(1-\tau)\left[\phi_{1}\left(\left\|\nabla f\left(a_{m_{k}}\right)-G\left(a_{m_{k}}\right)\right\|\right)+\phi_{2}\left(\left\|\nabla g\left(b_{m_{k}}\right)-K\left(b_{m_{k}}\right)\right\|\right)\right] \\
& +\frac{\rho}{2}\left\|T x_{m_{k}}-S u_{m_{k}}\right\|^{2} \\
\leq & \Theta_{m_{k}}-\Theta_{m_{k}+1}+\alpha_{m_{k}} M . \tag{3.68}
\end{align*}
$$

Taking the limit as $k \rightarrow \infty$ to both sides of (3.68), we derive

$$
\begin{gathered}
\lim _{k \rightarrow \infty}\left\|T x_{m_{k}}-S u_{m_{k}}\right\|^{2}=0, \\
\lim _{k \rightarrow \infty}\left\|\nabla f\left(a_{m_{k}}\right)-G\left(a_{m_{k}}\right)\right\|=\lim _{k \rightarrow \infty}\left\|\nabla g\left(b_{m_{k}}\right)-K\left(b_{m_{k}}\right)\right\|=0 .
\end{gathered}
$$

Following the method used in Case I, we get

$$
\begin{array}{r}
\lim _{k \rightarrow \infty}\left\|x_{m_{k}}-z_{m_{k}}\right\|=\lim _{k \rightarrow \infty}\left\|u_{m_{k}}-w_{m_{k}}\right\|=0, \\
\lim _{k \rightarrow \infty}\left\|y_{m_{k}}-z_{m_{k}}\right\|=\lim _{k \rightarrow \infty}\left\|v_{m_{k}}-w_{m_{k}}\right\|=0, \\
\lim _{k \rightarrow \infty}\left\|x_{m_{k}}-y_{m_{k}}\right\|=\lim _{k \rightarrow \infty}\left\|u_{m_{k}}-v_{m_{k}}\right\|=0 .
\end{array}
$$

Furthermore, following similar steps as in Case I, we obtain

$$
\lim _{k \rightarrow \infty}\left\|x_{m_{k}+1}-x_{m_{k}}\right\|=0, \quad \lim _{k \rightarrow \infty}\left\|u_{m_{k}+1}-u_{m_{k}}\right\|=0
$$

and

$$
\limsup _{k \rightarrow \infty} \Delta_{m_{k}+1} \leq 0
$$

Thus, from (3.47) and (3.67), we have

$$
\alpha_{m_{k}} \Theta_{m_{k}} \leq \Theta_{m_{k}}-\Theta_{m_{k}+1}+\alpha_{m_{k}} \Delta_{m_{k}+1} \leq \alpha_{m_{k}} \Delta_{m_{k}+1},
$$

which implies that

$$
\begin{equation*}
\Theta_{m_{k}} \leq \Delta_{m_{k}+1} \tag{3.69}
\end{equation*}
$$

Taking the limit on both sides of (3.69) as $k \rightarrow \infty$ and using the fact that $\lim \sup _{k \rightarrow \infty} \Delta_{m_{k}+1} \leq 0$, we get that the sequence $\Theta_{m_{k}} \rightarrow 0$ as $k \rightarrow \infty$. It follows from (3.66) that $\Theta_{m_{k}+1} \rightarrow 0$ as $k \rightarrow \infty$. Since $\Theta_{k} \leq \Theta_{m_{k}+1}$ for all $k \geq 0$, we have that $\Theta_{k} \rightarrow 0$ as $k \rightarrow \infty$. Thus, we have $\lim _{k \rightarrow \infty} D_{f}\left(\bar{x}, x_{k}\right)=0$ and $\lim _{k \rightarrow \infty} D_{g}\left(\bar{u}, u_{k}\right)=0$, which implies by Lemma 2.9 that $\lim _{k \rightarrow \infty} x_{k}=\bar{x}$ and $\lim _{k \rightarrow \infty} u_{k}=\bar{u}$.

Thus, we have shown, in Cases I and II, that the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm A, converges strongly to $(\bar{x}, \bar{u})=P_{\Upsilon}^{h}(x, u)$, and this completes the proof.

If $A$ and $B$ are uniformly continuous and monotone mappings, then the assumption that $A$ and $B$ are sequentially weakly continuous is not required and hence the following corollary follows.
Corollary 3.5. Assume that $A: C \rightarrow E_{1}^{*}$ and $B: D \rightarrow E_{2}^{*}$ are uniformly continuous and monotone mappings. If Conditions $(A 1)-(A 2),(B 2)-(B 4)$ and $(C 1)-(C 2)$ are satisfied, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon$, where $(\bar{x}, \bar{u})=P_{\Upsilon}^{h}(x, u)$.

If $x=0=u$, then Algorithm A can be used to locate an element of the solution with the minimum norm and hence we have the following corollary.
Corollary 3.6. Suppose that the Conditions (A1) - (A2), (B1) - (B4) and $(C 1)-(C 2)$ are satisfied. Then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$ with $x=0$ and $u=0$, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon$, where $(\bar{x}, \bar{u})=$ $P_{\Upsilon}^{h}(0,0)$.

## 4. Applications

## Condition 4.1.

This section deals with applications of the main result to some specific cases. The following are the assumptions that will be used in these cases.
(A3) Let $C$ and $D$ be nonempty, closed and convex subsets of a smooth, strictly convex, reflexive real Banach space $E$ with dual $E^{*}$;
(A4) Let $f, g: E \rightarrow \mathbb{R}$ be proper, lower semi-continuous, uniformly Fréchet differentiable, strongly convex, strongly coercive, Legendre functions which are bounded on bounded subsets. Let $f$ and $g$ have Lipschitz continuous gradients with the strong convexity constant of $f$ (respectively, $g$ ) greater than or equal to the Lipschitz constant of $\nabla f$ (respectively, $\nabla g$ );
(B5) Let $A, B: E \rightarrow E^{*}$ be uniformly continuous, pseudomonotone and sequentially weakly continuous on bounded subsets of $E$;
(B6) Let $G, K: E \rightarrow E^{*}$ be Bregman relatively $f$-nonexpansive and Bregman relatively $g$-nonexpansive mappings, respectively.
4.1. Common Solutions of Variational Inequality and $f, g$-Fixed Point

Problems. Let $E=E_{1}=E_{2}=E_{3}, T=S=I$ and let $f$ and $g$ be as in Condition 4.1 (A4). Then the split equality of variational inequality and $f, g-$ fixed point problems reduces to finding a common solution of two variational inequality and $f, g$-fixed point problems. This problem can be expressed as:
find $\bar{x} \in\left(V I(C, A) \cap F_{f}(G)\right)$ and $\bar{u} \in\left(V I(D, B) \cap F_{g}(K)\right)$ such that $\bar{x}=\bar{u}$.
Denote $\Sigma=\left\{(\bar{x}, \bar{u}) \in\left(V I(C, A) \cap F_{f}(G)\right) \times\left(V I(D, B) \cap F_{g}(K)\right): \bar{x}=\bar{u}\right\}$.
In this case, we have the following corollaries.

Corollary 4.1. Assume that $\Sigma \neq \emptyset$. If Conditions $(A 3)-(A 4),(B 5)-(B 6)$, $(C 1)-(C 2)$ are satisfied, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, with $E=E_{1}=E_{2}=E_{3}$ and $T=S=I$, converges strongly to $(\bar{x}, \bar{u}) \in \Sigma$, where $(\bar{x}, \bar{u})=P_{\Sigma}^{h}(x, u)$.

Corollary 4.2. Assume that $\Sigma \neq \emptyset$ and let $A: E \rightarrow E^{*}$ and $B: E \rightarrow E^{*}$ be uniformly continuous monotone mappings. If Conditions $(A 3)-(A 4),(B 6)$, $(C 1)-(C 2)$ are satisfied, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm A, with $E=E_{1}=E_{2}=E_{3}$ and $T=S=I$, converges strongly to $(\bar{x}, \bar{u}) \in \Sigma$, where $(\bar{x}, \bar{u})=P_{\Sigma}^{h}(x, u)$.
4.2. Split Equality of Null Point and $f, g$-Fixed Point Problems. Let $f$ and $g$ be as in (A2). If $C=E_{1}$ and $D=E_{2}$, then the split equality of variational inequality and $f, g$-fixed point problem reduces to the split equality of null point and $f, g$-fixed point problem which can be described as finding a point $(\bar{x}, \bar{u})$ with the property

$$
(\bar{x}, \bar{u}) \in\left(A^{-1}(0) \cap F_{f}(G)\right) \times\left(B^{-1}(0) \cap F_{g}(K)\right): T \bar{x}=S \bar{u},
$$

where $A^{-1}(0)=\left\{x \in E_{1}: 0 \in A x\right\}$ and $B^{-1}(0)=\left\{u \in E_{2}: 0 \in B u\right\}$.
Denote

$$
\Upsilon^{*}=\left\{(\bar{x}, \bar{u}) \in\left(A^{-1}(0) \cap F_{f}(G)\right) \times\left(B^{-1}(0) \cap F_{g}(K)\right): T \bar{x}=S \bar{u}\right\} .
$$

In this case, we have the following results:
Corollary 4.3. Assume that $\Upsilon^{*} \neq \emptyset$. If Conditions $(A 1)-(A 2),(B 1)-(B 3)$, with $C=E_{1}, D=E_{2},(C 1)-(C 2)$ are satisfied, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon^{*}$, where $(\bar{x}, \bar{u})=$ $P_{\Upsilon^{*}}^{h}(x, u)$.

Corollary 4.4. Let $A: E_{1} \rightarrow E_{1}^{*}$ and $B: E_{2} \rightarrow E_{2}^{*}$ be uniformly continuous monotone mappings and assume that $\Upsilon^{*} \neq \emptyset$. If Conditions (A1) $(A 2),(B 2)-(B 3)$ with $C=E_{1}$ and $D=E_{2},(C 1)-(C 2)$ are satisfied, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon^{*}$, where $(\bar{x}, \bar{u})=P_{\Upsilon^{*}}^{h}(x, u)$.
4.3. Split Equality Variational Inequality Problem. Let $f$ and $g$ be as in (A2). If, in Condition 3.2, $G x=\nabla f(x)$ for all $x \in C$ and $K u=\nabla g(u)$ for all $u \in D$, then the split equality of variational inequality and $f, g$-fixed point problems reduces to the split equality variational inequality problem which seeks to
find $\bar{x} \in V I(C, A)$ and $\bar{u} \in V I(D, B)$ such that $T \bar{x}=S \bar{u}$.
Denote $\Gamma=\{(\bar{x}, \bar{u}) \in V I(C, A) \times V I(D, B): T \bar{x}=S \bar{u}\}$.

One can easily show that $\nabla f$ and $\nabla g$ are Bregman $f$-relatively nonexpansive and Bregman $g$-relatively nonexpansive, respectively, and hence we have the following results.

Corollary 4.5. Assume that $\Gamma \neq \emptyset$. If Conditions $(A 1)-(A 2),(B 1)-(B 3)$, $(C 1)-(C 2)$ are satisfied with $G x=\nabla f(x)$ for all $x \in C$ and $K u=\nabla g(u)$ for all $u \in D$, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, converges strongly to $(\bar{x}, \bar{u}) \in \Gamma$, where $(\bar{x}, \bar{u})=P_{\Gamma}^{h}(x, u)$.

Corollary 4.6. Let $A: E_{1} \rightarrow E_{1}^{*}$ and $B: E_{2} \rightarrow E_{2}^{*}$ be uniformly continuous monotone mappings. Assume that $\Gamma \neq \emptyset$. If Conditions (A1)-(A2), (B2) (B3), (C1) - (C2) are satisfied with $G x=\nabla f(x)$ for all $x \in C$ and $K u=$ $\nabla g(u)$ for all $u \in D$, then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, converges strongly to $(\bar{x}, \bar{u}) \in \Gamma$, where $(\bar{x}, \bar{u})=P_{\Gamma}^{h}(x, u)$.
4.4. Split Equality $f$-Fixed Point Problem. If we have $A=0$ and $B=0$, in Algorithm A, then the split equality of variational inequality and $f, g$-fixed point problem reduces to the split equality $f, g$-fixed point problem which seeks to
find $\bar{x} \in F_{f}(G)$ and $\bar{u} \in F_{g}(K)$ such that $T \bar{x}=S \bar{u}$.
Denote $\Gamma^{*}=\left\{(\bar{x}, \bar{u}) \in F_{f}(G) \times F_{g}(K): T \bar{x}=S \bar{u}\right\}$.
Corollary 4.7. Assume that $\Gamma^{*} \neq \emptyset$. If Conditions (A1) - (A2), (B2) - (B3), $(C 1)-(C 2)$ are satisfied. then the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ generated by Algorithm $A$, with $A=B=0$, converges strongly to $(\bar{x}, \bar{u}) \in \Gamma^{*}$, where $(\bar{x}, \bar{u})=P_{\Gamma^{*}}(x, u)$.

## 5. Numerical Example

Under this section, a numerical example is given to demonstrate the convergence of the sequence generated by Algorithm A.
Example 5.1. Given $E_{1}=E_{2}=E_{3}=\mathbb{R}^{2}$. Let the norm and inner product on $\mathbb{R}^{2}$ be, respectively, given by $\|x\|=\sqrt{\langle x, x\rangle}=\sqrt{\sum_{i=1}^{2}\left|x_{i}\right|^{2}}$ and $\langle x, u\rangle=$ $\sum_{\text {by }}^{2} x_{i} u_{i}$ for all $x, u \in \mathbb{R}^{2}$. Define the mappings $A: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ and $B: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$

$$
\begin{gathered}
A x=A\left(x_{1}, x_{2}\right)=\left(\frac{5}{2}+\sqrt{x_{1}^{2}+x_{2}^{2}}\right)\left(x_{1}, x_{2}-1\right), \\
B u=B\left(u_{1}, u_{2}\right)=\sqrt{u_{1}^{2}+u_{2}^{2}}\left(u_{1}, u_{2}\right)
\end{gathered}
$$

and let $C$ and $D$ be given by $C=\left\{x \in \mathbb{R}^{2}:\|x\| \leq 2\right\}, D=\left\{u \in \mathbb{R}^{2}:\|u\| \leq 2\right\}$. Clearly, $C$ and $D$ are nonempty, closed and convex subsets of $\mathbb{R}^{2}$, and the mappings $A$ and $B$ are uniformly continuous and sequentially weakly continuous on $C$ and $D$, respectively. It can be easily shown that $B$ is pseudomonotone
on subsets of $D$. We only show here that $A$ is pseudomonotone on $\mathbb{R}^{2}$. To this end, let $\langle A x, y-x\rangle \geq 0$. Then we have,

$$
\left.\left(\frac{5}{2}+\sqrt{x_{1}^{2}+x_{2}^{2}}\right)\left(x_{1}, x_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle \geq 0
$$

this implies that

$$
\left(\frac{5}{2}+\sqrt{x_{1}^{2}+x_{2}^{2}}\right)\left\langle\left(x_{1}, x_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle \geq 0
$$

Since $\frac{5}{2}+\sqrt{x_{1}^{2}+x_{2}^{2}}>0$, we conclude that $\left\langle\left(x_{1}, x_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle \geq 0$.
Now, for $\left(y_{1}, y_{2}\right) \in \mathbb{R}^{2}$ we have

$$
\begin{aligned}
\langle A y, y-x\rangle & =\left(\frac{5}{2}+\sqrt{y_{1}^{2}+y_{2}^{2}}\right)\left\langle\left(y_{1}, y_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle \\
& \geq\left(\frac{5}{2}+\sqrt{y_{1}^{2}+y_{2}^{2}}\right)\left[\left\langle\left(y_{1}, y_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle\right] \\
& -\left(\frac{5}{2}+\sqrt{y_{1}^{2}+y_{2}^{2}}\right)\left[\left\langle\left(x_{1}, x_{2}-1\right),\left(y_{1}-x_{1}, y_{2}-x_{2}\right)\right\rangle\right] \\
& =\left(\frac{5}{2}+\sqrt{y_{1}^{2}+y_{2}^{2}}\right)\left(\left|y_{1}-x_{1}\right|^{2}+\left|y_{2}-x_{2}\right|^{2}\right) \\
& \geq \frac{5}{2}\|y-x\|^{2} \\
& \geq 0 .
\end{aligned}
$$

Therefore, $A$ is pseudomonotone on $\mathbb{R}^{2}$.
Let us define $T, S: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ by

$$
T x=T\left(x_{1}, x_{2}\right)=\left(5 x_{1}, 0\right) \text { and } S u=S\left(u_{1}, u_{2}\right)=\left(2 u_{1}, 3 u_{2}\right),
$$

where $\left(x_{1}, x_{2}\right),\left(u_{1}, u_{2}\right) \in \mathbb{R}^{2}$. Then $T$ and $S$ are bounded linear maps on $\mathbb{R}^{2}$ with adjoints $T^{*} x=\left(5 x_{1}, 0\right)$ and $S^{*} u=\left(2 u_{1}, 3 u_{2}\right)$, respectively. Now, we have $\left\langle A(0,1),\left(x_{1}, x_{2}\right)-(0,1)\right\rangle \geq 0$ for all $\left(x_{1}, x_{2}\right) \in C,\left\langle B(0,0),\left(u_{1}, u_{2}\right)-(0,0)\right\rangle \geq 0$ for all $\left(u_{1}, u_{2}\right) \in D$. So, $\bar{x} \in V I(C, A)$ and $\bar{u} \in V I(D, B)$, where $\bar{x}=(0,1)$ and $\bar{u}=(0,0)$. Let us define $G: C \rightarrow \mathbb{R}^{2}$ and $K: D \rightarrow \mathbb{R}^{2}$ by

$$
G x=G\left(x_{1}, x_{2}\right)=\left(0, x_{2}\right) \text { and } K u=K\left(u_{1}, u_{2}\right)=\left(u_{2}, u_{1}\right),
$$

and $f, g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ by $f(x)=\frac{1}{2}\|x\|^{2}$ and $g(u)=\frac{1}{2}\|u\|^{2}$. Then, we have $\nabla f(x)=x, \nabla g(u)=u$ and $J_{E}=I$, where $I$ is the identity mapping on $\mathbb{R}^{2}$. One can easily show that $G$ is Bregman relatively $f$-nonexpansive and $K$ is Bregman relatively $g$-nonexpansive. Moreover,

$$
G \bar{x}=G(0,1)=(0,1)=\nabla f(0,1) \text { and } K \bar{u}=K(0,0)=(0,0)=\nabla g(0,0) .
$$

From this, we conclude that $\bar{x} \in F_{f}(G)$ and $\bar{u} \in F_{g}(K)$. Moreover, $T(0,1)=$ $(0,0)=S(0,0)$. Thus,

$$
(\bar{x}, \bar{u}) \in\left(V I(C, A) \cap F_{f}(G)\right) \times\left(V I(D, B) \cap F_{g}(K)\right) \text { with } T \bar{x}=S \bar{u} .
$$

Now, taking $\alpha_{n}=\frac{1}{n+100000}$ for $n \geq 1$ and

$$
\gamma_{n}=\left\{\begin{array}{l}
\left(\frac{2}{5}\right) \frac{\left(5 x_{1 n}-2 u_{1 n}\right)^{2}+\left(3 u_{2 n}\right)^{2}}{\left(25 x_{1 n}-10 u_{1 n}\right)^{2}+\left(4 u_{1 n}-10 x_{1 n}\right)^{2}+\left(9 u_{2 n}\right)^{2}} \text { if } n \in \Omega, \\
\frac{1}{1000000} \text { if } n \notin \Omega .
\end{array}\right.
$$

then the Conditions $(A 1)-(A 2),(B 1)-(B 4)$, and $(C 1)-(C 2)$ are satisfied.
The figures below show that the error term sequence $\left\{E_{n}\right\}=\left\{\left(x_{n}, u_{n}\right)\right.$ $-(\bar{x}, \bar{u})\}$ converges strongly to zero for different choices of the parameter $l$ and different initial points.


Figure 1. Illustration of convergence rate of the sequence for different values of the parameter $l$.


Figure 2. Illustration of convergence rate of the sequence for different initial values.

The numerical experiments were carried out using MATLAB version R2020a and all programs were run on a 64 -bit OS PC with $\operatorname{Intel}(\mathrm{R})$ Core(TM) i7-8550U CPU@1.80GHz 1.99GHz and 16GB RAM.

Remark 5.2. Figure 1 shows that the convergence of the sequence generated by Algorithm A gets faster as $l$ gets closer to 1 . From Figure 2, we observe that for any choice of initial point, the sequence $\left\{\left(x_{n}, u_{n}\right)\right\}$ converges to a solution of the split equality of variational inequality and $f, g$-fixed point problem. That is, the ultimate convergence of the sequence does not depend on the choice of initial points.

## 6. Conclusions

In this paper, we have proposed a method for finding a solution of split equality of variational inequality and $f, g$-fixed point problems, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the $f, g$-fixed point problems are for Bregman relatively
$f, g$-nonexpansive mappings in reflexive Banach spaces. We have proved strong convergence of the algorithm using the Bregman distance approach. Finally, a numerical example is provided to show the applicability of the proposed algorithm. The results in this paper extend most of the results which are discussed in the literature in one or the other way. Specifically, the results of our method improve the result obtained by Wega and Zegeye [34] in the sense that it extends the result from finding common solution of variational inequality and $f$-fixed point problems to finding a solution of split equality of variational inequality and $f, g$-fixed point problems. Our result also extends the results obtained by Boikanyo and Zegeye [5] in the sense that it addresses $f, g$-fixed point problems on top of the split equality variational inequality problems.

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