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SOLVABILITY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SOME NONLINEAR INTEGRAL EQUATIONS RELATED TO CHANDRASEKHAR'S INTEGRAL EQUATION ON THE REAL HALF LINE

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Abstract. We investigate the existence and uniform attractivity of solutions of a class of functional integral equations which contain a number of classical nonlinear integral equations as special cases. Using the technique of measures of noncompactness and a fixed point theorem of Darbo type we prove the existence of solutions of these equations in the Banach space of continuous and bounded functions on the nonnegative real half axis. Our results extend and improve some known results in the recent literature. An example illustrating the main result is presented in the last section.

1. Introduction

Nonlinear functional-integral equations have wide application in many branches of sciences [14], [16] such as in the theory of optimal control, economics, engineering, mechanics, physics, optimization [7], queing theory and so on.

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The theory of integral equations is rapidly developing with the help of tools in functional analysis topology and fixed point theory. In this paper, we are interested with the following nonlinear functional-integral equations:

$$\begin{aligned} x(t) &= f_1\left(t, \left(T_1x\right)(t)\right) \\ &+ f_5\begin{pmatrix} t, f_2\left(t, \left(T_2x\right)(t)\right) \int_0^{+\infty} u\left(t, s, x(s)\right) ds, \\ f_3\left(t, \left(T_3x\right)(t)\right) \int_0^{+\infty} k(t,s) f_4(s, x(s)) ds \end{pmatrix}, \end{aligned} \tag{1.1}$$

where $t \geq 0,\ f_i : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}, i = 1, 2, 3.4, f_5 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}, k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $T_j : \mathbb{R}_+ \longrightarrow \mathbb{R}, j = 1, 2, 3$ are given while $x(t)$ is an unknown function.

In this study, we investigate a more general class of nonlinear integral equations which contain a number of classical nonlinear integral equations as particular cases. Some special cases of Eq.(1.1) have been investigated by various authors. In [1], Argarwal and O'Regan gave the existence of solutions for the nonlinear integral equation:

$$x(t) = \int_0^{+\infty} k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.2}$$

in the space of bounded and continuous functions $C_l\left[0, +\infty\right)$ which have limit at infinity. Meehan and O'Regan [19] discussed the existence of solutions for the nonlinear integral equation:

$$x(t) = h(t) + \mu \int_0^{+\infty} k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.3}$$

in the space $C_l\left[0, +\infty\right)$ and the existence of solutions for the nonlinear equation:

$$x(t) = h(t) + \int_0^{+\infty} k(t,s) \left[f(x(s)) + g(x(s))\right] ds, \quad t \in \mathbb{R}_+ \tag{1.4}$$

in the space $BC\left(\mathbb{R}_+, \mathbb{R}\right)$ of bounded and continuous functions on $\mathbb{R}_+$.

Later in 2001 [19], they established the existence of at least one positive solution for the nonlinear integral equation

$$x(t) = h(t) + \int_0^{+\infty} k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.5}$$

in the space $L^p\left(\mathbb{R}_+\right)$.

In 2004, Banaś and Poludniak [5] investigated the monotonic solutions for the nonlinear integral equation

$$x(t) = f(t) + \int_0^{+\infty} u(t, s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.6}$$

in the space of Lebesgue integrable functions on unbounded interval by using the Darbo fixed point theorem and the measure of noncompactness.

In 2006 Banaś and Martin [6] studied the existence and asymptotic stability of the solutions for the nonlinear equation

$$x(t) = g(t) + f(t, x(t)) \int_0^{+\infty} K(t,s) h(s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.7}$$

in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$.

In 2004, Cabellaro et al., [13], In 2008 [3] and in 2013 Darwish et al. [15] studied the existence of the solutions for the Urysohn integral equation defined on an unbounded interval

$$x(t) = a(t) + f(t, x(t)) \int_0^{+\infty} u(t, s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.8}$$

with the help of measure of noncompactness and a fixed point theorem in the space $BC(\mathbb{R}_+, \mathbb{R})$. Of course authors studied integral equation (1.8) under different assumptions and measure of noncompactness, also they have rather different existence theorems.

Very recently, in 2016 [23] and in 2017 [22], Ozdemir and Ilhan studied the existence and uniform attractivity of the solutions of a class of nonlinear integral equations on an unbounded interval

$$x(t) = (T_1 x)(t) + (T_2 x)(t) \int_0^{+\infty} u(t, s, x(s)) ds, \quad t \in \mathbb{R}_+ \tag{1.9}$$

with the help of measure of noncompactness and a fixed point theorem in the space $BC(\mathbb{R}_+, \mathbb{R})$.

Motivated by recent researches in this field, we study the more general nonlinear functional integral equation (1.1) .This equation encompasses many important integral and functional equations that arise in nonlinear analysis and its applications, in particular integral equations (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) , (1.7), (1.8), (1.9) see also [8, 9, 10, 11, 24, 25] for more other special cases and the references therein. Using the technique of a suitable measure of noncompactness, we prove an existence theorem and uniform attractivity of the solutions of (1.1) . We give an example satisfying the conditions of our results given in this paper.

2. Preliminaries

In this section, we give a collection of auxiliary facts which will be needed further on. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and $ConvX$ denote the closure and convex hull of X, respectively. With the symbols λX and $X + Y$, we denote the standard algebraic operations on sets. Moreover, we denote by M_E the family of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness presented bellow comes from [2].

Definition 2.1. ([2]) A mapping $\mu : M_E \to \mathbb{R}_+ = [0, +\infty[$ is said to be a measure of noncompactness in E if it satisfies following conditions:

(1) The family $\ker \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset N_E$,
(2) $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,
(3) $\mu(\overline{X}) = \mu(Conv X) = \mu(X)$,
(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)Y$, for $\lambda \in [0,1]$,
(5) If $\{X_n\}$ is a sequence of nonempty, bounded and closed subsets of E such that $X_{n+1} \subset X_n$ $(n = 1, 2, \cdots)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

Observe that the intersection set X_∞ belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. This simple remark will be crucial in our further considerations. For other facts concerning measures of noncompactness we refer to [2].

In the sequel, we will work in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions defined, continuous and bounded on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, \mathbb{R})$ is equipped with the standard norm

$$\|x\| = \sup\left\{|x(t| \ ; t \in \mathbb{R}_+\right\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. In order to define this measure, let us fix a nonempty subset X of $BC(\mathbb{R}_+,\mathbb{R})$. For $x \in X$, $\varepsilon \geq 0$ and $T > 0$ denoted by $\omega^T(x, \varepsilon)$ the modulus of continuity of function x, that is,

$$\omega^T(x,\varepsilon) = \sup\left\{|x(s) - x(t)| : t, s \in [0,T] \text{ and } |t-s| \leq \varepsilon\right\}.$$

Further let us set

$$\begin{aligned} \omega^T(X,\varepsilon) &= \sup\left\{\omega^T(x,\varepsilon), x \in X\right\}, \\ \omega_0^T(X) &= \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon) \end{aligned}$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X). \tag{2.1}$$

Moreover, if $t \in \mathbb{R}_+$ is a fixed number, let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\text{diam } X(t) = \sup\left\{|x(t) - y(t)| : x, y \in X\right\}.$$

With help of the above mappings, we define the following measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$ [2]

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \text{diam } X(t). \tag{2.2}$$

It can be shown [2] that the function μ is a sublinear measure of noncompactness with the maximum property in the space $BC(\mathbb{R}_+, \mathbb{R})$. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and they tend to zero at infinity uniformly with respect to the set X. This property of the kernel $\ker \mu$ allows us to characterize in terms of asymptotic behavior solutions of the functional integral equations (1.1).

Now we recall definitions of the concepts of global attractivity, local attractivity and asymptotic stability of the solutions of operator equations. Those definitions may be found in papers [4], [7], [12], [18]. Here we arrange those definitions and we establish relations among them.

Let Ω be a nonempty subset of the space $BC(\mathbb{R}_+, \mathbb{R})$ and Q be an operator acting from Ω into $BC(\mathbb{R}_+, \mathbb{R})$.

Let us consider the following operator equation:

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{2.3}$$

Definition 2.2. The solution $x = x(t)$ of Eq.(2.3) is said to be globally attractive if for each solution $y = y(t)$ of Eq.(2.3) we have that

$$\lim_{t\to+\infty}(x(t) - y(t)) = 0. \tag{2.4}$$

Other words we may that solutions of Eq.(2.3) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of this equation, condition is satisfied.

Definition 2.3. We say that solutions of Eq.(2.3) are locally attractive if there exists a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary solutions $x(t)$ and $y(t)$ of Eq.(2.3) belonging to $B(x_0, r) \cap \Omega$, condition (2.4) does hold.

In the case when the limit (2.4) is uniform with respect to the set $B(x_0, r) \cap \Omega$, that is, when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \varepsilon \tag{2.5}$$

for all $x(t)$, $y(t)$ of Eq.(2.3) from $B(x_0, r) \cap \Omega$ and for $t \ge T$, we will say that Eq.(2.3) are uniformly locally attractive. For more detail about the uniform global attractivity, the reader can see for instance [17].

Finally, we will make use of the following fixed point theorem [3] which is the main tool for our proof.

Theorem 2.4. *Let Q be a nonempty, bounded, closed and convex subset of a Banach space E and let*

$$F : Q \longrightarrow Q$$

be a continuous transformation such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where μ is a measure of noncompactness in E and $k \in [0,1[$ is a constant. Then F has a fixed point in the set Q.

Remark 2.5. Denote by Fix(F) the set of all fixed points of the operator F belonging to Q. It can be readily seen that the set Fix(F) belongs to the family $\ker \mu$, see [2]

The aim of this paper is to study the existence of solutions for Eq.(1.1) under suitable conditions. Tools used in this paper are the technique of measure of noncompactness and Darbo fixed point theorem [3]. We obtain some results about the asymptotic stability of solutions. Finally, an example illustrating the main result is presented in the last section.

3. Main results

Equation (1.1) will be studied under the following assumptions:

(1) The operators $T_i : BC(\mathbb{R}_+, \mathbb{R}) \longrightarrow BC(\mathbb{R}_+, \mathbb{R})$ are continuous and there exist continuous nondecreasing functions $d_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|(T_i x)(t)| \leq d_i(\|x\|), i = 1, 2, 3.$$

for all $x \in BC(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$.

(2) The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist functions $a, b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\lim_{t \to +\infty} a(t) = 0$, $\|b\|_1 = \int_0^{+\infty} |b(s)|\, ds < +\infty$ and a continuous nondecreasing function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|u(t, s, x)| \leq a(t) b(s) h(|x|)$$

for all $(t, s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$.

(3) There exists a continuous nondecreasing function $\varphi_{r_0} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which holds $\varphi_{r_0}(0) = 0$ and

$$|u(t_2, s, x) - u(t_1, s, x)| \leq \varphi_{r_0}(|t_2 - t_1|)\, \tau(s)$$

for all $t_2, t_1, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$, where τ is an element of the space $BC(\mathbb{R}_+, \mathbb{R}_+)$ such that $\int_0^{+\infty} \tau(s) ds < +\infty$.

(4) There exists a continuous nondecreasing function $\eta_{r_0} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which holds $\eta_{r_0}(0) = 0$ and

$$|u(t,s,x) - u(t,s,y)| \leq \eta_{r_0}(|x-y|)\, v(s)$$

for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0, |y| \leq r_0$, where v is an element of the space $BC(\mathbb{R}_+, \mathbb{R}_+)$ such that $\int_0^{+\infty} v(s)ds < +\infty$.

(5) $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist $l_i : \mathbb{R}_+ \to \mathbb{R}_+$ functions of the space $BC(\mathbb{R}_+, \mathbb{R}_+)$ with bounds L_i such that

$$|f_i(t,x) - f_i(t,y)| \leq l_i(t)\, |x-y|\, , i = 1, 2, 3$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$. $f_i(t,0)$ are elements of the space $BC(\mathbb{R}_+, \mathbb{R}_+)$ such that $\overline{f_i} = \sup_{t \in \mathbb{R}_+} |f_i(t,0)|$, $i = 1, 2, 3$.

(6) $f_5 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $l_i : \mathbb{R}_+ \to \mathbb{R}_+$ functions of the space $BC(\mathbb{R}_+, \mathbb{R}_+)$ with bounds L_i, $i = 5, 6$, such that

$$\begin{aligned} |f_5(t,a,x) - f_5(t,a,y)| &\leq l_5(t)\, |x-y|\, , \\ |f_5(t,a,x) - f_5(t,b,x)| &\leq l_6(t)\, |a-b|\, , \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $x, y, a, b \in \mathbb{R}$ where $f_5(t,0,0)$ is an element of the space $BC(\mathbb{R}_+, \mathbb{R})$ such that $\overline{f_5} = \sup_{t \in \mathbb{R}_+} |f_5(t,0,0)|$.

(7) There exists a nonnegative constant L such that

$$\max\{L_i, i = 1, 2, 3, 5, 6\} \leq L.$$

(8) $f_4 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous function on every rectangle of the form $\mathbb{R}_+ \times [-\alpha, \alpha]$ and there exists a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $d_4 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_4(t,x)| \leq \rho(t) d_4(|x|)$$

for any $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$.

(9) $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $p, q : \mathbb{R}_+ \to \mathbb{R}_+$ continuous functions such that $q(t)$ and $\rho(t)q(s)$ are integrable over $\mathbb{R}_+$ with the following inequality

$$|k(t,s)| \leq p(t)q(s)$$

for any $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t \longrightarrow +\infty} p(t) = 0.$$

Keeping in mind the above assumptions, we can easily infer that the constants Q, P are defined by the formulas:

$P = \sup\{p(t), t \geq 0\}$ and $Q = \int_0^{+\infty} \rho(s)q(s)ds$ are finite.

(10) There exists a positive real number r_0 satisfying the inequality

$$Ld_1(r_0) + \overline{f_1} + L\left\{Ld_2(r_0) + \overline{f_2}\right\} h(r_0)\|a\|\|b\|_1 + L\left\{Ld_3(r_0) + \overline{f_3}\right\} d_4(r_0) PQ + \overline{f_5} \leq r_0. \tag{3.1}$$

(11) There exist the nonnegative constants α_i for r_0 such that the inequality

$$\mu(T_i X) \leq \alpha_i \mu(X) \tag{3.2}$$

holds for all nonempty and bounded set X of the ball B_{r_0}, $(i = 1, 2, 3)$.

(12) $K = \max\left\{\alpha_1 L, L^2 h(r_0)\|a\|\|b\|_1 \alpha_2, PQd_4(r_0)\alpha_3\right\} < 1$.

Theorem 3.1. *Under assumptions* $(1)-(12)$*, there exists* $r_0 \in]0,1[$ *such that the equation* (1.1) *has at least one solution in* $B_{r_0} \subset BC(\mathbb{R}_+, \mathbb{R})$*. Also, these solutions are uniformly locally attractive.*

Proof. We define an operator F on $BC(\mathbb{R}_+, \mathbb{R})$ for $t \geq 0$ as follows

$$(Fx)(t) = f_1(t, (T_1x)(t)) + f_5\begin{pmatrix} t, f_2(t, (T_2x)(t)) \int_0^{+\infty} u(t, s, x(s))\,ds, \\ f_3(t, (T_3x)(t)) \int_0^{+\infty} k(t, s) f_4(s, x(s))ds \end{pmatrix}. \tag{3.3}$$

Notice that in view of assumptions $(1)-(9)$, the function $t \to (Fx)(t)$ is well-defined on the interval $\mathbb{R}_+$. At first we show that the function (Fx) is continuous on $\mathbb{R}_+$. To do this fix arbitrary $T > 0$ and $\varepsilon \geq 0$. Take arbitrary numbers $t, t_0 \in [0, T]$ with $|t - t_0| \leq \varepsilon$.

$$\begin{aligned} |(Fx)(t) - (Fx)(t_0)| &\leq |f_1(t, (T_1x)(t)) - f_1(t_0, (T_1x)(t_0))| \\ &+ \left| \begin{array}{l} f_5\begin{pmatrix} t, f_2(t, (T_2x)(t)) \int_0^{+\infty} u(t, s, x(s))\,ds, \\ f_3(t, (T_3x)(t)) \int_0^{+\infty} k(t, s) f_4(s, x(s))ds \end{pmatrix} \\ -f_5\begin{pmatrix} t_0, f_2(t_0, (T_2x)(t_0)) \int_0^{+\infty} u(t_0, s, x(s))\,ds, \\ f_3(t_0, (T_3x)(t_0)) \int_0^{+\infty} k(t_0, s) f_4(s, x(s))ds \end{pmatrix} \end{array} \right|. \end{aligned} \tag{3.4}$$

From (3.4), we get

$$\begin{aligned} |(Fx)(t) - (Fx)(t_0)| &\leq |f_1(t, (T_1x)(t)) - f_1(t_0, (T_1x)(t))| \\ &\quad + |f_1(t_0, (T_1x)(t)) - f_1(t_0, (T_1x)(t_0))| \\ &\leq \omega^T_{d_1(r_0)}(f_1, \varepsilon) + L\omega^T(T_1x, \varepsilon) \qquad \text{(I)} \\ &\quad + \left| \begin{array}{l} f_5\begin{pmatrix} t, f_2(t, (T_2x)(t)) \int_0^{+\infty} u(t, s, x(s))\,ds, \\ f_3(t, (T_3x)(t)) \int_0^{+\infty} k(t, s) f_4(s, x(s))ds \end{pmatrix} \\ -f_5\begin{pmatrix} t, f_2(t, (T_2x)(t)) \int_0^{+\infty} u(t, s, x(s))\,ds, \\ f_3(t, (T_3x)(t)) \int_0^{+\infty} k(t_0, s) f_4(s, x(s))ds \end{pmatrix} \end{array} \right| \qquad \text{(II)} \end{aligned}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t,s,x(s)\right)ds, \\ f_3\left(t,\left(T_3x\right)(t)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t,s,x(s)\right)ds, \\ f_3\left(t,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right| \qquad \text{(III)}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t,s,x(s)\right)ds, \\ f_3\left(t,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right| \qquad \text{(IV)}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right| \qquad \text{(V)}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right| \qquad \text{(VI)}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t_0,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right| \qquad \text{(VII)}$$

$$+\left|\begin{array}{l} f_5\left(\begin{array}{c} t, f_2\left(t_0,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t_0, f_2\left(t_0,\left(T_2x\right)(t_0)\right)\int_0^{+\infty}u\left(t_0,s,x(s)\right)ds, \\ f_3\left(t_0,\left(T_3x\right)(t_0)\right)\int_0^{+\infty}k(t_0,s)f_4(s,x(s))ds \end{array}\right) \end{array}\right|. \qquad \text{(VIII)}$$

Further, we have the following chain of estimates

$$\begin{aligned} \text{(II)} \quad &\leq \quad L\left|f_3\left(t,\left(T_3x\right)(t)\right)\right|\int_0^{+\infty}\left|k(t,s)-k(t_0,s)\right|\left|f_4(s,x(s))\right|ds \qquad (3.5) \\ &\leq \quad \left(L^2\left|\left(T_3x\right)(t)\right|+L\overline{f_3}\right)\int_0^{+\infty}\left|k(t,s)-k(t_0,s)\right|\rho(s)d_4(|x(s)|)ds \\ &\leq \quad \left(L^2\left|\left(T_3x\right)(t)\right|+L\overline{f_3}\right)d_4(\|x\|)\int_0^{+\infty}\left|k(t,s)-k(t_0,s)\right|\rho(s)ds. \end{aligned}$$

Furthermore we can obtain the following estimates

$$\begin{aligned}\int_0^{+\infty} |k(t,s)-k(t_0,s)|\,\rho(s)ds &\leq \int_0^{T} |k(t,s)-k(t_0,s)|\,\rho(s)ds \\ &\quad + \int_T^{+\infty} \left(|k(t,s)|+|k(t_0,s)|\right)\rho(s)ds \\ &\leq \rho_T T\omega^T(k,\varepsilon) \\ &\quad + \int_T^{+\infty} \left(|p(t)|+|p(t_0)|\right)q(s)\rho(s)ds \\ &\leq \rho_T T\omega^T(k,\varepsilon) + 2P_T\int_T^{+\infty} q(s)\rho(s)ds, \end{aligned} \tag{3.6}$$

where $P_T = \sup\{p(t) : t \geq T\}$ and $\rho_T = \sup\{\rho(t) : t \in [0,T]\}$.
Hence we get

$$(\mathrm{II}) \leq \left(L^2 d_3(\|x\|) + L\overline{f_3}\right) d_4(\|x\|)\left(\rho_T T\omega^T(k,\varepsilon) + 2P_T Q\right). \tag{3.7}$$

Further, we use assumptions (8) and (9) , we have

$$\begin{aligned}(\mathrm{III}) &\leq L\left|f_3(t,(T_3x)(t)) - f_3(t,(T_3x)(t_0))\right| \\ &\quad \times \int_0^{+\infty} k(t_0,s)\left|f_4(s,x(s))\right|ds \\ &\leq L^2\left|(T_3x)(t)-(T_3x)(t_0)\right| \\ &\quad \times \int_0^{+\infty} p(t_0)q(s)\rho(s)d_4(|x(s)|)ds \\ &\leq L^2\omega^T(T_3X,\varepsilon)PQd_4(\|x\|). \end{aligned} \tag{3.8}$$

Similarly, we get

$$\begin{aligned}(\mathrm{IV}) &\leq L\left|f_3(t,(T_3x)(t_0)) - f_3(t_0,(T_3x)(t_0))\right| \\ &\quad \times \int_0^{+\infty} k(t_0,s)\left|f_4(s,x(s))\right|ds \\ &\leq L\omega^T_{d_3(r_0)}(f_3,\varepsilon)PQd_4(\|x\|). \end{aligned} \tag{3.9}$$

Using assumption (4) and the fact that

$$\begin{aligned}\left|f_2(t,(T_2x)(t))\right| &\leq \left|f_2(t,(T_2x)(t)) - f_2(t,0)\right| + \left|f_2(t,0)\right| \\ &\leq L\left|(T_2x)(t)\right| + \overline{f_2}, \end{aligned}$$

we obtain

$$\begin{aligned}
(\mathrm{V}) &\leq L\left|f_2\left(t,\left(T_2x\right)(t)\right)\right| \int_0^{+\infty}\left|u(t,s,x(s))-u(t_0,s,x(s))\right| ds \\
&\leq L\left[L\left|\left(T_2x\right)(t)\right|+\overline{f_2}\right] \int_0^{+\infty}\left|u(t,s,x(s))-u(t_0,s,x(s))\right| ds \\
&\leq \left(L^2 d_2(\|x\|)+L\overline{f_2}\right) \varphi_{r_0}(\varepsilon)\|\tau\|_1 .
\end{aligned} \tag{3.10}$$

Using assumptions $(2), (5)$ and (7), we get

$$\begin{aligned}
(\mathrm{VI}) &\leq L\left|f_2\left(t,\left(T_2x\right)(t)\right)-f_2\left(t,\left(T_2x\right)(t_0)\right)\right| \\
&\quad\times \int_0^{+\infty}\left|u(t_0,s,x(s))\right| ds \\
&\leq L^2\left|\left(T_2x\right)(t)-\left(T_2x\right)(t_0)\right| \int_0^{+\infty}\left|u(t_0,s,x(s))\right| ds \\
&\leq L^2 \omega^T(T_2X,\varepsilon) h(\|x\|)\|a\|\|b\|_1 .
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
(\mathrm{VII}) &\leq L\left|f_2\left(t,\left(T_2x\right)(t_0)\right)-f_2\left(t,\left(T_2x\right)(t)\right)\right| \\
&\quad\times \int_0^{+\infty}\left|u(t_0,s,x(s))\right| ds \\
&\leq L\omega_{d_2(r_0)}^T(f_2,\varepsilon) h(\|x\|)\|a\|\|b\|_1 .
\end{aligned} \tag{3.12}$$

Next, we can write the last term as following

$$(\mathrm{VII}) = \omega_{A,B}^T(f_5,\varepsilon). \tag{3.13}$$

From (I)-(VIII) and (3.5)-(3.13), we obtain

$$\begin{aligned}
\left|(Fx)(t)-(Fx)(t_0)\right| \leq\ & \omega_{d_1(r_0)}^T(f_1,\varepsilon)+L\omega^T(T_1x,\varepsilon) \\
&+\left(L^2 d_3(\|x\|)+L\overline{f_3}\right) d_4(\|x\|) \\
&\quad\times\left(\rho_T T\omega^T(k,\varepsilon)+2P_TQ\right) \\
&+L^2\omega^T(T_3x,\varepsilon) PQ d_4(\|x\|) \\
&+L\omega_{d_3(r_0)}^T(f_3,\varepsilon) PQ d_4(\|x\|) \\
&+\left(L^2 d_2(\|x\|)+L\overline{f_2}\right) \varphi_{r_0}(\varepsilon)\|\tau\|_1 \\
&+L^2\omega^T(T_2x,\varepsilon) h(\|x\|)\|a\|\|b\|_1 \\
&+L\omega_{d_2(r_0)}^T(f_2,\varepsilon) h(\|x\|)\|a\|\|b\|_1 \\
&+\omega_{A_2,A_3}^T(f_5,\varepsilon),
\end{aligned} \tag{3.14}$$

where we denoted by
$\omega_d^T(f_i,\varepsilon)=\sup\left\{\left|f_i(t,x)-f_i(t_0,x)\right| : t,t_0\in[0,T], |t-t_0|\leq\varepsilon,\ x\in[-d,d]\right\}$,

$\omega^T(f_i, \varepsilon) = \sup\{|f_i(t) - f_i(t_0)| : t, t_0 \in [0,T], |t - t_0| \leq \varepsilon\}, i = 1,2,3,$
$\omega^T(k, \varepsilon) = \sup\{|k(t) - k(t_0)| : t, t_0 \in [0,T], |t - t_0| \leq \varepsilon\}$
and

$$\omega^T_{\|A_2\|,\|A_3\|}(f, \varepsilon) = \sup\left\{ \begin{array}{c} |f(t,x,y) - f(t_0,x,y)| : t, t_0 \in [0,T], |t - t_0| \leq \varepsilon, \\ x \in [-\|A_2\|, \|A_2\|], \, y \in [-\|A_3\|, \|A_3\|] \end{array} \right\}.$$

In view of our assumption, we infer that the functions f_i are uniformly continuous on $[0,T] \times [-d_i(r_0), d_i(r_0)], i = 1,2,3$ and the function f_5 is uniformly continuous on $[0,T] \times [-\|A_2\|, \|A_2\|] \times [-\|A_3\|, \|A_3\|]$. Hence, we deduce that $\omega^T(T_i x, \varepsilon), \omega^T(k, \varepsilon)$, convergent to 0 as $\varepsilon \longrightarrow 0$, where

$$\|A_2\| = \left\{Ld_2(\|x\|) + \overline{f_2}\right\} h(\|x\|) \|a\| \|b\|_1$$

and

$$\|A_3\| = \left\{Ld_3(\|x\|) + \overline{f_3}\right\} d_4(\|x\|) PQ.$$

Thus we have that (Fx) is continuous on $[0,T]$. We can choose T in such away the term appearing P_T becomes sufficiently small, (Fx) is continuous on $\mathbb{R}_+$.

Further, we show that (Fx) is bounded on $\mathbb{R}_+$. Indeed, by our assumptions, for arbitrary fixed $t \in \mathbb{R}_+$, setting

$$A_2(t) = f_2(t, (T_2 x)(t)) \int_0^{+\infty} u(t, s, x(s)) \, ds$$

and

$$A_3(t) = f_3(t, (T_3 x)(t)) \int_0^{+\infty} k(t,s) f_4(s, x(s)) ds.$$

We obtain

$$(Fx)(t) = f_1(t, (T_1 x)(t)) + f_5(t, A_2(t), A_3(t))$$

and

$$\begin{aligned} |A_2(t)| &\leq |f_2(t, (T_2 x)(t))| \int_0^{+\infty} |u(t,s,x(s))| \, ds \\ &\leq \{|f_2(t, (T_2 x)(t)) - f_2(t, 0)| + |f_2(t,0)|\} \\ &\quad \times \int_0^{+\infty} |u(t,s,x(s))| \, ds \\ &\leq \left\{L |(T_2 x)(t)| + \overline{f_2}\right\} \int_0^{+\infty} |u(t,s,x(s))| \, ds \\ &\leq \left\{Ld_2(\|x\|) + \overline{f_2}\right\} h(\|x\|) \|a\| \|b\|_1 . \end{aligned}$$

Therefore

$$\|A_2\| \leq \left\{Ld_2(\|x\|) + \overline{f_2}\right\} h(\|x\|) \|a\| \|b\|_1 . \tag{3.15}$$

Similarly, we have

$$
\begin{aligned}
|A_3(t)| &\leq |f_3(t,(T_3x)(t))| \int_0^{+\infty} k(t,s)\,|f_4(s,x(s))|\,ds \\
&\leq \left\{ Ld_3(\|x\|) + \overline{f_3} \right\} d_4(\|x\|)\, PQ \qquad (3.16)
\end{aligned}
$$

and

$$
\|A_3\| \quad \leq \quad \left\{ Ld_3(\|x\|) + \overline{f_3} \right\} d_4(\|x\|)\, PQ. \qquad (3.17)
$$

We derive From (3.15) and (3.17)

$$
\begin{aligned}
|(Fx)(t)| \quad &\leq \quad |f_1(t,(T_1x)(t)) - f_1(t,0)| + |f_1(t,0)| \qquad (3.18) \\
&\quad + |f_5(t,A_2(t),A_3(t))| \\
&\leq \quad L\,|(T_1x)(t)| + \overline{f_1} + |f_5(t,A_2(t),A_3(t)) - f_5(t,A_2(t),0)| \\
&\quad + |f_5(t,A_2(t),0) - f_5(t,0,0)| + \overline{f_5} \\
&\leq \quad Ld_1(\|x\|) + \overline{f_1} + L\,|A_3(t)| + L\,|A_2(t)| \\
&\leq \quad Ld_1(\|x\|) + \overline{f_1} + L\left\{ Ld_2(\|x\|) + \overline{f_2} \right\} h(\|x\|)\,\|a\|\,\|b\|_1 \\
&\quad + L\left\{ Ld_3(\|x\|) + \overline{f_3} \right\} d_4(\|x\|)\, PQ + \overline{f_5},
\end{aligned}
$$

which implies that the function (Fx) is bounded on $\mathbb{R}_+$. Combining this fact with the continuity of the function (Fx) on $\mathbb{R}_+$, we conclude that the operator F transforms the ball B_{r_0} into the space $BC(\mathbb{R}_+, \mathbb{R})$. The inequality (3.18) in conjuction with assumption (10) ensures that there exists a positive number r_0 for which the operator F transforms the ball B_{r_0} into itself. Further we shall prove the operator F is continuous on B_{r_0}. To do this, consider $\varepsilon > 0$ and take $x, y_0 \in B_{r_0}$ such that $\|x - y_0\| \leq \varepsilon$. Then, for arbitrary $t \in \mathbb{R}_+$, we get

$$
\begin{aligned}
|(Fx)(t) - (Fy_0)(t)| &\leq |f_1(t,(T_1x)(t)) - f_1(t,(T_1y_0)(t))| \\
&+ \left| \begin{array}{c} f_5\left(\begin{array}{c} t, f_2(t,(T_2x)(t)) \int_0^{+\infty} u(t,s,x(s))\,ds, \\ f_3(t,(T_3x)(t)) \int_0^{+\infty} k(t,s) f_4(s,x(s))ds \end{array} \right) \\ \\ -f_5\left(\begin{array}{c} t, f_2(t,(T_2y_0)(t)) \int_0^{+\infty} u(t,s,y_0(s))\,ds, \\ f_3(t,(T_3y_0)(t)) \int_0^{+\infty} k(t,s) f_4(s,y_0(s))ds \end{array} \right) \end{array} \right|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|(Fx)(t) - (Fy_0)(t)| &\leq |f_1(t,(T_1x)(t)) - f_1(t,(T_1y_0)(t))| \\
&+ \left| \begin{array}{c} f_5\left(\begin{array}{c} t, f_2(t,(T_2x)(t)) \int_0^{+\infty} u(t,s,x(s))\,ds, \\ f_3(t,(T_3x)(t)) \int_0^{+\infty} k(t,s) f_4(s,x(s))ds \end{array} \right) \\ \\ -f_5\left(\begin{array}{c} t, f_2(t,(T_2y_0)(t)) \int_0^{+\infty} u(t,s,x(s))\,ds, \\ f_3(t,(T_3x)(t)) \int_0^{+\infty} k(t,s) f_4(s,x(s))ds \end{array} \right) \end{array} \right|
\end{aligned}
$$

$$+\left|\begin{array}{c} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, x(s)\right) ds, \\ f_3\left(t,\left(T_3 x\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, x(s)) ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, y_0(s)\right) ds, \\ f_3\left(t,\left(T_3 x\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, x(s)) ds \end{array}\right) \end{array}\right|$$

$$+\left|\begin{array}{c} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, y_0(s)\right) ds, \\ f_3\left(t,\left(T_3 x\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, x(s)) ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, y_0(s)\right) ds, \\ f_3\left(t,\left(T_3 y_0\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, x(s)) ds \end{array}\right) \end{array}\right|$$

$$+\left|\begin{array}{c} f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, y_0(s)\right) ds, \\ f_3\left(t,\left(T_3 y_0\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, x(s)) ds \end{array}\right) \\ -f_5\left(\begin{array}{c} t, f_2\left(t,\left(T_2 y_0\right)(t)\right) \int_0^{+\infty} u\left(t, s, y_0(s)\right) ds, \\ f_3\left(t,\left(T_3 y_0\right)(t)\right) \int_0^{+\infty} k(t, s) f_4(s, y_0(s)) ds \end{array}\right) \end{array}\right|.$$

We get

$$\begin{aligned} \left|(Fx)(t)-\left(Fy_0\right)(t)\right| \leq\ & L\left|\left(T_1 x\right)(t)-\left(T_1 y_0\right)(t)\right| \\ & + L\left|f_2\left(t,\left(T_2 x\right)(t)\right)-f_2\left(t,\left(T_2 y_0\right)(t)\right)\right| \\ & \quad \times \int_0^{+\infty}\left|u\left(t, s, x(s)\right)\right| ds \\ & + L\left|f_2\left(t,\left(T_2 y_0\right)(t)\right)\right| \\ & \quad \times \int_0^{+\infty}\left|u\left(t, s, x(s)\right)-u\left(t, s, y_0(s)\right)\right| ds \\ & + L\left|f_3\left(t,\left(T_3 x\right)(t)\right)-, f_3\left(t,\left(T_3 y_0\right)(t)\right)\right| \\ & \quad \times \int_0^{+\infty}\left|k(t, s) f_4(s, x(s))\right| ds \\ & + L\left|f_3\left(t,\left(T_3 y_0\right)(t)\right)\right| \\ & \quad \times \int_0^{+\infty}|k(t, s)|\left|f_4(s, x(s))-f_4(s, y_0(s))\right| ds. \end{aligned}$$

This implies that

$$\begin{aligned}\left|(Fx)(t)-(Fy_0)(t)\right| \leq\; & L\left|(T_1x)(t)-(T_1y_0)(t)\right| \\ & +L^2\left|(T_2x)(t)-(T_2y_0)(t)\right| \\ & \times\int_0^{+\infty}\left|u(t,s,x(s))\right|ds \\ & +L\left\{L\left|(T_2x)(t)\right|+\overline{f_2}\right\} \\ & \times\int_0^{+\infty}\left|u(t,s,x(s))-u(t,s,y_0(s))\right|ds \\ & +L^2\left|(T_3x)(t)-(T_3y_0)(t)\right| \\ & \times\int_0^{+\infty}k(t,s)\left|f_4(s,x(s))\right|ds \\ & +L\left\{L\left|(T_3x)(t)\right|+\overline{f_3}\right\} \\ & \times\int_0^{+\infty}\left|k(t,s)\right|\left|f_4(s,x(s))-f_4(s,y_0(s))\right|ds.\end{aligned}$$

So, we use assumptions (8) and (9) we obtain

$$\begin{aligned}\left|(Fx)(t)-(Fy_0)(t)\right| \leq\; & L\left|(T_1x)(t)-(T_1y_0)(t)\right| \\ & +L^2\left|(T_2x)(t)-(T_2y_0)(t)\right| \\ & \times\int_0^{+\infty}a(t)b(s)h\left(\left|x(s)\right|\right)ds \\ & +L\left\{L\left|(T_2x)(t)\right|+\overline{f_2}\right\} \\ & \times\int_0^{+\infty}\eta_{r_0}\left(\left|x-y_0\right|\right)v(s)ds \\ & +L^2\left|(T_3x)(t)-(T_3y_0)(t)\right| \\ & \times\int_0^{+\infty}p(t)q(s)\rho(s)d_4\left(\left|x(s)\right|\right)ds \\ & +L\left\{L\left|(T_3x)(t)\right|+\overline{f_3}\right\} \\ & \times\int_0^{+\infty}p(t)q(s)\omega_{r_0}(\varepsilon)\,ds. \qquad (3.19)\end{aligned}$$

Hence we have

$$\begin{aligned}\left\|Fx-Fy_0\right\| \leq\; & L\left\|T_1x-T_1y_0\right\|+L^2\left\|T_2x-T_2y_0\right\|\left\|a\right\|\left\|b\right\|_1 h(r_0) \\ & +L\left\{Ld_2(r_0)+\overline{f_2}\right\}\eta_{r_0}\left(\left\|x-y_0\right\|\right)\left\|v\right\|_1 \\ & +L^2d_4(r_0)\left\|T_3x-T_3y_0\right\|PQ \\ & +L\left\{Ld_3(r_0)+\overline{f_3}\right\}P\left\|q\right\|_1\omega_{r_0}(\varepsilon), \qquad (3.20)\end{aligned}$$

where

$$\omega_{r_0}(\varepsilon) = \sup\{|f_4(s,x) - f_4(s,y_0)| : s \geq 0, x, y \in [-r_0, r_0], |x - y_0| \leq \varepsilon\}.$$

Observe that in view of assumption (8) we infer that $\omega_{r_0}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since the operators T_i are continuous for any $y_0 \in B_{r_0}$, there exist the number $\delta_i(\varepsilon) > 0$ with $\delta_i(\varepsilon) \leq \varepsilon$ such that we have

$$\|T_i x - T_i y_0\| \leq \varepsilon$$

for all x satisfying $\|x - y_0\| < \delta_i$. Let us take $\delta(\varepsilon) = \min\{\delta_i(\varepsilon), i = 1, 2, 3\}$. In this case if $\|x - y_0\| < \delta(\varepsilon)$, (3.20) becomes

$$\begin{aligned} \|Fx - Fy_0\| \leq{}& L\varepsilon + L^2\varepsilon \|a\| \|b\|_1 h(r_0) \\ &+ L\left\{Ld_2(r_0) + \overline{f_2}\right\} \eta_{r_0}(\varepsilon) \|v\|_1 \\ &+ L^2\varepsilon d_4(r_0) PQ + L\left\{Ld_3(r_0) + \overline{f_3}\right\} P \|q\|_1 \omega_{r_0}(\varepsilon). \end{aligned} \tag{3.21}$$

Therefore from (3.21) and assumption (4), we have that F is continuous on the ball B_{r_0}.

Further, we shall show that operator F satisfies the Darbo condition on the ball B_{r_0}. In order to do this, let us take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon \geq 0$, $T > 0$ and choose $x \in X$ and $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leq \varepsilon$. Then in view of (3.14) we have

$$\begin{aligned} \omega^T(FX, \varepsilon) \leq{}& L\omega^T(T_1x, \varepsilon) + 2\left(L^2 d_3(r_0) + L\overline{f_3}\right) d_4(r_0) P_T Q \\ &+ L^2\omega^T(T_3x, \varepsilon) PQ d_4(r_0) \\ &+ \left(L^2 d_2(r_0) + L\overline{f_2}\right) \varphi_{r_0}(\varepsilon) \|\tau\|_1 \\ &+ L^2\omega^T(T_2x, \varepsilon) h(r_0) \|a\| \|b\|_1 + \omega^T_{A_2,A_3}(f_5, \varepsilon). \end{aligned}$$

which yields by going to the limit as $T \to +\infty$

$$\begin{aligned} \omega_0(FX) \quad\leq\quad & L\omega_0(T_1X) + L^2\omega_0(T_3X) PQ d_4(r_0) \\ & + L^2\omega_0(T_2X) h(r_0) \|a\| \|b\|_1. \end{aligned} \tag{3.22}$$

Further, let us take a nonempty subset X of the ball B_{r_0}. For $x, y \in X$ and $t \in \mathbb{R}_+$, from estimate (3.19) we get that

$$\begin{aligned} \operatorname{diam}(FX)(t) \leq{}& \left\{L\operatorname{diam}(T_1(X)(t)) + L^2 \|a\| \|b\|_1 h(r_0) \operatorname{diam}(T_2(X)(t))\right\} \\ &+ L^2 PQ d_4(r_0) \operatorname{diam}(T_3(X)(t)). \end{aligned}$$

If we take the limit supremum as $t \to +\infty$ in the above inequality, we have the inequality

$$\limsup_{t\to+\infty} \operatorname{diam}(FX)(t) \leq L \limsup_{t\to+\infty} \operatorname{diam}\left(T_1\left(X\right)(t)\right) + L^2 \|a\| \|b\|_1 h(r_0) \limsup_{t\to+\infty} \operatorname{diam} T_2\left(X\right)(t) + L^2 PQ d_4(r_0) \limsup_{t\to+\infty} \operatorname{diam}\left(T_3\left(X\right)(t)\right) \tag{3.23}$$

By linking (3.22) and (3.23) and by assumption (11) we derive that

$$\mu\left(FX\right) \leq \max\left\{\alpha_1 L, L^2 h(r_0) \|a\| \|b\|_1 \alpha_2, PQd_4(r_0)\alpha_3\right\} \mu\left(X\right). \tag{3.24}$$

Now, let us observe that assumption (12) and (3.24) we have that F is a contraction with respect to the measure of noncompactness μ. Hence by Theorem 2.4, the operator F has a fixed point x in the ball B_{r_0}.

Obviously, every function $x = x(t)$ being a fixed point of the operator F is a solution to (1.1). Further, keeping in mind Remark 2.5, we conclude that the set $\operatorname{Fix}(F)$ of all fixed points of the operator F belonging to the ball $B_{r_0}$is a member of the $\ker\mu$. Hence, in view of the description of the $\ker\mu$ we infer that all of solutions all $x(t), y(t)$ of (1.1) we have that $\mu\left(X\right) = 0, X = \{x, y\} \subset BC\left(\mathbb{R}_+, \mathbb{R}\right)$ belonging to the ball B_{r_0} are uniformly attractive on $\mathbb{R}_+$. Indeed, we have in particular $\limsup_{t\to\infty} \operatorname{diam}\ X(t) = 0$, that is, for all $\epsilon > 0$ there exists $T > 0$ such that for all $t > T$ diam $|x(t) - y(t)| \leq \varepsilon$. Consequently, the solutions of (1.1) are asymptotically stable. This step completes the proof of our theorem. □

Now, we present some examples of classical integral and functional equations considered in nonlinear analysis which are particular cases of Eq.(1.1) and consequently, the existence of their solutions can be established using Theorem 2.4.

(1) By setting $f_1 = 0$, $f_3 = 1$, $f_5\left(t, x, y\right) = y$, Eq.(1.1) reduces to the nonlinear integral equation (1.2).
(2) By setting $f_1 = 0$, $f_3(t, a) = \mu$, $f_5\left(t, x, y)\right) = h(t) + y$, Eq.(1.1) reduces to the nonlinear integral equation (1.3) and if $\mu = 1$ we obtain (1.5).
(3) By setting $f_1 = 0$, $f_5\left(t, x, y\right) = y$, $T_3(x)(t) = x(t)$ and $f_4\left(s, x(s)\right) = f(x(s)) + g(x(s))$ Eq.(1.1) reduces to the nonlinear integral equation (1.4).
(4) By setting $f_1 = 0$, $f_3 = 0$, $f_2\left(t, b\right) = b$, $f_5\left(t, x, y\right) = a(t) + x, T_2(x)(t) = x(t)$ Eq.(1.1) reduces to the nonlinear integral equation (1.6).
(5) By setting $f_1 = 0$, $f_3\left(t, b\right) = f(t, b)$, $f_5\left(t, x, y\right) = g(t) + y$ and $T_3(x)(t) = x(t)$ Eq.(1.1) reduces to the nonlinear integral equation (1.7).

(6) By setting $f_1 = 0$, $f_2(t,b) = f(t,b)$, $f_5(t,x,y) = a(t)+y$ and $T_2(x)(t) = x(t)$ Eq.(1.1) reduces to the nonlinear integral equation (1.8).

(7) By setting $f_1(t,a) = a$, $f_2(t,b) = b$, $f_5(t,x,y) = y$, Eq.(1.1) reduces to the nonlinear integral equation (1.9). Note Chandrasekhar's integral equation, appears in the theory of radiative transfer, in the theory of neutron transport and in theory of traffic, is a special case of Eq.(1.1).

4. Example

Consider the following functional integral equation:

$$x(t) = \frac{1}{10}\left(\begin{array}{c} \frac{e^{-t}}{1+t^2} + \frac{t\sin x(t)}{3t+9} + \left(\frac{\arctan t}{2+t^2} + \frac{t^2x^2(t)}{3t^2+2}\right)\int_0^{+\infty}\frac{\arctan x(s)}{e^t(s^2+1)}ds \\ + \left(\frac{t}{25+t^2}x(t) + \frac{t}{16+t^2}\right)\int_0^{+\infty}\frac{te^{-(t+s)}}{1+t^2}\sqrt{|x(s)|} \end{array}\right) \tag{4.1}$$

where $t \in \mathbb{R}_+$.

Let $(T_1x)(t) = \frac{t\sin x(t)}{10(3t+9)}$, $(T_2x)(t) = \frac{t^2x^2(t)}{10(3t^2+2)}$, $(T_3x)(t) = \frac{t}{10(25+t^2)}x(t)$, $f_1(t,x) = \frac{e^{-t}}{10(1+t^2)} + x$, $f_2(t,x) = \frac{\arctan t}{10(2+t^2)} + x$, $f_3(t,x) = \frac{1}{10}\left(\frac{t}{16+t^2} + \frac{t}{25+t^2}x\right)$, $u(t,s,x(s)) = \frac{\arctan x(s)}{e^t(s^2+1)}$, $k(t,s) = \frac{te^{-s}}{1+t^2}$, $f_4(t,x) = e^{-t}\sqrt{|x|}$, $f_5(t,y,z) = f_2(t,x)\,y + f_3(t,x)\,z$, $a(t) = \frac{\pi}{2e^t}$, $b(s) = \frac{1}{1+s^2}$. Then the assumptions of Theorem 2.4 are satisfied. Indeed, $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, is continuous, further $|f_1(t,x) - f_1(t,y)| \leq \frac{e^{-t}}{10(1+t^2)}|x-y|$ for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. We put $l_1(t) = \frac{e^{-t}}{10(1+t^2)}, L_1 = 0,1$. $l_3(t) = \frac{t}{10(25+t^2)}, L_3 = \frac{1}{100}, q(s) = e^{-s}, p(t) = \frac{t}{1+t^2}, \rho(t) = t, d_4(r) = \sqrt{r}$ and $\rho(t) = e^{-t}$. Then this yields that $P = Q = \frac{1}{2}$ and $L = \frac{1}{10}$, $f_5 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies:

$$\begin{array}{rcl} |f_5(t,a,x) - f_5(t,a,y)| & \leq & \frac{1}{100}|x-y|, \\ |f_5(t,a,x) - f_5(t,b,x)| & \leq & \frac{\pi}{40}|a-b|. \end{array}$$

Also, we have $|f_1(t,0)| \leq \overline{f_1} = \frac{1}{10}$ and $|f_2(t,0)| \leq \overline{f_2} = \frac{\pi}{40}, |f_3(t,0)| \leq \overline{f_3} = \frac{1}{80}$ and $f_5(t,0,0) = 0$. It is easily verified that the assumptions of Theorem 2.4 are satisfied, first (5) and (6) are satisfied, T_1, T_2 and T_3 are continuous operators on the space $BC(\mathbb{R}_+,\mathbb{R})$. Further for all $t \in \mathbb{R}_+$ and $x \in BC(\mathbb{R}_+,\mathbb{R})$, we have

$$|(T_1x)(t)| \leq \frac{1}{30},\ |(T_2x)(t)| \leq \frac{x^2(t)}{30}, |(T_3x)(t)| \leq \frac{1}{100}|x(t)|. \tag{4.2}$$

Hence assumption (1) is satisfied with $d_1(x) = \frac{1}{30}$, $d_2(x) = \frac{x^2(t)}{30}$ and $d_3(x) = \frac{1}{100}x$. Further note that the function u is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$.

Moreover, we get

$$|u(t,s,x)| = \left|\frac{\arctan x(s)}{e^t(s^2+1)}\right| \leq \frac{\pi}{2e^t(s^2+1)} \tag{4.3}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Thus, according to assumption (2) we may put $a(t) = \frac{\pi}{2e^t}, b(s) = \frac{1}{(s^2+1)}$ and $h(x) = 1$. Further we get

$$\|a\| = \sup\left\{\frac{\pi}{2e^t}, t \geq 0\right\} = \frac{\pi}{2}, \|b\|_1 = \int_0^{+\infty} \frac{1}{(s^2+1)} ds = \frac{\pi}{2}$$

and obviously, we have that $a(t) \to 0$ as $t \to +\infty$. Additionally, without loss of generality that for all t_1, t_2 and $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$ we have

$$\begin{aligned} |u(t_1,s,x) - u(t_1,s,x)| &= \left|\frac{\arctan x}{e^{t_1}(s^2+1)} - \frac{\arctan x}{e^{t_2}(s^2+1)}\right| \\ &= \left|\frac{\arctan x}{s^2+1}\right| \frac{|e^{t_2} - e^{t_1}|}{e^{t_1+t_2}} \\ &\leq \left|\frac{\arctan x}{s^2+1}\right| \frac{e\xi|t_2 - t_1|}{e^{t_1+t_2}} \\ &\leq \frac{\pi}{2(s^2+1)} |t_2 - t_1|, \end{aligned}$$

where $\xi \in (t_1, t_2)$. If we put $\varphi_{r_0}(t) = t$ and $\tau(s) = \frac{\pi}{2(s^2+1)}$, the assumption (3) is satisfied. Without loss of generality assume that $x < y$, for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0$, $|y| \leq r_0$, we get

$$\begin{aligned} |u(t,s,x) - u(t,s,y)| &= \left|\frac{\arctan x - \arctan y}{e^t(s^2+1)}\right| \\ &\leq \frac{|x-y|}{e^t(\xi^2+1)(s^2+1)} \\ &\leq \frac{|x-y|}{s^2+1}, \end{aligned}$$

where $\xi \in (x, y)$. If we choose $\eta_{r_0}(t) = t$ and $v(s) = \frac{1}{s^2+1}$, the assumption (4) is satisfied.

Now notice that the inequality in assumption (9) has the form:

$$\frac{1}{10}\left(\frac{4}{3} + \frac{\pi^3}{16} + \left(\frac{1}{10}r_0 + \frac{1}{8}\right)\frac{\sqrt{r_0}}{4} + \frac{\pi^2 r_0^2}{12}\right) - r_0 \leq 0. \tag{4.4}$$

It can be easily verified if we define continuous function $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(r) = \frac{1}{10}\left(\frac{4}{3} + \frac{\pi^3}{16} + \left(\frac{1}{10}r + \frac{1}{8}\right)\frac{\sqrt{r}}{4} + \frac{\pi^2 r^2}{12}\right) - r$, then $\varphi(0) > 0$ and $\varphi(1) = 0.41 - 1 < 0$. The continuity of φ guarantees that there exists a number

$r_0 \in (0,1)$ such that $\varphi(r_0) = 0$. Also for $\varepsilon \geq 0, T > 0, \|x\| \leq r_0$ and $t, s \in [0,T]$ such that $|t-s| \leq \varepsilon$, we have

$$\begin{aligned}
|(T_1x)(t) - (T_1x)(s)| &= \frac{1}{10}\left|\frac{t\sin x(t)}{3t+9} - \frac{s\sin x(s)}{3s+9}\right| \\
&\leq \frac{t(s+3)\,|\sin x(t) - \sin x(s)| + 3\,|\sin x(s)|\,|t-s|}{30\,(t+3)\,(s+3)} \\
&\leq \frac{t}{30\,(t+3)}\,|x(t) - x(s)| + \frac{\varepsilon\,|\sin x(\varepsilon)|}{10\,(t+3)\,(s+3)} \\
&\leq \frac{1}{30}\,|x(t) - x(s)| + \frac{\varepsilon}{90}. \qquad (4.5)
\end{aligned}$$

Further, it can be seen that

$$\begin{aligned}
|(T_2x)(t) - (T_2x)(s)| &= \frac{1}{10}\left|\frac{t^2x^2(t)}{3t^2+2} - \frac{s^2x^2(s)}{3s^2+2}\right| \\
&\leq \frac{2r_0t^2\left(3s^2+2\right)|x(t) - x(s)| + 2r_0^2\,(t+s)\,(t-s)}{10\,(3t^2+2)\,(3s^2+2)} \\
&\leq \frac{r_0t^2}{5\,(3t^2+2)}\,|x(t) - x(s)| + \frac{r_0^2\,(t+s)\,\varepsilon}{5\,(3t^2+2)\,(3s^2+2)} \\
&\leq \frac{r_0}{15}\,|x(t) - x(s)| + \frac{r_0^2\varepsilon T}{10} \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
|(T_3x)(t) - (T_3x)(s)| &= \frac{1}{10}\left|\frac{tx(t)}{t^2+25} - \frac{sx(s)}{s^2+25}\right| \\
&\leq \frac{t\left(s^2+25\right)|x(t) - x(s)| + 25r_0\,(t-s)}{10\,(t^2+25)\,(s^2+25)} \\
&\leq \frac{1}{100}\,|x(t) - x(s)| + \frac{25r_0\varepsilon}{10\,(t^2+25)\,(s^2+25)} \\
&\leq \frac{1}{100}\,|x(t) - x(s)| + \frac{r_0\varepsilon}{10}. \qquad (4.7)
\end{aligned}$$

From (4.5), (4.6)and (4.7) and in view of (2.1), we get

$$\begin{aligned}
\omega_0(T_1X) &\leq \frac{1}{3}\omega_0(X), \\
\omega_0(T_2X) &\leq \frac{r_0}{15}\omega_0(X), \\
\omega_0(T_3X) &\leq \frac{1}{15}\omega_0(X). \qquad (4.8)
\end{aligned}$$

Now for $x, y \in X$, we get

$$\begin{aligned}
\left|\left(T_1 x\right)(t)-\left(T_1 y\right)(t)\right| & =\frac{1}{10}\left|\frac{t \sin x(t)}{3 t+9}-\frac{t \sin y(t)}{3 t+9}\right| \\
& \leq \frac{t|\sin x(t)-\sin y(t)|}{10(3 t+9)} \\
& \leq \frac{t}{10(3 t+9)}|x(t)-y(t)| \\
& \leq \frac{1}{30}|x(t)-y(t)| .
\end{aligned} \tag{4.9}$$

Using (4.9), we have

$$\limsup_{t \rightarrow+\infty} \operatorname{diam}\left(T_1(X)(t)\right) \leq \frac{1}{30} \limsup_{t \rightarrow+\infty} \operatorname{diam} X(t). \tag{4.10}$$

From (4.8) and (4.10), we get

$$\mu\left(T_1 X\right) \leq \frac{1}{30} \mu(X) . \tag{4.11}$$

For $x, y \in X$, we get

$$\begin{aligned}
\left|\left(T_2 x\right)(t)-\left(T_2 y\right)(t)\right| & =\frac{1}{10}\left|\frac{t^2 x^2(t)}{3 t^2+2}-\frac{t^2 y^2(t)}{3 s^2+2}\right| \\
& \leq \frac{2 r_0 t^2|x(t)-y(t)|}{10\left(3 t^2+2\right)} \\
& \leq \frac{r_0 t^2}{5\left(3 t^2+2\right)}|x(t)-y(t)| \\
& \leq \frac{r_0}{15}|x(t)-y(t)| .
\end{aligned} \tag{4.12}$$

From (4.8) and (4.12), we have

$$\mu\left(T_2 X\right) \leq \frac{r_0}{15} \mu(X) \tag{4.13}$$

and

$$\begin{aligned}
\left|\left(T_3 x\right)(t)-\left(T_3 y\right)(t)\right| & =\frac{1}{10}\left|\frac{t}{t^2+25} x(t)-\frac{t}{s^2+25} y(t)\right| \\
& \leq \frac{t|x(t)-y(t)|}{10\left(t^2+25\right)} \\
& \leq \frac{t}{10\left(t^2+25\right)}|x(t)-y(t)| \\
& \leq \frac{1}{100}|x(t)-y(t)| .
\end{aligned} \tag{4.14}$$

From (4.8) and (4.14), we obtain

$$\mu(T_3X) \leq \frac{1}{100}\mu(X). \tag{4.15}$$

Since $0 < r_0 < 1$, we have

$$K = \max\left(\frac{1}{30}, \sqrt{r_0}\frac{1}{400}, \frac{r_0\pi^2}{60}\right) < 1.$$

Hence, from (4.11), (4.13), (4.15) and (3.24) that

$$\mu(FX) \leq K\mu(X). \tag{4.16}$$

It follows from (4.16) that the assumptions (10) and (12) are satisfied.

Finally we conclude that the assumptions of Theorem 2.4 are satisfied. This implies that the functional integral equations (4.1) has at least one solution belonging to the ball B_{r_0} of the space $BC(\mathbb{R}_+, \mathbb{R})$. Taking into account Remark 2.5 and the measure of noncompactness μ given in (2.2), we infer easily that any solutions of (4.1) which belong to the ball B_{r_0} are asymptotically stable on $\mathbb{R}_+$ as defined in Definition 2.3.

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