Nonlinear Functional Analysis and Applications
Vol. 28, No. 1 (2023), pp. 11-36
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.01.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

QUANTUM CONTROL OF PARTICLES AT MATTER SURFACE OUTSIDE THE DOMAIN*

Quan-Fang Wang

(former) The Chinese University of Hong Kong, N. T. Hong Kong
e-mail: quanfangwang_office@yahoo.co.jp

Abstract. In this presentation, the particles at the matter surface (metal, crystal, nano) will be considered as the control target outside the physical domain. As is well known that control problems of quantum particles at surface had been investigated in various aspects in last couple of years, but the realization of control would become rather difficult than theoretical results. Especially, whether surface control would be valid? what kind of particles at what kind of matter surfaces can be controlled? so many questions still left as the mystery in the current research literature and papers. It means that the direct control sometime does not easy. On the other hands, control outside the physical domain is quite a interest consideration in mathematics, physics and chemistry. The main plan is to take the quantum systems operator (such as Laplacian Δ) in the form of fractional operator ($\Delta^s, 0 < s < 1$), then to consider the control outside of physical domain. Fortunately, there are many published articles in the field of applied mathematics can be referred for the achievement of control outside of domain. The external quantum control would be a fresh concept to do the physical control, first in the theoretic, second in the computational, final in the experimental issues.

1. Introduction

The topics of quantum control at surface had already been taken account into consideration past a long time (cf. [3], [5], [7], [8]), those investigation contained the control of different particles (molecule, atom, elementary particle) at different matter surface (metal, crystal, catalysis). Although several questions are inside the obtained results for theoretical study (cf. [18]), restrictedly,

[0]Received September 3, 2021. Revised April 19, 2022. Accepted October 13, 2022.
[0]2020 Mathematics Subject Classification: 34K37, 35R11, 26A33, 35Q40, 35Q41, 35Q93, 35J10, 35A15, 35K55, 49J20, 49R05, 93C20.
[0]Keywords: Quantum control, particles, surface, fractional differential equation.

the consequences could be a direction for the experimental research of control of quantum system in the physical chemistry area. More precisely, there is a numerical result for the quantum control of Klein-Gordon-Schrödinger dynamics system, it quite need the physical support for meaningful in the realistic sense. That is, whether neutrons and meson can be really arranged in a corral or circle? these connection between the mathematical study and experimental physics would be existing problems also for other attained results. Without lost of generalization, it give us the big opportunity to solve those problems in the future works.

In this paper, suppose the control of a quantum system take place outside the domain, then differential equation is changed its usual differential operator into a fractional operator. In the nonlinear fractional differential equation, apply optimal control theory, and seek the results as supplementary to aid the control at matter surface using the external force.

2. Fractional differential equation

First of all, we introduce the fractional differential equation as preparation in corresponding to the control outside the domain. For a very usual quantum system given by

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi,$$

where ψ is a wave function to represent the probability of particle, it is a complex valued function in complex Hilbert space (cf. [7]). Take most common differential operator $E = \Delta$, then to have $i\hbar\frac{\partial\psi}{\partial t} = \Delta\psi$, which is composed the famous Schrödinger equation. In here, $\hbar$ is the reduced Planck constant, and i is unit of imaginary part at complex space.

For such a differential operator Δ, assume its eigenvalue λ_i and eigenfunction w_i, then there is

$$\Delta w_i = \lambda w_i, \quad i = 1, 2, ...$$

Therefore, for differential operator Δ^s, $0 < s < 1$, its eigenvalue λ_i^s and eigenfunction w_i can be defined as

$$\Delta^s w_i = \lambda^s w_i, \quad i = 1, 2, ... \quad \text{for} \quad 0 < s < 1. \tag{2.1}$$

The definition (2.1) is well-posed and reasonable. The fractional square is appeared at the eigenvalue, and it is a valid and existed value to define the fractional square at the operator Δ (as well as E). Directly, that is to say, we define the fractional operator Δ^s by (2.1), that is, $\Delta^s w_i \doteq \lambda^s w_i$. It means that fractional differential operator make sense. One can consider the fractional differential equation as

$$i\hbar\frac{\partial\psi}{\partial t} = \Delta^s\psi, \quad 0 < s < 1$$

or

$$(i\hbar)^s\frac{\partial\psi}{\partial t} = \Delta^s\psi, \quad 0 < s < 1.$$

Mathematically, it is equivalent to each other. For generalization, take the form of quantum system as

$$i\hbar^s\frac{\partial\psi}{\partial t} = \Delta^s\psi, \quad 0 < s < 1. \tag{2.2}$$

For example, by (2.2), denote m as mass of particle, then the nonlinear fractional Schrödinger equation can be described as

$$(i\hbar)^s\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\Delta^s\psi + i\alpha^s\frac{\partial\psi}{\partial t} + \beta^s\psi.$$

It is equivalent to the formulation of

$$i\hbar^s\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\Delta^s\psi + i\alpha\frac{\partial\psi}{\partial t} + \beta\psi,$$

by the consideration of arbitrary of coefficients of α and β.

2.1. **Physics support.** For a particle at matter surface, the stationary status most interested in the chemistry field, which consider particle-particle, surface-particle reaction between particle and matter. At viewpoint of control field, this paper initially suppose that control process of particle focus on a particle at some one given surface, and received the external source outside the domain. Certainly, it can be realized at current experimental facility.

For the consideration of particle at surface, this paper is restricted in the two dimension plane for spatial space. Let Ω be a open set of $\mathbf{R}^N, N = 2$, then $\mathbf{x} = (x_1, x_2) \in \Omega$, and outside domain denoted as $\mathbf{R}^N/\Omega$. Γ denote the boundary of Ω, and $\Gamma = \partial\Omega$. Suppose the term $u(t)$ will be used to express the potential of external force f, outside the domain, which acting at the system. Since just one particle is considered in the surface, it can be taken as a quantum dot. For simplification, assume $u(t)$ depended on time variable t only. This term indicated the electronic field (shaped laser pulse, etc).

More precisely, set source is located at two spatial space point $\mathbf{x}_0 = (x_1^0, x_2^0)$. the external force f will be a emission at the point source, therefore, the interaction to the whole system can be expressed as the formulation $u(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0)$.

Due to the pointwise source outside the physics domain, it completely independent from the system, therefore, the problem assume that initial function $\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x})$ for ground state of wave function ψ at start time $t = 0$.

For other kind of outside value problems, such as outside distributed control, boundary value $\psi_\Gamma = \psi_{\partial\Omega}$ can be calculated at the boundary $\partial\Omega$ by given outside control $\mathbf{R}^N/\Omega$, it can avoid to take the initial guess of ψ.

2.2. **Fractional Schrödinger equation.** For $0 < s < 1$, set $Q = \Omega \times (0, T)$. For $(\mathbf{x}, t) \in Q$, the nonlinear fractional Schrödinger equation has the form of

$$\begin{cases} -i\hbar^s\psi = \Delta^s\psi + V(\mathbf{x}, t)\psi, & \text{in } Q, \\ \psi(\mathbf{x}, 0) = \psi(0), & \text{in } \Omega, \\ \psi(\mathbf{x}, t) = u(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0), & \text{in } \mathbf{R}^N/\Omega \times [0, T] \text{ for } \mathbf{x}_0 \in \mathbf{R}^N/\Omega, \end{cases} \tag{2.3}$$

where $\hbar$ is reduced Planck constant, take $N = 2$ for plane surface. Δ^s is Laplacian defined by (2.1). $V(\mathbf{x}, t)$ is a physical and chemical potential in domain Ω. Control variable $u(t)$ is depended on time varying only. Pointwise function

$$\delta(\mathbf{x} - \mathbf{x}_0) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{x}_0 \text{ for } \forall \mathbf{x} \in \mathbf{R}^N/\Omega, \\ 0 & \text{if } \mathbf{x} \neq \mathbf{x}_0 \text{ for } \forall \mathbf{x} \in \mathbf{R}^N/\Omega. \end{cases}$$

Note that $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, and $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right)$ as usual. Δ^s, ∇^s are used to express the fractional derivative of spatial variable $\mathbf{x}$, but the formulation are different from Δ and ∇.

Remark 2.1. It needs to explain the control term $u(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0)$, which mean that there is a control input $u(t)$ only at the point $\mathbf{x}_0$ outside the domain $\mathbf{R}^2/\Omega$, at other place its value be zero. It is point source for the system (2.3).

Remark 2.2. Mathematically, the domain of a parabolic partial differential equation is infinite both for inside and outside parts. Our problem is configured for a particle at matter surface. The domain mentioned in here is indicated the physics domain of surface, which located the particle. Such as $[0, L] \times [0, L]$ and $[-L, L] \times [-L, L]$. Therefore, the posed quantum system (2.3) is meaningful.

3. Mathematical setting

In this section, nonlinear fractional Schrödinger equation (2.3) as quantum system will be considered to do mathematical setting in Sobolev, and Hilbert spaces (cf. [1], [4], [10], [11], [13]). To concentrate to the fractional operator and outside control, this paper is restricted to take the value of real Hilbert space as wave function for simplification (cf. [16]). By the equivalent of norm at complex Hilbert spaces and real Hilbert spaces, it is easily to regard the calculation of term of complex wave functions by their real part function only, or the real part and imaginary part separately.

3.1. **Fractional derivative and norm.** To the practical problem (2.3), define two Hilbert spaces $H = L^2(\Omega), V = H_0^1(\Omega)$ with usual norm and inner product. Then (V, H) is a Gelfand triple spaces $V \hookrightarrow H \hookrightarrow V'$, in which two embeddings are continuous, dense and compact. For $0 < s < 1$, let the fractional spaces $L^s(\Omega), H^s(\Omega), W^{s,p}(\Omega)$ be the fractional L space, fractional Hilbert space, fractional Sobolev space, respectively. As usual $H^s(\Omega) = W^{s,2}(\Omega), p = 2$. s is the order of the derivative of a function ψ of $W^{s,2}(\Omega)$. For example $s = \frac{1}{2}$, and $L^{\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega), W^{\frac{1}{2},2}(\Omega)$.

An interaction operator $\mathcal{N}(s)$ had been used in contributed papers [2], here is quotation for a continuous mapping: $W^{s,2}(\Omega) \to L^2(\mathbf{R}^N/\Omega)$ using nonlocal normal derivative of order s by the definition

$$\mathcal{N}(s)\psi = C(N,s) \int_\Omega \frac{\psi(\mathbf{x}) - \psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y}, \quad \forall\, \mathbf{x} \in \mathbf{R}^N/\bar{\Omega},$$

where $C(N,s) = \dfrac{s 2^{2s} \Gamma(\frac{2s-N}{2})}{\pi^{\frac{s}{2}} \Gamma(1-s)}$, and Γ is special function, Gamma function.

At first, for $0 < s < 1$, cite [1, 4, 9], it need to introduce the fractional derivative and norm for a functional ψ at Sobolev space $W^{s,2}(\Omega)$. For $\psi \in W^{s,2}(\Omega)$, the fractional derivative can be given by the form of

$$\frac{\partial^s \psi}{\partial \mathbf{x}^s} := C(N,s) \int_\Omega \frac{\psi(\mathbf{x}) - \psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y} \quad \text{for all} \quad \psi \in W^{s,2}(\Omega), \tag{3.1}$$

where N is dimension of $\mathbf{R}^N$, take $N = 2$ in here. Denote $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$. Denote $\mathbf{x} = (x_1, x_2, \cdots, x_N)$, $\mathbf{y} = (y_1, y_2, \cdots, y_N)$ for $\mathbf{R}^N$, for detailed expression (cf. [9]) for the integration $d\mathbf{x} = dx_1 dx_2 \cdots dx_N$ of $\mathbf{R}^N$.

In system (2.3) and (3.1), additionally, for $\psi(\mathbf{x}, t) = u(t) \otimes (\mathbf{x} - \mathbf{x}^0)$, the integration of nonlocal normal derivative can be calculated as

$$\begin{aligned}
\int_\Omega \frac{\psi(\mathbf{x}) - \psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y} &= \int_\Omega \frac{u(t) \otimes \delta(\mathbf{x}-\mathbf{x}^0) - u(t) \otimes \delta(\mathbf{y}-\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y} \\
&= u(t) \otimes \int_\Omega \frac{\delta(\mathbf{x}-\mathbf{x}^0) - \delta(\mathbf{y}-\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y} \\
&= u(t) \otimes \int_\Omega \frac{\delta((\mathbf{x}-\mathbf{y}) - \mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{y} \\
&= -u(t) \otimes \int_\Omega \frac{\delta(\mathbf{z}-\mathbf{x}^0)}{|\mathbf{z}|^{N+2s}} d\mathbf{z} \quad \text{(set } \mathbf{z} = \mathbf{x}-\mathbf{y})
\end{aligned}$$

$$
\begin{aligned}
&= -u(t)\otimes\int_\Omega \Big(\delta(\mathbf{z}-\mathbf{x}^0)|\mathbf{z}|^{-N-2s}\Big)d\mathbf{z}\\
&= -u(t)\otimes\int_\Omega |\mathbf{x}^0|^{-(N+2s)}d\mathbf{z}\\
&= -u(t)|\mathbf{x}^0|^{-(N+2s)}\bar{\Omega}, \qquad (3.2)
\end{aligned}
$$

where $\bar{\Omega}$ is a measurement of domain Ω. Thus, by the definition of nonlocal normal derivative to find that $\mathcal{N}(s)\psi$ belong to $L^2(\mathbf{R}^N/\Omega)$, and depended only on time variable t.

Definition 3.1. The Sobolev space $W^{s,2}(\Omega)$ is defined as (cf. [1], [4])

$$W^{s,2}(\Omega)=\Big\{\psi\ \Big|\ \psi\in L^2(\Omega),\int_\Omega\int_\Omega\frac{|\psi(\mathbf{x})-\psi(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^{N+2s}}d\mathbf{x}'d\mathbf{x}<\infty,\ \forall\,\mathbf{x},\mathbf{x}'\in\Omega\Big\}.$$

The norm of y at space $W^{s,2}(\Omega)$ is given by

$$\|\psi\|_{W^{s,2}(\Omega)}=\Big(\|\psi\|^2_{L^2(\Omega)}+\int_\Omega\int_\Omega\frac{|\psi(\mathbf{x})-\psi(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}}d\mathbf{x}d\mathbf{y}\Big)^{\frac{1}{2}}, \qquad (3.3)$$

where N is the dimension of space $\mathbf{R}^N$, in here $N=2$ for particle at surface.

For $\psi,\varphi\in W^{s,2}(\Omega)$, the inner product of $W^{s,2}(\Omega)$ is given by

$$(\psi,\varphi)_{W^{s,2}(\Omega)}=(\psi,\varphi)_{L^2(\Omega)}+\int_\Omega\int_\Omega\frac{(\psi(\mathbf{x})-\psi(\mathbf{x}'))(\varphi(\mathbf{x})-\varphi(\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^{N+2s}}d\mathbf{x}'d\mathbf{x}. \qquad (3.4)$$

Because $\psi(\mathbf{x},t)=0$, a.e. $\mathbf{x}\in\mathbf{R}^N/\Omega$, for cover outside control, define Besov space

$$W_0^{s,2}(\bar{\Omega})=\Big\{\psi\in W^{s,2}(\mathbf{R}^N)\Big|\psi(\mathbf{x})=u(t)\otimes\delta(\mathbf{x}-\mathbf{x}^0)\text{ for }\mathbf{x}\in\mathbf{R}^N/\Omega,\mathbf{x}^0\in\mathbf{R}^N/\Omega\Big\}.$$

The norm of $W_0^{s,2}(\bar{\Omega})$ can be defined as

$$\|\psi\|_{W_0^{s,2}(\bar{\Omega})}=\Big(\|\psi\|^2_{L^2(\Omega)}+\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{|\psi(\mathbf{x})-\psi(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}}d\mathbf{x}d\mathbf{y}\Big)^{\frac{1}{2}}. \qquad (3.5)$$

By the definition (3.5) of fractional derivative, the difference term $\mathbf{x}-\mathbf{y}$ allow to take $\mathbf{x},\mathbf{y}$ outside the domain Ω. Therefore, as pointed out in [2], the normal Laplacian $-\Delta$ is a local operator, which just act at inside the domain Ω, the fractional operator $-\Delta^s$, is a nonlocal operator, which can act at outside of domain $\mathbf{R}^N/\Omega$, and also make ψ derivativiable. Hence, take fractional operator to do outside control is reasonable and well-posed. Additionally, By above Definition 3.1, the fractional gradient operator $\nabla^s=\Big(\frac{\partial^s}{\partial x_1^s},\frac{\partial^s}{\partial x_2^s},\cdots,\frac{\partial^s}{\partial x_N^s}\Big)$, $N=2$ in here. Response to the calculation of integration (3.2), define a local Besove space $W^{s,2}_{\text{local}}(\mathbf{R}^N/\Omega)$ if needed (cf. [2]).

Besov space is special Sobolev space on real space $\mathbf{R}^N$ with dimension N (cf. [1, 4]). To differ with Lebesgue space $L^p(\mathbf{R}^N)$, set $0 < s < 1$, define fractional space as

$$L^{s,p}(\mathbf{R}^N) = \Big\{\psi\Big|\ \mathbf{R}^N \to \mathbf{R}\ \text{ measurable, and } \int_{\mathbf{R}^N} \frac{|\psi(\mathbf{x})|}{(1+|\mathbf{x}|)^{N+2s}} d\mathbf{x} < \infty\Big\}.$$

In here, s is the order of derivative, and p just is a symbol for responding to later Sobolev space $W^{s,p}(\mathbf{R}^N)$, take $p = 1$ in here. Without specified in context, in general, the fractional order appeared at L space is to indicate the fractional Besov space, and the integral order appeared at L space is to indicate the Lebesgue space.

Next, define Besov space $W^{s,2}(\mathbf{R}^N)$ by (3.3) as following, and denote $H^s(\mathbf{R}^N) = W^{s,2}(\mathbf{R}^N)$ for $p = 2$. As to $\mathbf{x}$ in whole domain $\mathbf{R}^N$, for $\psi, \varphi \in W^{s,2}(\mathbf{R}^N)$, its inner product (cf. 3.4) is given by

$$\begin{aligned}(\psi,\varphi)_{W^{s,2}(\mathbf{R}^N)} = (\psi,\varphi)_{L^2(\mathbf{R}^N)} &+ \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\psi(\mathbf{x})-\psi(\mathbf{x}'))(\varphi(\mathbf{x})-\varphi(\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^{N+2s}} d\mathbf{x}' d\mathbf{x} \\ &+ \int_{\partial\mathbf{R}^N} \psi \mathcal{N}(s)\varphi d\mathbf{x}.\end{aligned}$$

The norm of $y \in W^{s,2}(\mathbf{R}^N)$ will be given by

$$\begin{aligned}&\|\psi\|_{W^{s,2}(\mathbf{R}^N)} \\ &= \Big(\|\psi\|^2_{L^2(\mathbf{R}^N)} + \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|\psi(\mathbf{x})-\psi(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^{N+2s}} d\mathbf{x}' d\mathbf{x} + \int_{\partial\mathbf{R}^N} \psi\mathcal{N}(s)\psi d\mathbf{x}\Big)^{\frac{1}{2}}.\end{aligned}$$

Thanks to the definition of $W^{s,2}(\Omega)$ in (3.3), it means that $W^{s,2}(\Omega) \subset L^2(\Omega)$. It is easily to let us take $\mathbf{H} = L^2(\Omega)$. Denote $W_0^{s,2}(\Omega) = \{\psi \,|\psi \in W^{s,2}(\Omega), \psi_\Gamma = \psi_{\partial\Omega} = 0\}$. Use that notation H^s for fractional Hilbert space $0 < s < 1$. Set $\mathbf{V} = H_0^s(\Omega) = W_0^{s,2}(\Omega)$, $\mathbf{H} = L^2(\Omega)$, and $\mathbf{V}' = H^{-s}(\Omega) = W^{-s,2}(\Omega)$. Thus, $(\mathbf{V}, \mathbf{H})$ is a Gelfand triple spaces $\mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}'$, in which two embeddings are continuous, dense and compact.

Remarkable that, not take $\mathbf{H} = L^s(\Omega)$ for the simplification. Due to fractional derivative $\dfrac{\partial^k}{\partial t^k}$, $0 < k < 1$ respect to variable time t is not involved in the equation (2.3), therefore, Sobolev space $W^{2s,s}(\Omega) = L^2(0,T; W^{s,2}(\Omega)) \cap L^2(\Omega; W^s(0,T))$ for two variable fractional derivative is exclusive in this paper.

3.2. **Bilinear form.** For the fractional operator $(-\Delta)^s$, $0 < s < 1$ to discuss the Bilinear form $a(\psi, \phi) = (\nabla^s\psi, \nabla^s\phi)$ as preparation to compose weak form.

Notice the boundary value $\psi_\Gamma = \psi_{\partial\Omega} = 0$ for $W_0^{s,2}(\Omega)$.

$$\begin{aligned} a(\psi,\phi) &= \langle -\Delta^s\psi, \phi\rangle_{\mathbf{V}',\mathbf{V}} = \langle -\Delta^s\psi,\phi\rangle_{H^{-s}(\Omega),H_0^s(\Omega)} \\ &= (\nabla^s\psi, \nabla^s\phi)_{L^2(\Omega)} = (\nabla^s\psi,\nabla^s\phi)_{\mathbf{H}} = (\psi,\phi)_{W^{s,2}(\Omega)} \\ &= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))(\nabla^s\phi(\mathbf{x}) - \nabla^s\phi(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \end{aligned} \tag{3.6}$$

for all $\psi, \phi \in D(A,s)$, where the value domain of operator $(-\Delta^s)$ can be defined as

$$\begin{aligned} D(A,s) &= \left\{ \psi \;\middle|\; \psi \in W_0^{2s,2}(\bar{\Omega}) \text{ s.t. } -i\hbar^s\psi = \Delta^s\psi + V\psi \right\} \\ &= W_0^{2s,2}(\bar{\Omega}) \cap W_0^{s,2}(\bar{\Omega}). \end{aligned}$$

In particular, to establish the connection between inside space $W^{s,2}(\Omega)$ and outside space $L^2(\mathbf{R}/\Omega)$. To avoid the concept of nonlocal normal derivative, in this paper, use the most common trace theorem as in [5] to describe the relationship of $W^{s,2}(\Omega)$ and $L^2(\mathbf{R}^N/\Omega)$ in Proposition 3.3.

For the domain Ω and its boundary $\partial\Omega = \Gamma$ (without confusion with Gamma function Γ), cite trace theorem in paper [9] of Wang, and [6] has the result.

Lemma 3.2. *For $0 < s < 1$, and the normal derivative for order j, denote as $\frac{\partial^j\psi}{\partial\eta^j}$, is a continuous mapping*

$$\frac{\partial^j}{\partial\eta^j} : \; W^{s,2}(\Omega) \to H^{s-j-\frac{1}{2}}(\partial\Omega).$$

Proposition 3.3. *For $0 < s < 1$, there is a continuous mapping from inside space $W^{s,2}(\Omega)$ to outside space $L^2(\mathbf{R}^N/\Omega)$.*

Proof. For parabolic differential equation, take $j = 2$ in Lemma 3.2. Hence $\|\cdot\|_{H^{s-j-\frac{1}{2}}(\Gamma)} \le C\|\cdot\|_{W^{s,2}(\Omega)}$ for constant C.

On the other hand, the outside space $\mathbf{R}^N/\Omega$ shared a boundary $\partial\Omega$ with inside domain Ω, therefore, the outside normal derivative is the minus of inside normal derivative at the same point, such as $\mathbf{x}' \in \partial\Omega$, its also belong to $\mathbf{x}' \in \partial(\mathbf{R}^N/\Omega)$. That is to say

$$\frac{\partial^j}{\partial\eta^j}\Big|_{\mathbf{x}=\mathbf{x}'\in\partial\Omega} = -\frac{\partial^j}{\partial\eta^j}\Big|_{\mathbf{x}=\mathbf{x}'\in\partial(\mathbf{R}^N/\Omega)}.$$

Let's use this relationship to connect inside Ω and outside $\mathbf{R}^N/\Omega$. Thus, it is easily to have

$$\psi(\mathbf{x}') \in H^{s-j-\frac{1}{2}}(\partial\Omega),\ \mathbf{x}' \in \partial\Omega \;\to\; \psi(\mathbf{x}') \in H^{s-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega)),\ \mathbf{x}' \in \partial(\mathbf{R}^N/\Omega).$$

For outside domain $\mathbf{R}^N/\Omega$, try to use the trace theorem for $L^2(\mathbf{R}^N/\Omega)$, and find that the mapping

$$-\frac{\partial^j}{\partial\eta^j}:\ L^2(\mathbf{R}^N/\Omega)\to\psi(\mathbf{x}')\in L^{-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega)),\quad \mathbf{x}'\in\partial(\mathbf{R}^N/\Omega)$$

is continuous. Hence $\|\cdot\|_{L^{-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}\le C\|\cdot\|_{L^2(\mathbf{R}^N/\Omega)}$ for constant $C>0$. Since $0<s<1$, we have

$$H^{s-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))\subset L^{s-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))\subset L^{-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))$$

and $\|\cdot\|_{L^{-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}\le\|\cdot\|_{H^{s-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}$. By the norm inequality of each space to know

$$\|\cdot\|_{L^2(\mathbf{R}^N/\Omega)}\le\|\cdot\|_{L^{-j-\frac{1}{2}}(\Gamma)}\le\|\cdot\|_{H^{s-j-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}\le C\|\cdot\|_{W^{s,2}(\Omega)}. \tag{3.7}$$

More precisely calculation in here for norm estimates, denote $\Gamma=\partial\Omega,\Sigma=\partial\Omega\times(0,T)$. For $t\in[0,T]$, $\gamma_\Sigma=\dfrac{\partial^n}{\partial\eta^n}$, $\gamma_\Sigma\in\mathcal{L}(W^{s,2}(\Omega),H^{s-\frac{1}{2}}(\partial\Omega))$ such that $\gamma_\Sigma\psi=\psi|_\Sigma$, its norm denote as $\|\gamma_\Sigma\|$. Then for $j=0$, the mapping $\dfrac{\partial^j}{\partial\eta^j}:W^{s,2}(\Omega)\to H^{s-\frac{1}{2}}(\partial\Omega)$ is continuous, and

$$\|\psi|_\Sigma\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\le\|\gamma_\Sigma\|\ \|\psi\|_{W^{s,2}(\Omega)}. \tag{3.8}$$

On the other hand, inverse mapping γ_Σ^* means $\gamma_\Sigma^*=-\gamma_\Sigma$, and the mapping $\gamma_\Sigma^*:H^{-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))\to L^2(\mathbf{R}^N/\Omega)$ is continuous, such that $\gamma_\Sigma^*\psi=\psi|_\Sigma$, that is, $\gamma_\Sigma\psi|_\Sigma=\psi$ for $\psi\in L^2(\mathbf{R}^N/\Omega)$. Then to have

$$\|\psi\|_{L^2(\mathbf{R}^N/\Omega)}\le\|\gamma_\Sigma\|\ \|\psi|_\Sigma\|_{H^{-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}. \tag{3.9}$$

For $0<s<1$ to know $\dfrac{-1}{2}<s-\dfrac{1}{2}<\dfrac{1}{2}$, and the shared boundary (e.g, regular enough) means $\partial\Omega=\partial(\mathbf{R}^N/\Omega)$ to get

$$H^{s-\frac{1}{2}}(\partial\Omega)\subset H^{-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega)).$$

That is,

$$\|\cdot\|_{H^{-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}\le C'\|\cdot\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \tag{3.10}$$

for constant $C'>0$. By (3.8), (3.9), (3.10) to get

$$\begin{aligned}\|\psi\|_{L^2(\mathbf{R}^N/\Omega)}&\le\|\gamma_\Sigma\|\ \|\psi|_\Sigma\|_{H^{-\frac{1}{2}}(\partial(\mathbf{R}^N/\Omega))}\\&\le C'\|\gamma_\Sigma\|\ \|\psi|_\Sigma\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\\&\le C'\|\gamma_\Sigma\|^2\ \|\psi\|_{W^{s,2}(\Omega)}.\end{aligned} \tag{3.11}$$

Therefore, set $C'' = C'\|\gamma_\Sigma\|^2$ to obtain (3.7) directly. Therefore, (3.11) show that there is a continuous mapping from inside space $W^{s,2}(\Omega)$ to outside space $L^2(\mathbf{R}^N/\Omega)$. This completes the proof. □

Furthermore, for fractional Laplacian $-\Delta^s$ and $j = 2$, the integration by part of bilinear form (3.6) can be deduced by inner product.

$$\int_{\mathbf{R}^N} (-\Delta)^s \psi\phi d\mathbf{x} = (-\Delta^s\psi, \phi)_{W^{s,2}(\mathbf{R}^N)} + (-\Delta^s\psi, \phi)_{W^{s,2}(\mathbf{R}^N/\Omega)}.$$

For $0 < s < 1$, define inner product of $L^s(\mathbf{R}^N)$ space for abstract calculation at shared boundary Γ.

$$(\psi, \phi)_{L^s(\mathbf{R}^N)} = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))(\phi(\mathbf{x}) - \phi(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y}, \text{ for } \mathbf{x}, \mathbf{y} \in \Omega. \tag{3.12}$$

Suppose control $u(t)$ belong to the $L^2(0, T)$ space. It is clarifying that time variable t in wave function will be belong to $L^2(0, T; \cdot)$ space although the spatial derivative for $\mathbf{x}$ is fractional order s. Notice the equivalent

$$L^2(0, T; H^s(\Omega)) = H^s(0, T; L^2(\Omega)) \quad \text{for} \quad 0 < s < 1.$$

Definition 3.4. The space $W(0, T; s)$ is called a solution space if it is defined by

$$\begin{aligned} W(0, T; , s) &= \left\{\psi \mid \psi \in L^2(0, T; \mathbf{V}), \frac{\partial^s\psi}{\partial\mathbf{x}^s} \in L^2(0, T; \mathbf{V}')\right\} \\ &= \left\{\psi \mid \psi \in L^2(0, T; H^s(\Omega)), \frac{\partial^s\psi}{\partial\mathbf{x}^s} \in L^2(0, T; H^{-s}(\Omega)\right\}. \end{aligned}$$

If $(\psi, \phi) \in W(0, T; s)$, for variable $\mathbf{x}$, then the inner product is defined by more detail expanding formula

$$\begin{aligned} (\psi, \phi)_{W(0,T;s)} = &(\psi(\mathbf{x}), \psi(\mathbf{x})_{L^2(\Omega)} \\ &+ \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))(\phi(\mathbf{x}) - \phi(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\ &+ (\nabla^s\psi(\mathbf{x}), \nabla^s\psi(\mathbf{x}))_{L^2(\Omega)} \\ &+ \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))(\nabla^s\phi(\mathbf{x}) - \nabla^s\phi(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N-2s}} d\mathbf{x}d\mathbf{y}, \end{aligned}$$

that is, $(\psi, \phi)_{W(0,T;s)} = (\psi, \phi)_{L^2(0,T;\mathbf{V})} + (\psi, \phi)_{L^2(0,T;\mathbf{V}')}$.

Then, inner product induced norm of solution space $W(0,T;s)$ can be defined

$$\|\psi\|_{W(0,T;s)} = \Big(\int_0^T \Big[\|\psi\|^2_{L^2(\Omega)} + \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|\psi(\mathbf{x})-\psi(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} + \|\nabla^s\psi\|^2_{L^2(\Omega)} + \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|\nabla^s\psi(\mathbf{x})-\nabla^s\psi(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{N-2s}} d\mathbf{x}d\mathbf{y}\Big]dt\Big)^{\frac{1}{2}}.$$

That is, $\|\psi\|^2_{W(0,T;s)} = \|\psi\|^2_{L^2(0,T;\mathbf{V})} + \|\psi\|^2_{L^2(0,T;\mathbf{V}')}$. Thus, $W(0,T;s)$ is a Hilbert space equipped by above norm and inner product.

3.3. **Weak solution.**

Definition 3.5. Let $T > 0$. A function ψ is called a weak solution of (2.3) if $\psi \in W(0,T;s)$ and satisfy the weak form:

$$\begin{aligned} &-\int_0^T\int_{\mathbf{R}^N} i\hbar^s \frac{\partial\psi}{\partial t}\bar{\eta} d\mathbf{x}dt \\ &+\int_0^T\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\nabla^s\psi(\mathbf{x})-\nabla^s\psi(\mathbf{y})(\nabla^s\bar{\eta}(\mathbf{x})-\nabla^s\bar{\eta}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y}dt \\ &= \int_0^T\int_{\mathbf{R}^N} V\psi\bar{\eta} d\mathbf{x}dt \end{aligned} \tag{3.13}$$

for all $\bar{\eta} \in C^1(0,T;\mathbf{V})$ in the sense of distribution on $(0,T)$, and $\eta(T) = 0$, *a.e.* $\mathbf{x} \in \Omega$, where $\bar{\eta}$ is a conjugate function of functional η.

3.4. **Existence of weak solution.**

Let $\mathbf{x}_0$ be a fix point at outside of domain Ω, that is, $\mathbf{x}_0 \in \mathbf{R}^2/\Omega$. For control variable u and outside control term $u(t) \otimes \delta(\mathbf{x}-\mathbf{x}_0)$, set $\mathcal{U} = L^2(0,T)$ is the control space, and $\mathcal{U}_{ad}$ is admissible subset of $\mathcal{U}$, then $u \in \mathcal{U}$.

For such a outside control problem, set

$$\mathcal{N}_\delta u(t) = u(t) \otimes \delta(\mathbf{x}-\mathbf{x}_0), \quad \mathbf{x} \in \mathbf{R}^2/\Omega, \quad t \in [0,T].$$

$\psi(\mathbf{x},t) = \mathcal{N}_\delta u(t)$ is a outside value of system (2.3) for $(\mathbf{x},t) \in \mathbf{R}^2/\Omega \times [0,T]$. Via control $u \in L^2(0,T)$ and $\delta(\mathbf{x}-\mathbf{x}_0) \le 1$, we get $\mathcal{N}_\delta u \in L^2(0,T;L^2(\mathbf{R}^2/\Omega))$. Then $\mathcal{N}_\delta$ is a continuous mapping $u \to \mathcal{N}_\delta u : L^2(0,T) \to L^2(0,T;L^2(\mathbf{R}^2/\Omega))$.

Theorem 3.6. *For given* $\psi_0 \in L^2(\Omega)$, $\mathbf{x}_0 = (x_1^0, x_2^0)$, *and outside value* $\psi(\mathbf{x},t) = u(t) \otimes \delta(\mathbf{x}-\mathbf{x}_0)$ *for* $\mathbf{x} \in \mathbf{R}^2/\Omega$, *if* $\mathcal{U} = L^2(0,T)$ *is bounded, then there exist a unique weak solution* $\psi(u)$ *of fractional Schrödinger system* (2.3) *in* $W(0,T;s)$ *and* $C(0,T;\mathbf{H})$, *such that a priori estimates*

$$\|\psi(u)\|^2_{L^2(0,T;\mathbf{H})} \le C_s(\|\psi(0)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T)}), \tag{3.14}$$

$$\|\psi(u)\|^2_{L^2(0,T;\mathbf{V})} \le C_s(\|\psi(0)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T)}) \tag{3.15}$$

are valid, where C_s *is bounded constant, which depended on fractional order* s *only.*

Proof. It is quite interest to prove the existence of weak solution for the nonlinear fractional Schrödinger state system (2.3).

In fractional Schrödinger equation (2.3), take $\bar{\eta} = \psi \in \mathbf{V} = H^s(\Omega)$ at weak form (3.13), then we get the formulation in spatial space for $\mathbf{x} \in \Omega$ as

$$\int_\Omega -i\hbar^s \frac{\partial \psi}{\partial t}\psi d\mathbf{x} + \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} = \int_\Omega V(\mathbf{x})\psi\psi d\mathbf{x}. \quad (3.16)$$

Denote bilinear form (3.6), and $a(\psi,\phi) = (\nabla^s\psi, \nabla^s\phi)_{\mathbf{H}} = (\psi,\phi)_{\mathbf{V}}$. Fractional differential operator is $-\Delta^s$ for spatial variable $\mathbf{x}$. The formulation (3.16) can be rewritten in the form of inner products, hence that

$$\frac{i\hbar^s}{2}\frac{d}{dt}(\psi,\psi)_{\mathbf{H}} + (\nabla^s\psi, \nabla^s\psi)_{\mathbf{H}} = (V(\mathbf{x})\psi,\psi)_{\mathbf{H}}. \quad (3.17)$$

By the equivalent of norm at complex space and real space, we deduce the formula for Schrödinger equation in the form of norm, that is, from (3.17) we find

$$\frac{\hbar^s}{2}\frac{d}{dt}\|\psi\|^2_{\mathbf{H}} + \|\nabla^s\psi\|^2_{\mathbf{V}} = (V(\mathbf{x})\psi,\psi)_{\mathbf{H}}. \quad (3.18)$$

Citing the definition of $L^s(\mathbf{R}^N)$ at (3.12), and consider $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$, then we calculate the integration of second term as

$$\begin{aligned}
&\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \Big[\int_\Omega\int_\Omega + \int_\Omega\int_{\mathbf{R}^N/\Omega} + \int_{\mathbf{R}^N/\Omega}\int_\Omega + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega}\Big] \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega} \frac{(\nabla^s\psi(\mathbf{x}) - \nabla^s\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^n/\Omega} \frac{[\nabla^s(u(t)\otimes\delta(\mathbf{x}-\mathbf{x}^0)) - \nabla^s(u(t)\otimes\delta(\mathbf{y}-\mathbf{x}^0))]^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega} \frac{(\nabla^s\delta(\mathbf{x}-\mathbf{x}^0) - \nabla^s\delta(\mathbf{y}-\mathbf{x}^0))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\|\delta(\mathbf{x}-\mathbf{x}^0)\|^2_{\mathbf{V}'(\mathbf{R^N}/\boldsymbol{\Omega})} \\
&\le \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\|\delta(\mathbf{x}-\mathbf{x}^0)\|^2_{\mathbf{V}(\mathbf{R}^N/\Omega)} \\
&\le \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\|1\|^2_{\mathbf{V}(\mathbf{R}^N/\Omega)} = \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\|1\|^2_{\mathbf{V}(\mathbf{R}^N)} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t)\otimes\|1\|^2_{\mathbf{V}(\Omega)} = \|\psi\|^2_{\mathbf{H}} + \bar{\Omega}^2 u(t),
\end{aligned}$$

where $\bar{\Omega}$ is a measurement of domain Ω. Hence

$$\|\nabla^s\psi\|^2_{\mathbf{V}} \le \|\nabla^s\psi\|^2_{\mathbf{H}} + \bar{\Omega}^2 u(t) \quad (3.19)$$

for $t \in [0,T]$. Notice that $H^1(\Omega) \subset H^s(\Omega) \subset L^2(\Omega)$, then

$$\|\cdot\|_{L^2(\Omega)} \leq \|\cdot\|_{H^s(\Omega)} \leq \|\cdot\|_{H^1(\Omega)}.$$

Due to $0 < s < 1$, take $p' = \dfrac{1}{s}, q' = \dfrac{1}{1-s}$, $p', q' > 1$, then, $\dfrac{1}{p'} = s$, $\dfrac{1}{q'} = 1-s$ such that $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. The nonlinear term in the right hand can be calculated by Hölder inequality as

$$\begin{aligned}
(V(\mathbf{x})\psi, \psi)_{\mathbf{H}} = (V(\mathbf{x})\psi, \psi)_{L^2(\Omega)} &= \int_\Omega V(\mathbf{x})\psi\psi d\mathbf{x} \\
&\leq \Big(\int_\Omega (V(\mathbf{x})\psi)^2 d\mathbf{x}\Big)^{\frac{1}{2}} \Big(\int_\Omega \psi^2 d\mathbf{x}\Big)^{\frac{1}{2}} \\
&= \Big(\int_\Omega V(\mathbf{x})^{\frac{2}{s}} d\mathbf{x}\Big)^{\frac{s}{2}} \Big(\int_\Omega \psi^{\frac{2}{1-s}} d\mathbf{x}\Big)^{\frac{1-s}{2}} \|\psi\|_{L^2(\Omega)} \\
&= \|V\|_{L^{\frac{2}{s}}(\Omega)} \|\psi\|_{L^{\frac{2}{1-s}}(\Omega)} \|\psi\|_{\mathbf{H}}
\end{aligned} \tag{3.20}$$

for $L^{\frac{2}{s}}(\Omega) \subset L^2(\Omega)$. Via Sobolev embedding theorem and space interpolation theorem $H^s(\Omega) \hookrightarrow L^{\frac{2}{1-s}}(\Omega)$ for $N = 2$, we know that $\|\cdot\|_{L^{\frac{2}{1-s}}(\Omega)} \leq \|\cdot\|_{H^s(\Omega)}$. And also, Young' s inequality and (3.20), we get for $\mathbf{V} = H_0^s(\Omega)$,

$$(V(\mathbf{x})\psi, \psi)_{\mathbf{H}} \leq \frac{1}{2}\|V\|^2_{L^{\frac{2}{s}}(\Omega)} \|\psi\|^2_{\mathbf{H}} + \frac{1}{2}\|\psi\|^2_{\mathbf{V}}. \tag{3.21}$$

That is, substituting (3.21) into equation (3.18), we get the evaluations

$$\frac{\hbar^s}{2}\frac{d}{dt}\|\psi\|^2_{\mathbf{H}} + \frac{1}{2}\|\psi\|^2_{\mathbf{V}} \leq \frac{1}{2}\|V\|^2_{L^{\frac{2}{s}}(\Omega)} \|\psi\|^2_{\mathbf{H}}. \tag{3.22}$$

Owing to the positivity of norm of ψ in space $\mathbf{V}$, the formula (3.22) can be converted to inequality (3.23) for ψ as

$$\frac{d}{dt}\Big(\|\psi\|^2_{\mathbf{H}}\Big) \leq \frac{1}{\hbar^s}\|V\|^2_{L^{\frac{2}{s}}(\Omega)} \|\psi\|^2_{\mathbf{H}}. \tag{3.23}$$

Since $\mathbf{H} = L^2(\Omega)$, applying Bellman-Gronwall inequality to (3.23), we get

$$\|\psi(t)\|^2_{L^2(\Omega)} \leq \|\psi(0)\|^2_{L^2(\Omega)} \exp\Big(\frac{1}{\hbar^s}\int_0^t \|V(\tau)\|^2_{L^{\frac{2}{s}}(\Omega)} d\tau\Big). \tag{3.24}$$

Denote $C(t,s) = \exp\Big(\dfrac{1}{\hbar^s}\displaystyle\int_0^t \|V(\tau)\|^2_{L^{\frac{2}{s}}(\Omega)} d\tau\Big)$, hence (3.24) to get $\|\psi(t)\|^2_{\mathbf{H}} \leq C(t,s)\|\psi(0)\|^2_{\mathbf{H}}$. Then, by $0 < s < 1$ to know that $2 < \dfrac{2}{s} < \infty$, and $L^\infty(\Omega) \subset$

$L^{\frac{2}{s}}(\Omega) \subset L^2(\Omega)$,

$$\begin{aligned}\int_0^T \|\psi(t)\|^2_{L^2(\Omega)} dt &\leq \|\psi(0)\|^2_{L^2(\Omega)} \exp\Big(\frac{1}{i\hbar^s}\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega)}\Big) \\ &\leq \|\psi(0)\|^2_{\mathbf{H}} \exp\Big(\frac{1}{\hbar^s}\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega))}\Big). \end{aligned} \tag{3.25}$$

It means that $\psi \in L^2(0,T;\mathbf{H})$ and its norm is bounded. Set $C_s = \exp\Big(\frac{1}{\hbar^s}\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega))}\Big)$, then from (3.25), we have

$$\|\psi\|^2_{L^2(0,T;\mathbf{H})} \leq C_s\|\psi(0)\|^2_{\mathbf{H}}. \tag{3.26}$$

It is (3.14). By inequality (3.22), we obtain that

$$\frac{d}{dt}\Big(\|\psi\|^2_{\mathbf{H}}\Big) + \frac{1}{\hbar^s}\|\psi\|^2_{\mathbf{V}} \leq \frac{1}{\hbar^s}\|V\|^2_{L^{\frac{2}{s}}(\Omega)}\|\psi\|^2_{\mathbf{H}}.$$

Take integration for both sides at $[0,t], t < T$, then we get

$$\hbar^s(\|\psi(t)\|^2_{\mathbf{H}} - \|\psi(0)\|^2_{\mathbf{H}}) + \|\psi\|^2_{L^2(0,t;\mathbf{V})} \leq \int_0^t \|V\|^2_{L^{\frac{2}{s}}(\Omega)}\|\psi\|^2_{\mathbf{H}} dt.$$

Similarly, by the positivity of norm ψ at space $\mathbf{H}$, we know that

$$\|\psi\|^2_{L^2(0,t;\mathbf{V})} \leq \int_0^t \|V\|^2_{L^{\frac{2}{s}}(\Omega)}\|\psi\|^2_{\mathbf{H}} dt + \hbar^s\|\psi(0)\|^2_{\mathbf{H}}. \tag{3.27}$$

Take derivative respect to variable t for both side of (3.27), then we get

$$\|\psi(t)\|^2_{\mathbf{V}} \leq \|V\|^2_{L^{\frac{2}{s}}(\Omega)}\|\psi\|^2_{\mathbf{H}} \leq C(t,s)\|V\|^2_{L^{\frac{2}{s}}(\Omega)}\|\psi(0)\|^2_{\mathbf{H}}, \quad \text{for } 0 < s < 1.$$

Set $C_s(t) = C(t,s)\|V\|^2_{L^{\frac{2}{s}}(\Omega)}$. Thus, the estimate of norm of ψ at space $\mathbf{V}$ is

$$\|\psi(t)\|^2_{\mathbf{V}} \leq C_s(t)\|\psi(0)\|^2_{\mathbf{H}}. \tag{3.28}$$

It means that $\psi \in L^2(\mathbf{V})$, and its norm is bounded. Let

$$C_s(t) = \exp\Big(\frac{1}{\hbar^s}\|V\|^2_{L^2(0,t;L^{\frac{2}{s}}(\Omega))}\Big)\|V(t)\|^2_{L^{\frac{2}{s}}(\Omega)}$$

for $0 < s < 1$. Then (3.28) implies that

$$\begin{aligned}\|\psi\|^2_{L^2(0,T;\mathbf{V})} &\leq \int_0^T C_s(t) dt \|\psi(0)\|^2_{\mathbf{H}} \\ &\leq \exp\Big(\frac{1}{\hbar^s}\|V\|^2_{L^2(0,t;L^{\frac{2}{s}}(\Omega))}\Big)\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega))}\|\psi(0)\|^2_{\mathbf{H}}. \end{aligned} \tag{3.29}$$

Set another $\mathbf{C}_s = \exp\Big(\frac{1}{\hbar^s}\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega))}\Big)\|V\|^2_{L^2(0,T;L^{\frac{2}{s}}(\Omega))}$, then (3.29) implies that

$$\|\psi\|^2_{L^2(0,T;\mathbf{V})} \leq \mathbf{C}_s\|\psi(0)\|^2_{\mathbf{H}}. \tag{3.30}$$

Furthermore, by definition of norm and (3.19), we know that

$$\int_0^T \|\psi\|_{\mathbf{V}}^2 dt \leq \int_0^T \|\psi\|_{\mathbf{H}}^2 dt + \bar{\Omega}\|u\|_{L^2(0,T)}^2.$$

This means that

$$\|\psi\|_{L^2(0,T;\mathbf{V})}^2 \leq \|\psi\|_{L^2(0,T;\mathbf{H})}^2 + \bar{\Omega}\|u\|_{L^2(0,T)}^2. \tag{3.31}$$

Thus, by (3.26), (3.30) and (3.31), we can get the inequality (3.15) directly.

By the definition of norm at $W^{s,2}(\Omega)$ and $W_0^{s,2}(\bar{\Omega})$ (corresponding space $L_0^s(\bar{\Omega}), H_0^s(\bar{\Omega})$), we know that their norm are equivalent at the domain Ω and different appeared at the outside domain $\mathbf{R}^N/\Omega$. Therefore obtained inequality (3.23) is hold for domain $\bar{\Omega}$ and $\mathbf{R}^N$.

By Corollary 6.1 in Appendix, we know that initial value can be chosen at fractional space $L_0^s(\Omega)$. Currently, take $\psi(0) \in L^2(\Omega)$. Clearly, due to take the $\mathbf{H} = L^2(\Omega)$ and without take $L^s(\Omega), 0 < s < 1$, it is easily to have argument. It is a suggest choice to take $L^s(\Omega)$ as $\mathbf{H}$ space. Here, it still keep the fashion for a normal selection as usual.

By the norm equivalent at inside and outside domain, the estimates (3.25) and (3.27) evident that ψ belong to $L^2(0,T;\mathbf{H})$ and $L^2(0,T;\mathbf{V})$, that is,

$$\psi \in L^2(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\Omega)),$$

and their norm is bounded at each space.

Now, to prove rest part of Theorem 3.6, using Faedo-Galerkin method, we prove the existence of weak solution in the following.

(1) We will construct an approximate solution for the system (2.3). Since $\mathbf{V} \hookrightarrow \mathbf{H}$ is compact, there exists orthogonal basis of $\mathbf{H}$, $\{\omega_j\}_{j=1}^{\infty}$ consisting of eigenfunctions of $A^s = \Delta^s$, such that for $0 < s \leq 1$,

$$A^s \omega_j = \lambda_j^s \omega_j$$

for all j, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$, as $j \to \infty$. Denote by $\bar{P}_n$ the orthogonal projection of $\mathbf{H}$(or $\mathbf{V}$) onto the space spanned by $\{\omega_1, \omega_2, \cdots, \omega_n\}$. For each $n \in N$, an approximate solution is defined for fractional Schrödinger system (2.3) by

$$\psi^n(t) = \sum_{j=1}^{n} a_{jn}(t)\omega_j, \tag{3.32}$$

where $a_{jn}(t)$ is real-valued coefficient function. Then approximate real-valued solution $\psi^n(t)$ in (3.32) satisfy the ordinary differential equation $(1 \leq j \leq n)$

given by

$$\begin{cases} \int_{\mathbf{R}^N} i\hbar^s \frac{\partial \psi^n}{\partial t} \bar{\omega}_j d\mathbf{x} = \int_{\mathbf{R}^N} \Delta^s \psi^n \bar{\omega}_j d\mathbf{x} + \int_{\mathbf{R}^N} V(\mathbf{x},t)\psi^n \bar{\omega}_j d\mathbf{x}, \quad \text{in} \quad Q, \\ \psi^n(0) = \psi_0^n \quad \text{in } \Omega, \\ \phi^n(\mathbf{x},t) = u(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0), \quad \text{for} \quad \mathbf{x} \in \mathbf{R}^N/\Omega, \end{cases} \tag{3.33}$$

where, $\bar{\omega}_j$ is the conjecture function of base function ω_j for each j. Therefore, the standard theory of ODE ensure that the obtained system (3.33) had unique local solution ψ^n for $n = 1, 2, 3, \cdots, N$.

(2) For given $\psi(0) = \psi_0 \in L^2(\Omega)$, set $\psi^n(0) = 0$, then there exists $\psi_0^n \in L^2(\Omega)$ such that

$$\psi_0^n \to \psi_0 \quad \text{in} \quad L^2(\Omega) \tag{3.34}$$

as $n \to \infty$. By two estimates (3.25) and (3.27) imply that

$$\psi^n \quad \text{is bounded in} \quad L^2(0,T;\mathbf{H}),$$
$$\psi^n \quad \text{is bounded in} \quad L^2(0,T;\mathbf{V}).$$

That is, for a function $\psi \in L^2(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$, there exists a subsequence $\{\psi^{n_k}\}$ of $\{\psi^n\}$ such that

$$\psi^{n_k} \to \psi \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}), \tag{3.35}$$
$$\psi^{n_k} \to \psi \quad \text{weakly in} \quad L^2(0,T;\mathbf{V}) \tag{3.36}$$

as $n_k \to \infty$.

(3) Suppose that $\{\psi^j\}$ and $\{\psi^k\}$ are two sets of solutions to (2.3) corresponding to initial value $\{\psi^j(0)\}$ and $\{\psi^k(0)\}$ for $j, k = 1, 2, \cdots, \infty$, respectively. For $0 < s < 1$, by calculating its difference as

$$\frac{i\hbar^s}{2} \frac{\partial}{\partial t}(\psi^j(t) - \psi^k(t)) = \Delta^s(\psi^j(t) - \psi^k(t)) + V(\mathbf{x},t)(\psi^j(t) - \psi^k(t)). \tag{3.37}$$

By the multiplying $(\psi^j - \psi^k)$ to the weak form of (3.37), we get inner product form of

$$\begin{aligned} &\frac{i\hbar^s}{2}\Big(\frac{\partial}{\partial t}(\psi^j(t) - \psi^k(t)), \psi^j(t) - \psi^k(t)\Big)_{\mathbf{H}} \\ &= \Big\langle \Delta^s(\psi^j(t) - \psi^k(t)), \psi^j(t) - \psi^k(t)\Big\rangle_{\mathbf{V}',\mathbf{V}} \\ &\quad + \Big(V(\mathbf{x},t)(\psi^j(t) - \psi^k(t)), \psi^j(t) - \psi^k(t)\Big)_{\mathbf{H}}. \end{aligned} \tag{3.38}$$

Suppose ψ^j, ψ^k are corresponding to different controls u^j, u^k for j, k. Then, using the estimate (3.19) for $\psi^j(t) - \psi^k(t)$, we get

$$\|\nabla^s \psi^j(t) - \nabla^s \psi^k(t))\|_{\mathbf{V}}^2 \le \|\nabla^s \psi^j(t) - \nabla^s \psi^k(t)\|_{\mathbf{H}}^2 + \bar{\Omega}^2(u^j(t) - u^k(t)). \tag{3.39}$$

For $0 < s < 1$, due to $2 < \dfrac{2}{s} < \infty$, and $L^\infty(\Omega) \subset L^{\frac{2}{s}}(\Omega) \subset L^2(\Omega)$, $\|\cdot\|_{L^{\frac{2}{s}}(\Omega)} \leq \|\cdot\|_{L^2(\Omega)}$, by using the same estimate (3.21), we get

$$\begin{aligned}&\Big(V(\mathbf{x},t)(\psi^j(t)-\psi^k(t)),\psi^j(t)-\psi^k(t)\Big)_{\mathbf{H}}\\ &\leq \frac{1}{2}\|V(t)\|^2_{L^{\frac{2}{s}}(\Omega)}\|(\psi^j(t)-\psi^k(t))\|^2_{\mathbf{H}}+\frac{1}{2}\|(\psi^j(t)-\psi^k(t))\|^2_{\mathbf{V}}.\end{aligned} \tag{3.40}$$

Substitute (3.39) and (3.40) into (3.38), we obtain that

$$\begin{aligned}&\frac{\hbar^s}{2}\frac{d}{dt}\|\psi^j(t)-\psi^k(t))\|^2_{\mathbf{H}}+\|\nabla^s\psi^j(t)-\nabla^s\psi^k(t)\|^2_{\mathbf{H}}\\ &\leq \frac{1}{2}\|V(t)\|^2_{L^{\frac{2}{s}}(\Omega)}\|(\psi^j(t)-\psi^k(t))\|^2_{\mathbf{H}}\\ &\quad+\frac{1}{2}\|\psi^j(t)-\psi^k(t)\|^2_{\mathbf{V}}+\bar{\Omega}^2(u^k(t)-u^j(t)).\end{aligned} \tag{3.41}$$

From $\|\nabla^s\cdot\|_{\mathbf{H}} = \|\cdot\|_{\mathbf{V}}$, and (3.41), we get

$$\begin{aligned}&\frac{\hbar^s}{2}\frac{d}{dt}\|\psi^j(t)-\psi^k(t))\|^2_{\mathbf{H}}+\frac{3}{2}\|\psi^j(t)-\psi^k(t)\|^2_{\mathbf{V}}\\ &\leq \frac{1}{2}\|V(t)\|^2_{L^{\frac{2}{s}}(\Omega)}\|(\psi^j(t)-\psi^k(t))\|^2_{\mathbf{H}}+\bar{\Omega}^2(u^k(t)-u^j(t)).\end{aligned} \tag{3.42}$$

Let $I^{jk}(t) = \|\psi^j(t)-\psi^k(t)\|^2_{\mathbf{H}} + \|\psi^j(t)-\psi^k(t)\|^2_{\mathbf{V}}$. Then (3.42) convert to

$$\hbar^s\frac{d}{dt}I^{jk}(t) \leq C_s(t)I^{jk}(t)+D(t), \tag{3.43}$$

where $C_s(t) = 3+\|V(t)\|^2_{L^{\frac{2}{s}}(\Omega)}$ for $0 < s < 1$ and $D(t) = 2\bar{\Omega}^2(u^k(t)-u^j(t))$. Apply Bellman-Gronwall inequality to (3.43), we obtain that

$$I^{jk}(t) \leq I^{jk}(0)\exp\Big(\int_0^t C_s(\tau)d\tau\Big)+\int_0^t D(\mathbf{t})\exp\big(-\int_0^{\mathbf{t}} C_s(\tau)d\tau\big)d\mathbf{t}$$

for $t \in [0,T]$. By (3.34), we know that $\psi^j(0)-\psi^k(0) \to \psi_0-\psi_0 = 0$ as $j,k\to\infty$, then $I^{jk}(0) = 0$. By the calculation (3.19) of ψ norm at $H^s(\mathbf{R}^N)$, we know $u^j, u^k \to u^*$ as $j,k\to\infty$. That is, $D(t)\to 0$ for $j,k\to\infty$. Therefore, by the convergence of ψ^n at each spaces of $L^2(\Omega)$ and $H^s(\Omega)$ in (3.35) and (3.36), we get $I^{jk}(t) \to 0$ as $j,k\to\infty$ for $0 < s < 1$. That is, there exist $\bar{\psi}\in\mathbf{H}$ and $\bar{\psi}\in\mathbf{V}$ such that

$$\psi^n \to \bar{\psi} \ \text{ in } \ L^2(0,T;L^2(\Omega)), L^2(0,T;H^s(\Omega)).$$

By uniqueness of limit $\bar{\psi} = \psi$, that is $\psi^n \to \psi$ in $L^2(0,T;\mathbf{H})$ and $L^2(0,T;\mathbf{V})$ as $n\to\infty$. By the inclusive of continuous space $C(0,T;\mathbf{H})$, we show that $\psi \in C(0,T;L^2(\Omega))$. This completes the proof. □

4. Quantum Control Outside Surface

In this section, it needs to consider control take place outside (disjoint) the physics domain of matter surface. For control variable $u \in \mathcal{U} = L^2(0,T)$ at the admissible set $\mathcal{U}_{ad} \subset \mathcal{U}$, by virtual of the existence of weak solution theorem, there is a continuous mapping from control space $\mathcal{U}$ to solution space

$$u \to \psi(u) : \mathcal{U} \to W(0,T;s).$$

The pointwise control outside the domain is taken the formula of $u(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0)$ for $\mathbf{x} \in \mathbf{R}^n/\Omega$. The objective function for the measurement of cost is defined by

$$J(u,s) = \frac{1}{2}\|\psi - \psi_d\|^2_{L^2(0,T;\mathbf{H})} + \frac{1}{2}\|u - u_d\|^2_{L^2(0,T)} \text{ for } 0 < s < 1. \tag{4.1}$$

where ψ is the wave function to represent the probability of the motion of particle at the surface, $\psi_d \in L^2(0,T;\mathbf{H})$ is the desired state of ψ at each time for duration $t \in [0,T]$. Certainly, for one particle at matter surface, it is easily to take distributed control at whole control process. That is to say, we can measure the motion of particle at all the time. Assume that $u_d \in L^2(0,T)$ is desired control for each time point t. This means to solve optimal pairing (ψ^*, u^*), (optimal state, optimal control) by the minimization of cost function in optimization at meantime. Particularly, s is the fractional number, as a parameter appeared at state system (2.3), and can be adjust at control process to get different control results if needing to do computational approach. By the existence of parameter s at the operator term, the total system structure can be confined, therefore, it is possible to select appropriate system for getting better control results.

4.1. **Control theory for fractional Schrödinger equation.** To do control at outside domain, it is necessary to clarify its physical meaning of quantum control. At the standing point of physics and chemistry realm, control input can not be directly executed, hence, it need to control the system described by Schrödinger equation indirectly. The external control variable outside the physical surface of existing particle, the control and system is separated by each others. It would be cucriual if symbolic calculation for addition control formulation to system equation, it must occurred "dramatic solution". Therefore, system equation must be changed, that is the idea of fractional Schrödinger equation, which cited the contributed papers at the field of mathematics [2] for PDEs. These theoretical conclusion provide us the ideal tool to investigate the quantum control problems outside its physical domain.

As is well known, quantum optimal control is to solve the following two fundamental problems:

(1) Find an element u^* such that

$$\inf_{u\in\mathcal{U}_{ad}} J(u,s) = J(u^*,s) \ \text{ for } 0<s<1.$$

(2) Characterization of u^*.

Such a u^* is called quantum optimal control for nonlinear fractional Schrödinger control system (2.3) subject to cost function (4.1).

4.2. **Existence of optimal control.**

Theorem 4.1. *Given $\psi_0 \in L^2(\Omega)$, $\psi(\mathbf{x},t) = u(t)\otimes\delta(\mathbf{x}-\mathbf{x}_0)$ for $\mathbf{x}\in\mathbf{R}^2/\Omega$. If $\mathcal{U}_{ad}\subset\mathcal{U}=L^2(0,T)$ is bounded closed convex set, then there is at least one quantum optimal control u^* of fractional Schrödinger system* (2.3) *subject to cost function* (4.1)*.*

Proof. For fractional $0<s<1$, the full proof will be given for quantum system (2.3). Set $J=\inf\limits_{u\in\mathcal{U}_{ad}} J(u,s)$, since $\mathcal{U}_{ad}$ is nonempty, there is a sequence $\{u_n\}$ at $\mathcal{U}_{ad}$ such that

$$\inf_{u\in\mathcal{U}_{ad}} J(u,s) = \lim_{n\to\infty} J(u_n,s) = J.$$

Obviously, $\{J(u_n,s)\}$ is bounded in $\mathbf{R}^+$. Since $\mathcal{U}_{ad}$ is bounded, closed and convex set of $\mathcal{U}$, there is a subsequence $\{u_{n_k}\}$ can be selected from $\{u_n\}$, and there exist $u^*\in\mathcal{U}_{ad}$ such that

$$u_{n_k}\to u^* \ \text{ weakly in } \mathcal{U} \ \text{ as } n_k\to\infty. \tag{4.2}$$

It need mentioned that at the outside domain $\psi(\mathbf{x},t) = u(t)\otimes\delta(\mathbf{x}-\mathbf{x}_0)$ for all $\mathbf{x}\in\mathbf{R}^N/\Omega$, and $\mathbf{x}_0 = (x_1^0, x_2^0)\in\mathbf{R}^N/\Omega$. By the calculation of $\|\psi(0)\|_{L^2(0,T;L_0^s(\bar\Omega))}$ in Corollary 6.1 in Appendix to know that the norm is composed of $\psi(0)$ norm value at domain Ω and the value of norm of $u(0)$. Therefore, the functional

$$\psi(\mathbf{x},t) = \begin{cases} \psi(\mathbf{x},t), & \mathbf{x}\in\Omega,\\ u(t)\otimes\delta(\mathbf{x}-\mathbf{x}^0), & \mathbf{x}\in\mathbf{R}^N/\Omega \end{cases} \tag{4.3}$$

for $\mathbf{x}\in\mathbf{R}^N$ and $t\in[0,T]$ is meaningful and equivalent to each other at outside domain, inside domain, respectively. For $\psi=\psi(u)$, the estimate (3.14) in Theorem 3.6 implies that

$$\|\psi\|^2_{W(0,T;s)} \le C_s\|\psi_0\|^2_{\mathbf{H}}. \tag{4.4}$$

For control u, the boundedness of $\mathcal{U}_{ad}$ and (4.4) to find that $\psi(u)$ is bounded at $W(0,T;s)$. Then, there exist a subsequence $\psi(u_{n_k})$ of $\psi(u_n)$ and a function $\bar\psi$ of $W(0,T;s)$ such that

$$\psi(u_{n_k})\to\bar\psi \ \text{ weakly in } W(0,T;s) \tag{4.5}$$

as $n_k \to \infty$. Denote $\psi^{n_k} = \psi(u_{n_k})$ for simplify. Since the embedding $\mathbf{V} \hookrightarrow \mathbf{H}$ is compact, by the Aubin-Lions-Temam compactness embedding Theorem, we obtain that

$$\psi^{n_k} \to \bar{\psi} \ \text{ strongly in } L^2(0,T;L^2(\Omega))$$

as $n_k \to \infty$. Then (4.4) imply that

$$\begin{aligned} &\psi_t^{n_k} \to \bar{\psi}_t \ \text{ weakly in } L^2(0,T;H^{-s}(\Omega)) \quad \text{for } 0<s<1, \\ &\nabla^s\psi^{n_k} \to \nabla^s\bar{\psi} \text{ weakly in } L^2(0,T;L^2(\Omega)) \quad \text{for } 0<s<1 \end{aligned} \tag{4.6}$$

as $n_k \to \infty$. By the expanded formulation (4.3), we know that

$$\|\psi\|^2_{W^{s,2}(\mathbf{R}^N)} \le \|\psi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\mathbf{R}^N/\Omega))}.$$

Since $\psi = u(t)\otimes\delta(\mathbf{x}-\mathbf{x}^0)$ belong to $L^2(0,T;L^2(\mathbf{R}^N/\Omega))$, the convergence (4.5) to show (4.6) is valid for $\mathbf{R}^N$, that is,

$$\begin{aligned} &\psi_t^{n_k} \to \bar{\psi}_t \ \text{ weakly in } L^2(0,T;H^{-s}(\mathbf{R}^N)) \ \text{ for } 0<s<1, \\ &\nabla^s\psi^{n_k} \to \nabla^s\bar{\psi} \ \text{ weakly in } L^2(0,T;L^2(\mathbf{R}^N)) \ \text{ for } 0<s<1. \end{aligned} \tag{4.7}$$

The definition of weak solutions for ψ^{n_k} has the form of

$$\begin{aligned} &-\int_0^T\int_{\mathbf{R}^N} i\hbar^s\psi^{n_k}\bar{\eta}_t d\mathbf{x}dt \\ &+\int_0^T\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{(\nabla^s\psi^{n_k}(\mathbf{x})-\nabla^s\psi^{n_k}(\mathbf{y}))(\bar{\eta}(\mathbf{x})-\bar{\eta}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}}d\mathbf{x}d\mathbf{y}dt \\ &=\int_0^T\int_{\mathbf{R}^N} V(\mathbf{x},t)\psi^{n_k}\bar{\eta}d\mathbf{x}dt. \end{aligned} \tag{4.8}$$

Therefore, using (4.2) and (4.7), take $n_k \to \infty$ in (4.8), then we obtain

$$\begin{aligned} &\int_0^T\int_{\mathbf{R}^N} -i\hbar^s\bar{\psi}\bar{\eta}_t d\mathbf{x}dt \\ &=\int_0^T\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{(\nabla^s\bar{\psi}(\mathbf{x})-\nabla^s\bar{\psi}(\mathbf{y}))(\bar{\eta}(\mathbf{x})-\bar{\eta}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{N+2s}}d\mathbf{x}d\mathbf{y}dt \\ &\quad+\int_0^T\int_{\mathbf{R}^N} V(\mathbf{x},t)\bar{\psi}\bar{\eta}d\mathbf{x}dt \end{aligned} \tag{4.9}$$

for all $\eta \in C(0,T;\mathbf{V})$. Thus, by the standard manipulation, we have that the limit $\bar{\psi}$ satisfy (4.9) for all $\psi \in \mathbf{V}$ in the sense of $\mathcal{D}'(0,T)$, which is distribution on $(0,T)$. From the uniqueness of weak solution for fractional Schrödinger system (2.3), we confirm that $\bar{\psi} = \psi(u^*)$. It follows from approximate solutions convergenceness that

$$\psi(u_{n_k}) \to \psi(u^*) \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{4.10}$$

$$\psi_d(u_{n_k}) \to \psi_d(u^*) \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \tag{4.11}$$

as $n_k \to \infty$. Since norm $\|\cdot\|_{L^2(0,T;L^2(\Omega)}$ is lower semi-continuous for weak topology of space $\mathbf{H} = L^2(\Omega)$, respectively. (4.10) and (4.11) imply that

$$\liminf_{n_k\to\infty} \|\psi(u_{n_k},t) - \psi_d(t)\|^2_{\mathbf{H}} \geq \|\psi(u^*,t) - \psi_d(t)\|^2_{\mathbf{H}}.$$

Similarly, we have

$$\liminf_{n_k\to\infty} (u_{n_k}, u_{n_k})_{\mathcal{U}} \geq (u^*, u^*)_{\mathcal{U}}.$$

On the other hand, it follows from the weak convergenceness (4.2) that

$$\|u^{n_k} - u_d\|^2_{L^2(0,T)} \geq \|u^* - u_d\|^2_{L^2(0,T)}.$$

From weakly lower semi-continuity of J, we get $J = \liminf_{n_k\to\infty} J(u^{n_k}, s) \geq J(u^*, s)$. Resultantly, $J(u^*, s) = \inf_{u\in\mathcal{U}_{ad}} J(u, s)$. It means that u^* is quantum optimal control subject to cost function (4.1). This completes the proof. □

4.3. **Optimality system.**

Theorem 4.2. *For, $\psi_0 \in L^2(\Omega)$, let $\psi(\mathbf{x},t) = \mathcal{N}_\delta u(t) = u(t) \otimes \delta(\mathbf{x} - x^0)$ for $\mathbf{x} \in \mathbf{R}^N/\Omega$. If $\mathcal{U}_{ad} \subset \mathcal{U} = L^2(0,T)$ is a bounded closed convex set, then quantum optimal control u^* for fractional Schrödinger system* (2.3) *subject to cost function* (4.1) *is characterized by the equations and inequality, called an optimal system (Euler-Lagrange system):*

$$\begin{cases} i\hbar^s\psi_t = \Delta^s\psi + V(\mathbf{x},t)\psi \quad \text{in } \mathbf{R}^N \times [0,T], \\ \psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \quad \text{in } \Omega, \\ \psi(\mathbf{x},t) = u^*(t) \otimes \delta(\mathbf{x} - \mathbf{x}_0) \quad \text{in } \forall\mathbf{x} \in \mathbf{R}^2/\Omega \text{ for } \mathbf{x}_0 \in \mathbf{R}^2/\Omega. \end{cases} \tag{4.12}$$

$$\begin{cases} i\hbar^s p_t = \Delta^s p + V^*(\mathbf{x},t)\psi + (\psi(u^*) - \psi_d) \quad \text{in } \mathbf{R}^N \times [0,T], \\ ip(\mathbf{x},T) = 0, \quad \text{in } \Omega. \end{cases} \tag{4.13}$$

$$(u^*, u - u^*)_{\mathcal{U}} + \int_0^T (\mathcal{N}^*_\delta p(u^*), u - u^*)_{W^{s,2}_0(\bar{\Omega})} dt \geq 0, \ \forall u \in \mathcal{U}_{ad}, \tag{4.14}$$

where $p \in W(0,T;s)$ is a weak solution of adjoint system (4.13)*, which corresponding to ψ in state system* (4.12)*, $V^*, \mathcal{N}^*_\delta$ are the conjugate operators of V, $\mathcal{N}_\delta$, respectively.*

It is well known that (4.14) is necessary optimality condition. If $J(u,s)$ is convex, then (4.14) is sufficient condition. The proof of Theorem 4.2 can be obtained by citing [17].

Particularly, we know that $\mathcal{N}_\delta$ is a mapping $u \to \psi(\mathbf{x},t)$: $L^2(0,T) \to L^2(0,T;L^2(\mathbf{R}^N/\Omega))$ from $t \in [0,T]$ to $(\mathbf{x},t) \in \mathbf{R}^n/\Omega$. At the point $\mathbf{x}^0$, the value of ψ is $\psi(\mathbf{x}^0,t) = u(t)$. That is, for a given t, $\psi(\mathbf{x},t)|_{\mathbf{x}=\mathbf{x}^0} = \mathcal{N}_\delta u(t) =$

$\psi(\mathbf{x}^0,t)$ at point $\mathbf{x}^0$. Then $\mathcal{N}_\delta$ is a value given operator at point $\mathbf{x}_0$. In contrast, $\mathcal{N}_\delta^*$ is a $\mathbf{x}$ value taken operator such that

$$(p(u^*),\mathcal{N}_\delta(u-u^*))_{W^{s,2}(\Omega)} = (\mathcal{N}_\delta^* p(u^*), u-u^*)_{W^{s,2}(\Omega)},$$

where $\mathcal{N}_\delta^*\psi(\mathbf{x},t) = \psi(\mathbf{x}^0,t)$ for $(\mathbf{x},t)\to t$ from $\mathbf{R}^N/\Omega$ to $[0,T]$, it is a continuous mapping of space $L^2(0,T;L^2(\mathbf{R}^N/\Omega))$ to $L^2(0,T)$.

For physical and chemical potential function, consider conjugate functional V^* of V for variables $(\mathbf{x},t)$. By estimate of nonlinear term (3.20), we know that $V\in L^{\frac{2}{s}}(\Omega)$ for $0<s<1$.

(1) We estimate nonlinear coefficient function V^* at adjoint system to ensure the system (4.13) has a weak solution $p(\mathbf{x},t)$. By the state system (4.12), we know that $(\psi(u^*)-\psi_d)\in W^{s,2}(\Omega)$ and $V^*\in L^{\frac{2}{s}}(\Omega)$. Moreover, $L^\infty(\Omega)\subset L^{\frac{2}{s}}(\Omega)\subset L^2(\Omega)$ for $0<s<1$, and

$$\|V^*\|_{L^2(\Omega)} \le \|V^*\|_{L^{\frac{2}{s}}(\Omega)} \le \|V^*\|_{L^\infty(\Omega)}.$$

(2) On the other hands, V^* is the conjugate operator function of V, by $V\in L^{\frac{2}{s}}(\Omega)$, since $p_0=\dfrac{2}{s}, q_0=\dfrac{2}{2-s}$ such that $\dfrac{1}{p_0}+\dfrac{1}{q_0}=1$, by the conjugate space $L^{\frac{2}{2-s}}(\Omega)$ of $L^{\frac{2}{s}}(\Omega)$, we obtain that $V^*\in L^{\frac{2}{2-s}}(\Omega)$. Moreover, $L^2(\Omega)\subset L^{\frac{2}{2-s}}(\Omega)\subset L^1(\Omega)$ for $0<s<1$, and

$$\|V^*\|_{L^1(\Omega)} \le \|V^*\|_{L^{\frac{2}{2-s}}(\Omega)} \le \|V^*\|_{L^2(\Omega)}.$$

Therefore, by (1) and (2), because $\dfrac{2}{2-s}<\dfrac{2}{s}$ for $0<s<1$, it implies that $L^{\frac{2}{s}}(\Omega)\subset L^{\frac{2}{2-s}}(\Omega)$, and

$$\|V^*\|_{L^{\frac{2}{2-s}}(\Omega)} \le \|V^*\|_{L^{\frac{2}{s}}(\Omega)}, \quad \text{for } 0<s<1.$$

We know that for all $s\in(0,1)$, the conjugate space $L^{\frac{2}{2-s}}(\Omega)$ is much large than estimate space $L^{\frac{2}{s}}(\Omega)$. It means that estimate space $L^{p_0}(\Omega)$ is a subspace of conjugate space $L^{q_0}(\Omega)$. Its norm can be used to estimate the potential function V^*. Especially, we have

$$\begin{cases} L^{\frac{2}{s}}(\Omega)\to L^2(\Omega) & \text{as } s\uparrow 1^-, \\ L^{\frac{2}{s}}(\Omega)\to L^\infty(\Omega) & \text{as } s\downarrow 0^+. \end{cases}$$

It is clearly that for fractional Schrödinger equation (2.3), take $s\in(0,1)$, as s goes to 1^-, the system (2.3) approximate to general parabolic differential equation of Δ, then potential function belong to $L^2(\Omega)$ is enough for getting weak solution both state system and adjoint system. As s goes to 0^+, system

(2.3) tended to a degenerated parabolic differential equation, or an ordinary differential equation of t, with Δ^s goes to vanish. To grantee a weak solution, the nonlinear term need much more better continuity, such as $L^\infty(\Omega)$. Notice that if suppose $V \in L^\infty(\Omega)$ at the beginning, other kinds of deduction can be also obtained easily. Certainly, current discussion to domain Ω is also valid for space $\mathbf{R}^N$.

Consequently, it ensure that for $0 < s < 1$, the adjoint system has a weak solution $p(\mathbf{x}, t) \in W^{s,2}(\Omega)$ for terminal value $p(\mathbf{x}, T) = 0$.

4.4. **Bang-Bang principle.** For fractional $0 < s < 1$, consider Bang-Bang principle from necessary optimality condition without running cost in objective function (4.1). For simplifying, take control space $\mathcal{U} = L^2(0, T)$ and take non-empty admissible space as $\mathcal{U}_{ad} = \{u \mid u_a \leq u(t) \leq u_b,\ a.e.\ t \in [0, T],\ u \in L^2(0, T)\}$, with $u_a, u_b \in L^2(0, T)$. For real-valued functional p, the system cost in (4.14) is

$$\int_0^T (\mathcal{N}_\delta^* p(u^*), (u - u^*))_{W^{s,2}(\bar{\Omega})} dt \geq 0, \qquad \forall u \in \mathcal{U}_{ad}. \tag{4.15}$$

By Lebesgue convergence theorem and (4.15), we compute

$$(p(u^*), (u - u^*) \otimes \delta(\mathbf{x} - \mathbf{x}^0))_{L^2(\mathbf{R}^N/\Omega)} \geq 0 \quad a.e.\ (\mathbf{x}, t) \in \mathbf{R}^N/\Omega \times [0, T],$$

for $u \in \mathcal{U}_{ad}$ and $\mathbf{x}_0 \in \mathbf{R}^N/\Omega$. By $0 \leq \delta(\mathbf{x} - \mathbf{x}^0) \leq 1$, we can convert the property of u^* as follows:

$$\begin{aligned} u^*(t) &= u_a, \quad \text{if } p(u^*) > 0,\ a.e.\ t \in [0, T], \\ u^*(t) &= u_b, \quad \text{if } p(u^*) < 0,\ a.e.\ t \in [0, T]. \end{aligned}$$

As is well known, it is Bang-Bang principle of quantum optimal control u^*.

5. Conclusion And Discussion

In this paper, the fractional Schrödinger equation had been considered at Sobolev spaces as the particle at surface taking the external force outside physics domain. The full proof had been provided for both existence of weak solution and existence of quantum optimal control. Definitely, there are numerous unsolved problem will be arising upon this direction. For example, why not take complex Hilbert spaces for mathematical setting? the direct answer is the Schrödinger equation is a complex equation, without loss of generality, taking complex space is correct choice mathematically.

Other problems will be there, such as

(1) whether use another operator to replace the Δ^s as a new operator? Answer is no, it will be not clear than fractional operator.

(2) Beside Δ, can other differential operator to be fractional operator? Answer is yes, cite our paper [18] and book [12] for Cahn-Hillard equation, that is fourth order integer operator for bilinear form $a(\psi, \phi) = (\Delta^2\psi, \Delta^2\phi)$.
(3) Factional operator appeared at other equation, not at Schrödinger equation? Yes, in mathematical field, it had already been taken in other kind of equations (cf. [2]).
(4) The definition of fractional operator by its fractional eigenvalue is good consideration, its also for $s > 1$ no integer number.
(5) Quantum system described by Schrödinger equation can be fractional operator for outside control at domain, its also can be applied the density function theory for fractional operator.
(6) Currently, no indicates the particle type, and surface matter type, this is a theoretical work as a attempt.
(7) The equation (2.3) is a simplification equation, one can take other formulation for the needs of physics, chemistry and other areas (cf. [15, 17]).
(8) Particle-surface reaction has not been considered in the discussion.

Most interesting issues would be the numerical approximate in two dimensions spatial space. The physical experimental for fitting the outside control would also be attractive research in the future.

6. Appendix

Corollary 6.1. *Initial value ψ_0 can be taken in fractional space $L_0^s(\Omega)$.*

Proof. In additional, since $\psi(\mathbf{x}, t) = u(t) \otimes \delta(\mathbf{x} - \mathbf{x}^0)$, the function $\psi = 0$ at the point of $\mathbf{x} \neq \mathbf{x}_0 \in \mathbf{R}^N/\Omega$, it means that it take value at domain Ω and $\mathbf{x}_0$ only. To expand the integration at $L^s(\mathbf{R}^N)$ space similarly

$$
\begin{aligned}
\|\psi\|^2_{L^s(\mathbf{R}^N)} &= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \Big[\int_\Omega\int_\Omega + \int_\Omega\int_{\mathbf{R}^N/\Omega} + \int_{\mathbf{R}^N/\Omega}\int_\Omega + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega}\Big] \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega} \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega} \frac{(u(t)\otimes\delta(\mathbf{x}-\mathbf{x}^0) - u(t)\otimes\delta(\mathbf{y}-\mathbf{x}^0))^2}{|\mathbf{x}-\mathbf{x}^0|^{N+2s}} d\mathbf{x}d\mathbf{y} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \int_{\mathbf{R}^N/\Omega}\int_{\mathbf{R}^N/\Omega} \frac{(\delta(\mathbf{x}-\mathbf{x}^0) - \delta(\mathbf{y}-\mathbf{x}^0))^2}{|\mathbf{x}-\mathbf{y}|^{N+2s}} d\mathbf{x}d\mathbf{y}
\end{aligned}
$$

$$
\begin{aligned}
&= \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \|\delta(\mathbf{x}-\mathbf{x}^0)\|^2_{\mathbf{V}(\mathbf{R^N}/\mathbf{\Omega})} = \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \|1\|^2_{\mathbf{V}(\mathbf{R^N}/\mathbf{\Omega})} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \|1\|^2_{\mathbf{V}(\mathbf{R^N})} = \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \|1\|^2_{\mathbf{V}(\mathbf{\Omega})} \\
&= \|\psi\|^2_{\mathbf{H}} + u(t) \otimes \int_\Omega \int_\Omega 1 dx_1 dx_2 \ (\text{N=2}) \\
&= \|\psi\|^2_{L^2(\Omega)} + \bar{\Omega}\ u(t).
\end{aligned}
$$

The definition of extended norm is equivalent to each other for outside pointwise source. It make sense at outside domain. Aided by the definition of norm at $L^s_0(\bar{\Omega})$ similarly to $W^{s,2}_0(\Omega)$, now let's calculate $\|\psi(0)\|_{\mathbf{H}}$ using the outside value $\psi(\mathbf{x},t) = u(t)\otimes\delta(\mathbf{x}-\mathbf{x}_0)$. Similarly, denote $\bar{\Omega}$ is the measurement of domain Ω. Notice the equivalence of norm at $L^s(\Omega)$ and $L^s_0(\bar{\Omega})$ for pointwise function $\delta(\mathbf{x}-\mathbf{x}^0)$, then we have

$$
\begin{aligned}
&\|\psi(0)\|^2_{L^s_0(\bar{\Omega})} \\
&= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \frac{(\psi(x_1,0)-\psi(x_2,0))^2}{|x_1-x_2|^{N+2s}} dx_1 dx_2 \\
&= \|\psi(0)\|^2_{\mathbf{H}} + \int_{\mathbf{R}^N/\Omega} \int_{\mathbf{R}^N/\Omega} \frac{(u(0)\otimes\delta(x_1-x_1^0) - u(0)\otimes\delta(x_2-x_2^0))^2}{|x_1-x_2|^{N+2s}} dx_1 dx_2 \\
&= \|\psi(0)\|^2_{\mathbf{H}} + u(0) \otimes \int_{\mathbf{R}^N/\Omega} \int_{\mathbf{R}^N/\Omega} \frac{(\delta(x_1-x_1^0) - \delta(x_2-x_2^0))^2}{|x_1-x_2|^{N+2s}} dx_1 dx_2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\|\psi(0)\|^2_{L^2(0,T;L^s_0(\bar{\Omega}))} &\le \|\psi(0)\|^2_{L^2(0,T;\mathbf{H})} + \|u(0)\|^2_{L^2(0,T)} \|\delta(\mathbf{x}-\mathbf{x}_0)\|^2_{L^s_0(\bar{\Omega})} \\
&= \|\psi(0)\|^2_{L^2(0,T;L^s(\Omega))} + \|u(0)\|^2_{L^2(0,T)} \|\delta(\mathbf{x}-\mathbf{x}_0)\|^2_{L^s(\Omega)} \\
&\le \|\psi(0)\|^2_{L^2(0,T;L^s(\Omega))} + \bar{\Omega}^2 \|u(0)\|^2_{L^2(0,T)}.
\end{aligned}
$$

Above discussion show that initial value belong to fractional space $L^s_0(\Omega)$ is a selection if needed. □

Acknowledgments: The author appreciate the 2021 American Chemical Society National Meeting (Fall) for virtual poster [14].

Specially, this paper is dedicated to Supervisor Professor Shin-ichi Nakagiri (Kobe University, Japan) for his birthday at February 28 of year 2023.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. Antil, *External optimal control of nonlocal PDEs*, Inverse Problems **35**(8) (2019), 084003.

[3] R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 5, Evolution Problems I, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[4] A. Fursikov, *Optimal Control of Distributed System: Theory and Applications*, Translations of Mathematical Monographs 187, American Mathematical Society, 2000.

[5] J.L. Lions, *Optimal control of systems governed by partial differential equations*, Grundlehren der Mathematischen Wissenschaften, Vol. 170, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[6] J.L. Lions and E. Magenes, *Hon-Homogeneous Boundary Value Problems and Application I. II.*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[7] S.A. Rice and M. Zhao, *Optical control of molecular dynamics*, Wiley, New York, 2000.

[8] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Second Edition, Appl. Math. Sci., Vol. 68, Springer-Verlag, Berlin-Heidelberg-New York, 1997.

[9] Q.F. Wang, *On trace theorem in Sobolev spaces for initial-boundary control of nonlinear system*, 23rd Chinese Control Conference, (2004), 104–108.

[10] Q.F. Wang, *Optimal control for nonlinear parabolic distributed parameter systems: with numerical analysis*, Lambert Academic Publishing (LAP), Germany, 2011.

[11] Q.F. Wang, *Practical application of optimal control theory: computational approach*, Lambert Academic Publishing (LAP), Germany, 2011.

[12] Q.F. Wang, *Optimal Control for Cahn-Hilliard Issues: basics, concepts and tutorials*, Lambert Academic Publishing(LAP), Germany, 2014.

[13] Q.F. Wang, *Identification in inverse problems: parabolic partial differential equation*, Lambert Academic Publishing(LAP), Germany, 2015.

[14] Q.F. Wang, *Quantum control of particles at matter surface outside the domain*, ACS National Meeting 2021 (Fall), Aug. 22 $\sim$ 26, Virtual Poster (2021).

[15] Q.F. Wang, *Quantum numerical control of nuclei*, Inter. J. Atomic Nuclear Phy., **2021: 6:024**(1) (2021), 1-25, doi: 10.35840/2631-5017/2524.

[16] Q.F. Wang, *Quantum Control Theory and Application*, Lambert Academic Publishing (LAP), Germany, 2021.

[17] Q.F. Wang, *Quantum control of nanoparticles at low temperature*, Cybernetic And Phy., **11**(1) (2022), 37–46.

[18] Q.F. Wang and S. Nakagiri, *Optimal control of distributed parameter system given by Cahn-Hilliard equation*, Nonlinear Funct. Anal. Appl., **19**(1) (2014), 19–33.