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SOME BOUNDS FOR THE ZEROS OF POLYNOMIALS

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Abstract. In this paper, we find a bound for all the zeros of a polynomial in terms of its coefficients similar to the bound given by Montel (1932) and Kuneyida (1916) as an improvement of Cauchy's classical theorem. In fact, we use a generalized version of Hölder's inequality for obtaining various interesting bounds for all the zeros of a polynomial as function of their coefficients.

1. Introduction

One of the classical results on the zeros of a polynomial having complex coefficients is due to Cauchy [5].

Theorem 1.1. *All the zeros of the polynomial $F(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ lie in the circle*

$$|z| \leq 1 + M, \tag{1.1}$$

where $M = \max\{|a_j|\}_{j=0}^{n-1}$.

Montel [9] and Kuneyida [3] have proved the following result by using the Hölder's inequality as an improvement of Theorem 1.1.

Theorem 1.2. *If $F(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ is a polynomial of degree n, then for any p and q such that $p > 1, q > 1$ with $\frac{p+q}{pq} = 1$, all the zeros of $F(z)$ lie in*

$$|z| < (1 + n^{\frac{q}{p}} M^q)^{\frac{1}{q}}, \tag{1.2}$$

where $M = \max\limits_{0 \leq j \leq n-1} |\frac{a_j}{a_n}|$.

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Although various results concerning the bounds for zeros of polynomials are available in literature [6, 11, 12, 13, 14], but the remarkable property of the bound in (1.2) which distinguishes it from other such bounds is its simplicity of computations. Kuniyeda [3] by using Hölder's inequality obtained the following result.

Theorem 1.3. *For any* $p, q \in (1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *the polynomial* $G(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *has all the zeros in the circle*

$$|z| \leq \left(1 + A_p^q\right)^{\frac{1}{q}}, \tag{1.3}$$

where $A_p = \left[\sum\limits_{j=0}^{n-1} |a_j|^p\right]^{\frac{1}{p}}$.

Another improvement in this direction was obtained by Mohammad [7] by using well-known Hölder's inequality as follows.

Theorem 1.4. *If* $F(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ *is a polynomial of degree* n, *then all the zeros of* $F(z)$ *lie in the circle*

$$|z| < \left[\frac{1}{2}\left\{1 + \left(1 + 4\alpha_p^q\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{q}}, \tag{1.4}$$

where $\alpha_p = \left\{\sum_{r=1}^{n} |\frac{a_{n-1}a_{n-r} - a_n a_{n-r-1}}{a_n^2}|^p\right\}^{\frac{1}{p}}$ *with* $p > 1$ *and* $\frac{p+q}{pq} = 1$.

2. MAIN RESULTS

In this paper, we have obtained certain bounds similar to that in Theorem 1.3 for the zeros of polynomials as functions of their coefficients. In fact we use a generalized version of Hölder's inequality [10] to prove the following.

Theorem 2.1. *Let* $P(z) = z^n + \sum\limits_{j=1}^{n} a_{n-j}z^{n-j}$ *be a polynomial with non zero coefficients, then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \max\{L(p, q, t), L^{\frac{1}{n}}(p, q, t)\}, \tag{2.1}$$

where
$L(p, q, t) = \left[\sum\limits_{j=0}^{n-1} |a_j|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum\limits_{j=0}^{n-1} |a_j|^{1-t}\right]^{\frac{1}{q}}, t \in [0, 1]$ *and* $p, q \in (1, \infty)$ *with* $\frac{p+q}{pq} = 1$.

The bound in Theorem 2.1 is sharp and the same can be seen for

$$P(z) = z^n - \frac{1}{n}(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

Remark 2.2. Taking $t = 1$ in (2.1), we get all the zeros of $P(z)$ lie in

$$|z| \le \max\{L_p, L_p^{\frac{1}{n}}\}, \tag{2.2}$$

where $L_p = n^{\frac{1}{q}}\left(\sum_{j=0}^{n-1}|a_j|^p\right)^{\frac{1}{p}}$ & $p, q \in (1, \infty)$ with $\frac{p+q}{pq} = 1$.

The bound in (2.2) is due to Q. G. Mohammad [8].

Remark 2.3. Taking $t = 0$ in (2.1), we get all the zeros of $P(z)$ lie in

$$|z| \le \max\left\{\sum_{j=0}^{n-1}|a_j|, \left(\sum_{j=0}^{n-1}|a_j|\right)^{\frac{1}{n}}\right\}. \tag{2.3}$$

The bound in (2.3) is an improvement of a result due to Montel [9].

Example 2.4. Let $P(z) = z^4 + 0.01z^3 + 0.01z^2 + 0.01z + 1$. Then by Theorem 1.1, all the zeros of $P(z)$ lie in $|z| \le 2$, whereas if we use Theorem 2.1 with $t = 0, p = q = 2$, all the zeros of $P(z)$ lie in $|z| \le 1.03$.

Theorem 2.5. *If* $P(z) = z^n + \sum_{j=1}^{n} a_{n-j}z^{n-j}$ *is a monic polynomial of degree* n *with non zero coefficients, then all the zeros of* $P(z)$ *lie in*

$$|z| < \left[1 + \Big(A(p,t)\Big)^q M^{1-t}\right]^{\frac{1}{q}}, \tag{2.4}$$

where $M = \max_{j\in\{0,1,\dots,n-1\}}|a_j|$, $A(p,t) = \left[\sum_{j=0}^{n-1}|a_j|^{1-t+pt}\right]^{\frac{1}{p}}$, $0 < t \le 1$ *and* $p, q \in (1, \infty)$ *with* $p^{-1} + q^{-1} = 1$.

Remark 2.6. For $t = 1$, (2.4) reduces to a result due to Kuniyeda [3], which in turn for $p = 2 = q$ reduces to a result due to Carmichael [4], Kelleher [2], and Fujiwara [1].

Remark 2.7. Taking $t = \frac{1}{pq}$ in (2.4) and letting $q \to \infty$, we get the result due to Cauchy [5].

Remark 2.8. By taking $t \to 0^+$ in (2.4), the bound therein takes the form

$$|z| \le \left[1 + \left(\sum_{j=0}^{n-1} |a_j|\right)^{\frac{q}{p}} M\right]^{\frac{1}{q}} \tag{2.5}$$
$$< \left[1 + \left(\sum_{j=0}^{n-1} |a_j|\right)^{\frac{q}{p}} M\right]$$
$$\le 1 + M; \quad \text{if } \sum_{j=0}^{n-1} |a_j| \le 1.$$

This shows that the bound in (2.5) is an improvement of the result due to Cauchy [5] in the case when the sum of moduli of non leading coefficients of $P(z)$ in Theorem 2.5 does not exceed 1.

Remark 2.9. Next by letting $q \to \infty$ in (2.5), it follows that all the zeros of the polynomial $P(z)$ in Theorem 2.5 lie in

$$|z| \le \max\left\{\sum_{j=0}^{n-1} |a_j|, 1\right\},$$

which is a bound due to Montel [9].

Example 2.10. Let $P(z) = z^3 + 0.003z^2 + 0.002z + 1$. Then by Theorem 1.1, all the zeros of polynomial $P(z)$ lie in $|z| \le 2$, whereas by using Theorem 2.5 with $t = 0, p = q = 2$, all the zeros of $P(z)$ lie in $|z| < 1.415$.

3. Proof of Theorems

Proof of Theorem 2.1. For a polynomial $P(z) = z^n + \sum_{j=1}^{n} a_{n-j} z^{n-j}$ with non zero coefficients, we have

$$\begin{aligned}
|P(z)| &= |z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0| \\
&= |z^n \left\{1 + a_{n-1}\frac{1}{z} + a_{n-2}\frac{1}{z^2} + ... + a_1\frac{1}{z^{n-1}} + a_0\frac{1}{z^n}\right\}| \\
&\ge |z|^n \left[1 - \left\{|a_{n-1}|\frac{1}{|z|} + |a_{n-2}|\frac{1}{|z|^2} + ... + |a_1|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}\right\}\right] \\
&= |z|^n \left[1 - \sum_{j=1}^{n} \left|\frac{a_{n-j}}{z^j}\right|\right].
\end{aligned}$$

Using the generalized version of Hölder's inequality [10], for every $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}|P(z)| &\geq |z|^n \left[1 - \left[\sum_{j=1}^{n} \left|\frac{a_{n-j}}{z^j}\right|^{1-t} |a_{n-j}|^{pt}\right]^{\frac{1}{p}} \left[\sum_{j=1}^{n} \left|\frac{a_{n-j}}{z^j}\right|^{1-t} \left|\frac{1}{z^j}\right|^{qt}\right]^{\frac{1}{q}}\right] \\ &= |z|^n \left[1 - \left[\sum_{j=1}^{n} \left|\frac{1}{z^j}\right|^{1-t} |a_{n-j}|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum_{j=1}^{n} \left|\frac{1}{z^j}\right|^{1-t+qt} |a_{n-j}|^{1-t}\right]^{\frac{1}{q}}\right],\end{aligned}$$

which further gives

$$P(z) \geq \begin{cases} |z|^n \left[1 - \frac{1}{|z|}\left[\sum\limits_{j=1}^{n} |a_{n-j}|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum\limits_{j=1}^{n} |a_{n-j}|^{1-t}\right]^{\frac{1}{q}}\right]; & \text{if } |z| \geq 1 \\ |z|^n \left[1 - \frac{1}{|z|^n}\left[\sum\limits_{j=1}^{n} |a_{n-j}|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum\limits_{j=1}^{n} |a_{n-j}|^{1-t}\right]^{\frac{1}{q}}\right]; & \text{if } |z| < 1\ . \end{cases}$$

Hence it follows that $|P(z)| > 0$ if

$$|z| > \max\{L(p, q, t), L^{\frac{1}{n}}(p, q, t)\},$$

where

$$L(p, q, t) = \left[\sum_{j=1}^{n} |a_{n-j}|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum_{j=1}^{n} |a_{n-j}|^{1-t}\right]^{\frac{1}{q}}.$$

Hence it follows that all the zeros of $P(z)$ lie in

$$|z| \leq \max\left\{L(p, q, t), L^{\frac{1}{n}}(p, q, t)\right\},$$

where

$$L(p, q, t) = \left[\sum_{j=1}^{n} |a_{n-j}|^{1-t+pt}\right]^{\frac{1}{p}} \left[\sum_{j=1}^{n} |a_{n-j}|^{1-t}\right]^{\frac{1}{q}}.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.5. Let $P(z) = z^n + \sum\limits_{j=1}^{n} a_{n-j} z^{n-j}$ be a monic polynomial of degree n with non zero coefficients. Then we have

$$\begin{aligned}
|P(z)| &= |z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0| \\
&= |z^n \left\{1 + a_{n-1}\frac{1}{z} + a_{n-2}\frac{1}{z^2} + ... + a_1\frac{1}{z^{n-1}} + a_0\frac{1}{z^n}\right\}| \\
&\geq |z|^n \left[1 - \left\{|a_{n-1}|\frac{1}{|z|} + |a_{n-2}|\frac{1}{|z|^2} + ... + |a_1|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}\right\}\right] \\
&= |z|^n\left[1 - \sum_{j=1}^{n}\left|\frac{a_{n-j}}{z^j}\right|\right].
\end{aligned}$$

Using the generalized version of Hölder's inequality [10], for every $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
|P(z)| &\geq |z|^n\left[1 - \sum_{j=1}^{n}\left|\frac{a_{n-j}}{z^j}\right|\right] \\
&= |z|^n\left[1 - \sum_{j=1}^{n}\frac{|a_{n-j}|}{|z^j|^{1-k}}\frac{1}{|z^j|^k}\right] \\
&\geq |z|^n\left[1 - \left[\sum_{j=1}^{n}\left|\frac{a_{n-j}}{z^j}\right|^{1-t}\left(\frac{|a_{n-j}|}{|z^j|^{1-k}}\right)^{pt}\right]^{\frac{1}{p}}\left[\sum_{j=1}^{n}\left|\frac{a_{n-j}}{z^j}\right|^{1-t}\left(\frac{1}{|z^j|^k}\right)^{qt}\right]^{\frac{1}{q}}\right] \\
&= |z|^n\left[1 - \left[\sum_{j=1}^{n}\frac{|a_{n-j}|^{1-t+pt}}{|z^j|^{1-t+(1-k)pt}}\right]^{\frac{1}{p}}\left[\sum_{j=1}^{n}\frac{|a_{n-j}|^{1-t}}{|z^j|^{1-t+kqt}}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Take $k = \frac{q-1+t}{qt}$, so that $1 - t + (1-k)pt = 0$ and $1 - t + kqt = q$. With this choice of k, the above inequality takes the form

$$|P(z)| \geq |z|^n\left[1 - \left[\sum_{j=0}^{n-1}|a_j|^{1-t+pt}\right]^{\frac{1}{p}}\left[\sum_{j=1}^{n}\frac{|a_{n-j}|^{1-t}}{|z^j|^q}\right]^{\frac{1}{q}}\right],$$

which further for $|z| > 1$, $M = \max_{j\in\{0,1,...,n-1\}}|a_j|$ and $A(p,t) = \left[\sum_{j=0}^{n-1}|a_j|^{1-t+pt}\right]^{\frac{1}{p}}$ gives

$$\begin{aligned}
|P(z)| &\geq |z|^n\left[1 - A(p,t)M^{\frac{1-t}{q}}\left[\sum_{j=1}^{n}\left(\frac{1}{|z|^q}\right)^j\right]^{\frac{1}{q}}\right] \\
&> |z|^n\left[1 - A(p,t)M^{\frac{1-t}{q}}\left[\sum_{j=1}^{\infty}\left(\frac{1}{|z|^q}\right)^j\right]^{\frac{1}{q}}\right] \\
&= |z|^n\left[1 - A(p,t)M^{\frac{1-t}{q}}\left(\frac{1}{|z|^q - 1}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This gives $|P(z)| > 0$, if $1 - A(p,t)M^{\frac{1-t}{q}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}} \geq 0$. That is, if

$$|z| \geq \left[1 + \left(A(p,t)\right)^q M^{1-t}\right]^{\frac{1}{q}}.$$

Hence it follows all the zeros of $P(z)$ lie in $|z| < \left[1 + \left(A(p,t)\right)^q M^{1-t}\right]^{\frac{1}{q}}$, which proves Theorem 2.5 completely. □

4. Conclusion

The results obtained in this paper give better bounds for the zeros of polynomials as compared to the results available in literature. These results can be further extended for polynomials with Quaternionic variables and to other fields, hence has very good scope for further research. Hölder's inequality has been used by mathematicians to solve various mathematical problems, the application of generalized form of Holders inequality can serve as a tool for researchers to obtain new results. Besides, the zero bounds of polynomials has applications in various subjects like Algebraic Number Theory, Fourier Analysis, Computer Science, Cryptography etc.

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