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A FIXED POINT THEOREM ON PARTIAL METRIC SPACES SATISFYING AN IMPLICIT RELATION

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Abstract. Popa [14] proved the common fixed point theorem using implicit relations. Saluja [17] proved a fixed point theorem on complete partial metric spaces satisfying an implicit relation. In this paper, we prove a fixed point theorem on complete partial metric space satisfying another implicit relation.

1. Introduction and Preliminaries

Metric spaces have been generalized in various ways. Among them, the notion of a partial metric space was introduced in 1992 by Matthews [7] to model computation over a metric space. His goal was to study the reality of finding closer and closer approximations to a given number and showing that contractive algorithms would serve to find these approximations.

There exist many generalizations of the well-known Banach contraction mapping principle in the literature. In particular, Matthews [7], [8] proved the Banach fixed point theorem in partial metric spaces and after that, fixed point results in partial metric spaces have been studied by many authors([1], [3], [4], [5], [6], [9], [10], [12], [13], [15], [16], [17]).

The study of common fixed point theorems using implicit relations was introduced by Popa [14]. Recently, Saluja [17] proved some fixed point theorems on complete partial metric spaces satisfying implicit relations.

In this paper, we introduce very simple implicit relation which is different with Popa and Saluja and we prove a fixed point theorem for a contraction mapping on complete partial metric space satisfying a new implicit relation. Our results extend and generalize several results from the existing literature.

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Now, we start with the following definition:

Definition 1.1. Let X be a non-empty set. Then a mapping $d : X \times X \longrightarrow [0, \infty)$ is called *a partial metric* if for any $x, y, z \in X$, the following conditions hold:

(pm1) $d(x, x) \leq d(x, y)$,
(pm2) $d(x, y) = d(y, x)$,
(pm3) if $d(x, x) = d(x, y) = d(y, y)$, then $x = y$, and
(pm4) $d(x, z) + d(y, y) \leq d(x, y) + d(y, z)$.

In this case, (X, d) is called *a partial metric space.*

Example 1.2. (1) Let $X = [0, \infty)$ and $d(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, d) is a partial metric space.
(2) Let $X = \{[a, b] \mid a, b \in \mathbb{R} \textit{ with } a \leq b\}$ and $d([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, d) is a partial metric space.

Let (X, d) be a partial metric space. For any $x \in X$ and $\epsilon > 0$, let

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) - d(x, x) < \epsilon\}.$$

Lemma 1.3 ([8])**.** *Let (X, d) be a partial metric space. Then we have the following:*

(1) *$\{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$ is a base for some topology τ_d,*
(2) *(X, τ_d) is a T_0-space, and*
(3) *a sequence $\{x_n\}$ converges to x in (X, τ_d) if and only if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$.*

Let (X, d) be a partial metric space. A sequence $\{x_n\}$ in (X, d) is called *Cauchy* if $\lim_{n,m\to\infty} d(x_m, x_n)$ exists and is finite and (X, d) is called *complete* if every Cauchy sequence $\{x_n\}$ in (X, d) converges to x in (X, τ_d) such that

$$\lim_{n\to\infty} d(x_n, x) = d(x, x) = \lim_{n,m\to\infty} d(x_m, x_n).$$

Lemma 1.4 ([11])**.** *Let (X, d) be a partial metric space. Then a sequence $\{x_n\}$ converges to x in (X, τ_d) with $d(x, x) = 0$ if and only if for any $y \in X$, $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.*

First, the well-known Banach contraction theorem [2] is stated as follows:

Theorem 1.5 ([2])**.** *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a contraction mapping, that is, there exists $\lambda \in [0, 1)$ such that*

$$d(fx, fy) \leq \lambda d(x, y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Matthews [8] proved the following Banach fixed point theorem in partial metric spaces.

Theorem 1.6 ([8])**.** *Let* (X, d) *be a complete partial metric space and let* $f : X \longrightarrow X$ *be a contraction mapping, that is, there exists* $\lambda \in [0, 1)$ *such that*

$$d(fx, fy) \leq \lambda d(x, y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point* $u \in X$ *with* $d(u, u) = 0$.

Now, let Σ be the set of all functions $\beta : [0, \infty) \longrightarrow [0, 1)$ which satisfies

$$\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

Dukic, Kadelburg, and Radenovic [5] proved the following fixed point theorem for Geraghty contractions :

Theorem 1.7 ([5])**.** *Let* (X, d) *be a complete partial metric space and* $f : X \longrightarrow X$ *be a mapping. Suppose that there is a* $\beta \in \Sigma$ *such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y) \tag{1.1}$$

for all $x, y \in X$. *Then* f *has a unique fixed point* $u \in X$ *with* $d(u, u) = 0$.

Altun and Sadarangani [1] proved the following fixed point theorem for Geraghty contractions :

Theorem 1.8 ([1])**.** *Let* (X, d) *be a complete partial metric space. Suppose that* $f : X \longrightarrow X$ *is a mapping such that there is a* $\beta \in \Sigma$ *with*

$$d(fx, fy) \leq \beta(A(x, y)) \max\{d(x, y), d(x, fx), d(y, fy)\}$$

for all $x, y \in X$, *where*

$$A(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(fx, y)]\Big\}.$$

Then f *has a unique fixed point* u *in* X.

Theorem 1.9 ([17])**.** *Let* (X, d) *be a complete partial metric space and* $f : X \longrightarrow X$ *be a mapping satisfying the inequality*

$$d(fx, fy) \leq \psi\Big(d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(fx, y)]\Big) \tag{1.2}$$

for all $x, y \in X$, *where* $\psi : [0, \infty)^3 \longrightarrow [0, \infty)$ *is a real valued continuous and non-decreasing function in the first argument for three variables. For some* $\mu \in [0, 1)$, ψ *satisfies the following conditions:*

(Ir1) *If* $y \le \psi\Big(x, \frac{x+y}{2}, \frac{x+y}{2}\Big)$, *then* $y \le \mu x$.
(Ir2) *If* $y \le \psi(y, 0, y)$, *then* $y = 0$.
(Ir3) *If* $y \le \psi\Big(0, \frac{y}{2}, \frac{y}{2}\Big)$, *then* $y = 0$.

Then f *has a unique fixed point in* X.

2. Main Results

In this section, we introduce a new implicit relation and a fixed point theorem on complete partial metric spaces satisfying a new implicit relation.

Let Φ be the family of real valued continuous non-decreasing functions $\phi : [0, \infty)^4 \longrightarrow [0, \infty)$ such that

$$b < \phi(a, b, b, a) \Rightarrow \phi(a, b, b, a) = a. \tag{2.1}$$

Lemma 2.1. *Let* $\phi \in \Phi$. *Then we have the following:*

(1) *if* $a \le \phi(a, a, a, a)$, *then* $\phi(a, a, a, a) = a$,
(2) *if* $a \le \phi(a, 0, 0, a)$, *then* $\phi(a, 0, 0, a) = a$, *and*
(3) *if* $b \le \phi(a, a, b, \frac{a+b}{2})$, *then* $b \le a$.

Proof. (1) Suppose that $a \le \phi(a, a, a, a)$. If $a < \phi(a, a, a, a)$, then by (2.1), $a < a$ which is a contradition. Hence one has result.

(2) Suppose that $a \le \phi(a, 0, 0, a)$. If $a = 0$, then by (1), $\phi(a, 0, 0, a) = \phi(0, 0, 0, 0) = 0$. If $0 < a$, then $0 < \phi(a, 0, 0, a)$ and by (2.1), $\phi(a, 0, 0, a) = a$.

(3) Suppose that $b \le \phi(a, a, b, \frac{a+b}{2})$. Assume that $a < b$. Then

$$\frac{a+b}{2} < b \le \phi\Big(a, a, b, \frac{a+b}{2}\Big) \le \phi\Big(\frac{a+b}{2}, b, b, \frac{a+b}{2}\Big)$$

and by (2.1), we have $\frac{a+b}{2} < b \le \frac{a+b}{2}$ which is a contradiction. □

Now, we will prove a fixed point theorem on complete partial metric spaces satisfying the implicit relation (2.1).

Theorem 2.2. *Let* (X, d) *be a complete partial metric space and* $f : X \longrightarrow X$ *be a mapping. Suppose that there are* $\beta \in \Sigma$ *and* $\phi_1 \in \Phi$ *such that*

$$d(f(x), f(y)) \le \beta(\phi_1(x, y))\phi_1(x, y) \tag{2.2}$$

for all $x, y \in X$, *where*

$$\phi_1(x, y) = \phi\Big(d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(fx, y)]\Big).$$

Then there is a unique fixed point u *of* f *with* $d(u, u) = 0$.

Proof. Let $x \in X$. For any $n \in \mathbb{N}$, let $f^{n+1}x = ff^n x$, $x_n = f^n x$, and $\alpha_n = d(x_n, x_{n+1})$. If $\alpha_m = 0$ for some $m \in \mathbb{N}$, then $x_m = x_{m+1}$, that is, x_m is a fixed point of f and $d(x_m, x_m) = 0$.

Suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since $0 \le \beta(t) < 1$ for all $t \ge 0$, by (2.2) and (pm4),

$$\begin{aligned}\alpha_{n+1} &\le \beta(\phi_1(x_{n+1}, x_n))\phi\Big(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{1}{2}[\alpha_n + \alpha_{n+1}]\Big)\\ &\le \phi\Big(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{1}{2}[\alpha_n + \alpha_{n+1}]\Big)\end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Hence by Lemma 2.1, $\{\alpha_n\}$ is a bounded below real decreasing sequence. Thus there is a non-negative real number α with $\lim_{n\to\infty} \alpha_n = \alpha$. Since ϕ is continuous, by (2.3), we have

$$\alpha \le \lim_{n\to\infty} \beta(\phi_1(x_{n+1}, x_n))\phi(\alpha, \alpha, \alpha, \alpha) \le \phi(\alpha, \alpha, \alpha, \alpha) \tag{2.4}$$

and by Lemma 2.1, we have

$$\lim_{n\to\infty} \beta(\phi_1(x_{n+1}, x_n))\alpha = \alpha. \tag{2.5}$$

If $\lim_{n\to\infty} \beta(\phi_1(x_{n+1}, x_n)) = 1$, then $\lim_{n\to\infty} \phi_1(x_{n+1}, x_n) = \alpha = 0$ and hence

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.6}$$

because $0 \le \lim_{n\to\infty} \beta(\phi_1(x_{n+1}, x_n)) \le 1$. If $\lim_{n\to\infty} \beta(\phi_1(x_{n+1}, x_n)) < 1$, then by (2.5), we have (2.6).

Now, we will show that $\{x_n\}$ is a Cauchy sequence in (X, d). Enough to show that $\lim_{n,m\to\infty} d(x_m, x_n) = 0$. Suppose that $\lim_{n,m\to\infty} d(x_m, x_n) \ne 0$. Then there is an $\epsilon > 0$ and there are subsequences $\{x_{m(k)}\}$, $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k) > n(k)$,

$$d(x_{m(k)}, x_{n(k)}) \ge \epsilon, \tag{2.7}$$

and

$$d(x_{m(k)-1}, x_{n(k)}) < \epsilon \tag{2.8}$$

for all $k \in \mathbb{N}$. By (2.7) and (2.8), we have

$$\begin{aligned}\epsilon &\le d(x_{m(k)}, x_{n(k)})\\ &\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)-1})\\ &< d(x_{m(k)}, x_{m(k)-1}) + \epsilon\end{aligned} \tag{2.9}$$

for all $k \in \mathbb{N}$. By (2.6) and (2.9), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.10}$$

Since $d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})$, for all $k \in \mathbb{N}$, by (2.6) and (2.10),

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \leq \epsilon. \tag{2.11}$$

and since $d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$ for all $k \in \mathbb{N}$, by (2.6) and (2.7),

$$\epsilon \leq \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}). \tag{2.12}$$

By (2.11) and (2.12), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.13}$$

By (2.8), we have

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k)}) \\
&\leq \beta(\phi_1(x_{m(k)-1}, x_{n(k)-1}))\phi_1(x_{m(k)-1}, x_{n(k)-1}) \\
&\leq \beta(\phi_1(x_{m(k)-1}, x_{n(k)-1}))\phi\Big(d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), \\
&\quad d(x_{n(k)}, x_{n(k)-1}), \frac{1}{2}[\epsilon + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})]\Big)
\end{aligned}$$

and since ϕ is continuous, by (2.6), (2.10), and (2.13),

$$\epsilon \leq \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \lim_{k\to\infty} \beta(\phi_1(x_{m(k)-1}, x_{n(k)-1}))\phi(\epsilon, 0, 0, \epsilon)$$

By Lemma 2.1, we get $\lim_{k\to\infty} \beta(\phi_1(x_{m(k)-1}, x_{n(k)-1})) = 1$.

Since $\beta \in \Sigma$, $\lim_{k\to\infty} \phi_1(x_{m(k)-1}, x_{n(k)-1}) = 0$ and so $\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = 0$ which is a contradiction. Thus $\lim_{n,m\to\infty} d(x_m, x_n) = 0$ and so $\{x_n\}$ is a Cauchy sequence in (X, d).

Since (X, d) is a complete partial metric space, there is an u in X such that $\lim_{n\to\infty} d(x_n, u) = d(u, u) = \lim_{n,m\to\infty} d(x_n, x_m)$ and hence

$$\lim_{n\to\infty} d(x_n, u) = d(u, u) = 0. \tag{2.14}$$

By Lemma 1.4 and (2.14), we have

$$\lim_{n\to\infty} d(x_n, fu) = d(u, fu) = 0 \tag{2.15}$$

and by (pm1) and (2.15), $d(u, u) = d(u, fu) = d(fu, fu)$. By (pm4), $fu = u$ and thus u is a fixed point of f.

To prove the uniqueness of u, let v be another fixed point of f with $d(v,v) = 0$. By (2.2) and Lemma 2.1,

$$\begin{aligned} d(u,v) = d(fu,fv) &\le \beta(\phi_1(u,v))\phi(d(u,v),d(u,u),d(v,v),d(u,v)) \\ &= \beta(\phi(d(u,v),0,0,d(u,v)))\phi(d(u,v),0,0,d(u,v)) \\ &= \beta(d(u,v))d(u,v). \end{aligned}$$

Since $0 \le \beta(d(u,v)) < 1$, $u = v$. □

3. Applications

By Theorem 2.2 which is our main result, we have the following Theorem :

Theorem 3.1. *Let (X,d) be a complete partial metric space and $f : X \longrightarrow X$ be a mapping. Let $\phi \in \Phi$ and $\psi : [0,\infty)^4 \longrightarrow [0,\infty)$ be a mapping satisfying*

$$\psi(a_1,a_2,a_3,a_4) \le \phi(a_1,a_2,a_3,a_4) \tag{3.1}$$

for all $a_1,a_2,a_3,a_4 \in [0,\infty)$. Suppose that there is a $\beta \in \Sigma$ with

$$\begin{aligned} d(f(x),f(y)) &\le \beta\Big(\max\Big\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(fx,y)}{2}\Big\}\Big) \\ &\psi\Big(d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(fx,y)}{2}\Big) \end{aligned}$$

for all $x,y \in X$. Then f has a unique fixed point $u \in X$ with $d(u,u) = 0$.

Let $\phi(a,b,c,d) = \max\{a,b,c,d\}$ and $\psi(a,b,c,d) = a, d, \frac{b+c}{2}$, or $\max\{a, \frac{b+c}{2}, d\}$. Then $\phi \in \Phi$ and ϕ, ψ satisfy (3.1). Then, using Theorem 2.2 and Theorem 3.1, we have the following corollary :

Corollary 3.2. *Let (X,d) be a complete partial metric space and $f : X \longrightarrow X$ be a mapping. Suppose that there is a $\beta \in \Sigma$ such that*

$$\begin{aligned} &d(f(x),f(y)) \\ &\le \beta\Big(\max\Big\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(fx,y)}{2}\Big\}\Big)M(x,y) \end{aligned}$$

for all $x,y \in X$, where

$$\begin{aligned} M(x,y) &= d(x,y), \frac{d(x,fy)+d(x,fy)}{2}, \frac{d(x,fx)+d(y,fy)}{2}, \\ \text{or } &\max\Big\{d(x,y),\frac{d(x,fx)+d(y,fy)}{2},\frac{d(x,fy)+d(fx,y)}{2}\Big\}. \end{aligned}$$

Then f has a unique fixed point $u \in X$ with $d(u,u) = 0$.

Let $\phi(a,b,c,d) = a, d, \frac{b+c}{2}, \max\{a,b,c\}, \max\{a,b,c,d\}, \max\{a, \frac{b+c}{2}\}$,or $\max\{a, \frac{b+c}{2}, d\}$. Then $\phi \in \Phi$. By choosing ψ appropriately for each ϕ, we can get a lot of results, including previous fixed point theorems. For examples, letting $\phi(a,b,c,d) = \psi(a,b,c,d) = a$ and $\beta(t) = \lambda(0 \le \lambda < 1)$ in Theorem 3.1, we have Theorem 1.6 and letting $\phi(a,b,c,d) = \psi(a,b,c,d) = a$ in Theorem 3.1, we have Theorem 1.7 for all $\beta \in \Sigma$. Further, letting $\phi(a,b,c,d) = \max\{a,b,c,d\}$ and $\psi(a,b,c,d) = \max\{a,b,c\}$ in Theoremy 3.1, we have Theorem 1.8 for all $\beta \in \Sigma$.

Now, for some real valued continuous non-decreasing function ψ that does not satisfy the implicity relation of Theorem 1.9, we present an example that can solve the fixed point problem using Theorem 2.2.

Example 3.3. Let $X = \{0,1,2\}$ and $d : X \times X \longrightarrow [0,\infty)$ be a map defined by

$$d(0,0) = d(1,1) = 0,\ d(0,1) = d(1,0) = 1,$$
$$d(0,2) = d(2,0) = \frac{3}{2},\ d(1,2) = d(2,1) = \frac{8}{5},\ d(2,2) = \frac{1}{2}.$$

Then clearly (X,d) is a partial metric space. Now, let $\{x_n\}$ be a Cauchy sequence in (X,d). Then there is a $k \in \mathbb{N}$ such that for any $n,m \ge k$,

$$d(x_n, x_m) < \frac{1}{5}$$

and by the definition of d, for any $n,m \ge k$,

$$x_n = x_m.$$

Hence $\{x_n\}$ converges to x_k in (X,d) and thus (X,d) is a complete partial metric space .

Define a mapping $\psi : [0,\infty)^3 \longrightarrow [0,\infty)$ by $\psi(a,b,c) = \max\{a,b,c\}$ and a mapping $f : X \longrightarrow X$ by

$$f(0) = f(1) = 0, f(2) = 1.$$

Then we have

$$d(fx,fy) \le \psi\Big(d(x,y), \frac{d(x,fx)+d(y,fy)}{2}, \frac{d(x,fy)+d(fx,y)}{2}\Big) \tag{3.2}$$

but ψ does not satisfy (Ir1) in Theorem 1.9 . Hence we can not apply Theorem 1.9 for (3.2). Define a mapping $\phi : [0,\infty)^4 \longrightarrow [0,\infty)$ by $\phi(a,b,c,d) = \max\{a,b,c,d\}$ and define a mapping $\beta : [0,\infty) \longrightarrow [0,1)$ by $\beta(t) = \frac{2}{3}$. Then $\phi \in \Phi$ and by (3.2), we have

$$d(fx, fy) \leq \beta\Big(\max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(fx,y)}{2}\Big\}\Big)$$
$$\max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(fx,y)}{2}\Big\}$$

for all $x, y \in X$, because $\max\Big\{a, \frac{b+c}{2}, d\Big\} \leq \max\{a,b,c,d\}$. By Theorem 2.2 , f has a unique fixed point with $d(u,u) = 0$.

4. Conclusion

In this paper, we introduce a new implicit relation that is different from the existing implicit relation and we prove Theorem 2.2 which is a fixed point theorem for a contraction mapping on complete partial metric space satisfying a new implicit relation. As its applications, using this fixed point theorem, we prove existing fixed-point theorems and finally, for some real valued continuous non-decreasing function ψ that does not satisfy the implicity relation of Theorem 1.9, we present an example that can solve the fixed point problem using Theorem 2.2.

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