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(CO)HOMOLOGY OF A GENERALIZED MATRIX BANACH ALGEBRA

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Abstract. In this paper, we show that bounded Hochschild homology and cohomology of associated matrix Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ to a Morita context $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$ are isomorphic to those of the Banach algebra $\mathfrak{A}$. Consequently, we indicate that the n-amenability and simplicial triviality of $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ are equivalent to the n-amenability and simplicial triviality of $\mathfrak{A}$.

1. Introduction

Topological homology actualized from the questions with respect to the Wedderburn structure of some extensions of Banach algebras by H. Kamowitz [12], who developed the Banach type of Hochschild (co)homology groups. The first results about "the homology groups being trivial in certain situations" were obtained by B. E. Johnson [11] and have been extensively advanced by A. Ya. Helemskii and several members of his academy [10, 18]. In recent years, there has been a particular interest in the computation of (co)homology groups. However, most of the literature has focused on the pure algebraic case; there have also been papers contending with the calculation of the Banach algebra version of these groups, especially for C^*-algebras [10, 19]. We calculate the simplicial (co)homology of associated matrix Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ to a Morita context $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$. The proofs will be connected with the particular case of Morita theory methods for Banach algebras, evolved by Niels Gronbaek [7]. The outline of this paper is briefly presented as follows. In Section 2, we express some essential preliminaries as a fundament for our work. After that, in Section 3, we provide conditions based on an associated matrix Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ to a Morita context $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$, is

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Morita equivalent to $\mathfrak{A}$. In Section 4, we articulate the theorem on Morita invariance of bounded Hochschild (co)homology. We obtain some related results, including the relation between the n-amenability and simplicial triviality of $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ and $\mathfrak{A}$ and we illustrate the applications with two examples.

2. Preliminaries

Let V and W be Banach spaces. The Banach space, which is the completed projective tensor product of V and W, is denoted by $V\hat{\otimes}W$; for $u \in V\hat{\otimes}W$, there are sequences $(v_n) \in V$ and $(w_n) \in W$ such that $\sum_{n=1}^{\infty} \|v_n\| \|w_n\| < \infty$ and $u = \sum_{n=1}^{\infty} v_n \otimes w_n$. The universal property of $V\hat{\otimes}W$ is that, for each bounded bilinear map $\varphi : V \times W \to Z$ into Banach space Z, there is a unique bounded bilinear map $\psi : V\hat{\otimes}W \to Z$ with $\|\varphi\| = \|\psi\|$ and $\psi(v \otimes w) = \varphi(v, w)\,(v \in V, w \in W)$ [10, 17].

Let $\mathfrak{A}$ be a Banach algebra, and let V be a Banach space. V is said to be a Banach left $\mathfrak{A}$-module if V is a left $\mathfrak{A}$-module and also satisfies the axiom: there exists a positive constant K such that $\|av\| \le K \|a\| \|v\|\,(a \in \mathfrak{A}, v \in V)$. A similar definition applies to Banach right $\mathfrak{A}$-module, and a Banach $\mathfrak{A}$-bimodule is an $\mathfrak{A}$-bimodule that is both a Banach left $\mathfrak{A}$-module and a Banach right $\mathfrak{A}$-module. For a Banach algebra $\mathfrak{A}$, we use the notation **$\mathfrak{A}$-mod** (resp. **mod-$\mathfrak{A}$**) for the category of left (resp. right) Banach $\mathfrak{A}$-modules. If $\mathfrak{B}$ is also a Banach algebra, we use the notation **$\mathfrak{A}$-mod-$\mathfrak{B}$** for the category of Banach ($\mathfrak{A}$-$\mathfrak{B}$)-bimodules. A Banach left $\mathfrak{A}$-module V is said to be unit linked (unital) if $\mathfrak{A}$ has a unit element $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}}v = v\,(v \in V)$. There are similar definitions for unit linked Banach right $\mathfrak{A}$-modules, and Banach $\mathfrak{A}$-bimodules [1]. The full subcategory of **$\mathfrak{A}$-mod** consisting of unit linked modules of $\mathfrak{A}$ is denoted by **$\mathfrak{A}$-unmod**. Let $V \in$ **$\mathfrak{A}$-mod**. We write

$$\mathfrak{A} \cdot V = \{a \cdot v : a \in \mathfrak{A}, v \in V\}, \quad \mathfrak{A}V = \mathbf{lin}\ \mathfrak{A} \cdot V.$$

It can be easily checked that $\mathfrak{A}V = V$, when $\mathfrak{A}$ is unital and $V \in$ **$\mathfrak{A}$-unmod**. Clearly, each closed left ideal of $\mathfrak{A}$ is a Banach left $\mathfrak{A}$-module, and each closed right ideal is a Banach right $\mathfrak{A}$-module, again with the product of $\mathfrak{A}$ giving the module multiplications.

Let V and W be Banach spaces and let $L : V \longrightarrow W$ be a bounded linear map. L is called admissible if $\ker L$ is complemented in V and $\mathrm{im}\ L$ is closed and complemented in W.

Let $V \in \mathfrak{A}$**-mod**. Then V is called (left) projective if, whenever $Y, Z \in \mathfrak{A}$**-mod**, $q : Y \to Z$ is both admissible and epimorphism and $\varphi : V \to Z$ is a morphism, then there exists $\tilde{\varphi} : V \to Y$ such that $q \circ \tilde{\varphi} = \varphi$. There is a similar definition for right projectivity. Let $\mathfrak{A}$ be a Banach algebra, let $V \in$ **mod-**$\mathfrak{A}$, and let $W \in \mathfrak{A}$**-mod**. Set N be the closure of

$$\mathbf{lin}\ \{(v \cdot a) \otimes w - v \otimes (a \cdot w)\ : a \in \mathfrak{A}, v \in V, w \in W\}$$

in $V \hat{\otimes} W$, and let $V \hat{\otimes}_{\mathfrak{A}} W = V \hat{\otimes} W / N$. The universal property of $V \hat{\otimes}_{\mathfrak{A}} W$ is that, for each bounded balanced bilinear map $\varphi : V \times W \to Z$ into Banach space Z, there is a unique bounded balanced bilinear map $\psi : V \hat{\otimes}_{\mathfrak{A}} W \to Z$ with $\|\varphi\| = \|\psi\|$ and $\psi(v \otimes_{\mathfrak{A}} w) = \varphi(v, w)\, (v \in V, w \in W)$.

A module $V \in \mathfrak{A}$**-mod** is called (left) flat, if the associated complex

$$0 \longrightarrow X \hat{\otimes}_{\mathfrak{A}} V \longrightarrow Y \hat{\otimes}_{\mathfrak{A}} V \longrightarrow Z \hat{\otimes}_{\mathfrak{A}} V \longrightarrow 0$$

of every admissible complex

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathfrak{A}$**-mod**, is exact. There is a similar definition for right flatness [10].
As is elementary homology, every projective $\mathfrak{A}$**-mod** is flat (also for **mod-**$\mathfrak{A}$ and $\mathfrak{A}$**-mod-**$\mathfrak{A}$) [17, Example 5.3.9].

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and let $R \in \mathfrak{A}$**-mod-**$\mathfrak{B}$ and $S \in \mathfrak{B}$**-mod-**$\mathfrak{A}$. Two bounded balanced bilinear maps $\{,\} : R \times S \to \mathfrak{A}$ and $[,] : S \times R \to \mathfrak{B}$ are called compatible pairings if they implement bounded bimodule homomorphisms $\rho : R \hat{\otimes}_{\mathfrak{B}} S \to \mathfrak{A}, \nu : S \hat{\otimes}_{\mathfrak{A}} R \to \mathfrak{B}$ and

$$\{r, s\}r' = r[s, r'],\ [s, r]s' = s\{r, s'\}; \quad r, r' \in R\ s, s' \in S.$$

This information is collected in a Morita context $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$.
To $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$ is associated a Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ consisting of 2×2 matrices

$$\begin{bmatrix} a & r \\ s & b \end{bmatrix}\ ;\ \ a \in \mathfrak{A}, r \in R, s \in S, b \in \mathfrak{B}$$

with the product defined by means of module multiplication and compatible pairings. The Banach algebra $\mathbf{G} := \mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ is called a generalized matrix Banach algebra [13]. If $S = 0$, then $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ becomes the well-known triangular Banach algebra, $\mathcal{T}(\mathfrak{A}, R, \mathfrak{B})$, that has been studied by Forrest and Marcoux [3, 4]; see also [5, 15]. We denote by ${}_i\mathbf{G}\, (i = 1, 2)$ the rows of $\mathbf{G}$ and by $\mathbf{G}_j\, (j = 1, 2)$ the columns of $\mathbf{G}$. We can identify each ${}_i\mathbf{G}$ ($\mathbf{G}_j$) as a closed right (left) ideal for $\mathbf{G}$ and

$\mathbf{G} = \bigoplus_{i=1,2} {}_i\mathbf{G}$ ($\mathbf{G} = \bigoplus_{j=1,2} \mathbf{G}_j$). When $\mathfrak{A}$ and $\mathfrak{B}$ are unital, let $e_1 = \begin{bmatrix} 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathfrak{B}} \end{bmatrix}$. Clearly, e_1 is the left identity for ${}_1\mathbf{G}$, and the right identity for $\mathbf{G}_1$, also e_2 is the left identity for ${}_2\mathbf{G}$ and the right identity for $\mathbf{G}_2$.

3. Morita Equivalence

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two unital Banach algebras. We call $\mathfrak{A}$ and $\mathfrak{B}$ Morita equivalent if $\mathfrak{A}$**-unmod** and $\mathfrak{B}$**-unmod** are equivalent, i.e., if there are covariant functors

$$\phi : \mathfrak{A}\textbf{-unmod} \longrightarrow \mathfrak{B}\textbf{-unmod}$$
$$\psi : \mathfrak{B}\textbf{-unmod} \longrightarrow \mathfrak{A}\textbf{-unmod}$$

such that $\phi\psi$ and $\psi\phi$ are isomorphic to the identity functors on $\mathfrak{A}$**-unmod** and $\mathfrak{B}$**-unmod**, respectively [7, Definition 2.3]. Now, we state the following theorem, which we use for the characterization of Morita equivalence.

Theorem 3.1 ([7, Corollary 3.4])**.** *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two unital Banach algebras. $\mathfrak{A}$ and $\mathfrak{B}$ are Morita equivalent if there are unit linked modules $V \in \mathfrak{B}$**-mod-**$\mathfrak{A}$ and $W \in \mathfrak{A}$**-mod-**$\mathfrak{B}$ so that $V \hat{\otimes}_{\mathfrak{A}} W \cong \mathfrak{B}$ and $W \hat{\otimes}_{\mathfrak{B}} V \cong \mathfrak{A}$, where the isomorphisms are implemented by bounded bilinear balanced module maps.*

Corollary 3.2. *There is an equivalence between the categories of unit linked bimodules given by $V \in \mathfrak{A}$**-unmod-**$\mathfrak{B}$ and $W \in \mathfrak{B}$**-unmod-**$\mathfrak{A}$, so that the equivalence functor*

$$\mathfrak{B}\textbf{-unmod-}\mathfrak{B} \longrightarrow \mathfrak{A}\textbf{-unmod-}\mathfrak{A}$$

has the form

$$R \longrightarrow V \hat{\otimes}_{\mathfrak{B}} R \hat{\otimes}_{\mathfrak{B}} W.$$

*In particular, V is right flat as a module in **unmod-**$\mathfrak{A}$ and left flat as a module in $\mathfrak{B}$**-unmod** , and W is right flat as a module in $\mathfrak{B}$**-unmod** and left flat as a module in **unmod-**$\mathfrak{A}$.*

Proposition 3.3. *Let $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ be a generalized matrix Banach algebra. Suppose that the pairings both implement epimorphisms. Then the Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$ are Morita equivalent.*

Proof. Let $\rho : R\hat{\otimes}_{\mathfrak{B}}S \to \mathfrak{A}$ and $\nu : S\hat{\otimes}_{\mathfrak{A}}R \to \mathfrak{B}$ be epimorphisms implemented by compatible pairings. Without loss of generality, we suppose that R and S are unit linked modules; otherwise, we replace R and S with the unit linked modules $\mathfrak{A}\hat{\otimes}_{\mathfrak{A}}R\hat{\otimes}_{\mathfrak{B}}\mathfrak{B}$ and $\mathfrak{B}\hat{\otimes}_{\mathfrak{B}}S\hat{\otimes}_{\mathfrak{A}}\mathfrak{A}$.

Note that if that is the case, we have a commutative diagram

$$\begin{array}{ccc} R\hat{\otimes}_{\mathfrak{B}}S\hat{\otimes}_{\mathfrak{A}}R\hat{\otimes}_{\mathfrak{B}}S & \xrightarrow{\ 1\ } & \mathfrak{A}\hat{\otimes}_{\mathfrak{A}}\mathfrak{A} \\ \Big\downarrow{\scriptstyle 2} & & \Big\downarrow{\scriptstyle 3} \\ R\hat{\otimes}_{\mathfrak{B}}\mathfrak{B}\hat{\otimes}_{\mathfrak{B}}S & \xrightarrow{\ 4\ } & \mathfrak{A} \end{array}$$

that 1, 2 and 3 are epimorphism, so is 4. Consequently

$$(\mathfrak{A}\hat{\otimes}_{\mathfrak{A}}R\hat{\otimes}_{\mathfrak{B}}\mathfrak{B})\hat{\otimes}_{\mathfrak{B}}(\mathfrak{B}\hat{\otimes}_{\mathfrak{B}}S\hat{\otimes}_{\mathfrak{A}}\mathfrak{A}) \longrightarrow \mathfrak{A}$$
$$(\mathfrak{B}\hat{\otimes}_{\mathfrak{B}}S\hat{\otimes}_{\mathfrak{A}}\mathfrak{A})\hat{\otimes}_{\mathfrak{A}}(\mathfrak{A}\hat{\otimes}_{\mathfrak{A}}R\hat{\otimes}_{\mathfrak{B}}\mathfrak{B}) \longrightarrow \mathfrak{B},$$

are epimorphisms. We show that ρ is injective. Since ρ is surjective and $\mathfrak{A}$ is unital, there is $\sum_j r_j \otimes_{\mathfrak{B}} s_j \in R\hat{\otimes}_{\mathfrak{B}}S$ that $\sum_j \{r_j, s_j\} = 1_{\mathfrak{A}}$. Suppose $x = \sum_j a_j \otimes_{\mathfrak{B}} b_j \in \ker\rho$, then $\rho(x) = \sum_j \{a_j, b_j\} = 0$. We have

$$\begin{aligned} x = x \cdot 1_{\mathfrak{A}} &= (\sum_i a_i \otimes_{\mathfrak{B}} b_i) \cdot 1_{\mathfrak{A}} = (\sum_i a_i \otimes_{\mathfrak{B}} b_i)(\sum_j \{r_j, s_j\}) \\ &= \sum_i \sum_j a_i \otimes_{\mathfrak{B}} b_i\{r_j, s_j\} = \sum_i \sum_j a_i \otimes_{\mathfrak{B}} [b_i, r_j]s_j \\ &= \sum_i \sum_j a_i[b_i, r_j] \otimes_{\mathfrak{B}} s_j = \sum_i \sum_j \{a_i, b_i\}r_j \otimes_{\mathfrak{B}} s_j = 0. \end{aligned}$$

Therefore ρ is an isomorphism. Similarly, ν is an isomorphism. Hence $\mathfrak{A}$ and $\mathfrak{B}$ are Morita equivalent by implementing modules R and S. □

Remark 3.4. From here onwards, we consider in Morita context $\mathfrak{M}(\mathfrak{A}, R, S, \mathfrak{B}, \{\ \}, [\ ])$, the pairings both implement epimorphisms.

Theorem 3.5 ([16, Theorem 4.4])**.** *If $\mathfrak{A}$ has a bounded left approximate identity and $Y \in \mathfrak{A}$-**mod**, then $\mathfrak{A}\hat{\otimes}_{\mathfrak{A}}Y \longrightarrow Y$ is an isometric isomorphism of modules.*

Notably, for a unital Banach algebra $\mathfrak{A}$ and a unit linked $\mathfrak{A}$-**mod** Y, the canonical map $\pi_Y : \mathfrak{A}\hat{\otimes}_{\mathfrak{A}}Y \longrightarrow Y$ given by $\pi_Y(a\otimes_{\mathfrak{A}}y) = a \cdot y$ is an isometric isomorphism of modules and plays a vital role.

The following is one of the fundamental theorems in this article.

Theorem 3.6. *A generalized matrix Banach algebra* $\mathbf{G} := \mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ *is Morita equivalent to the Banach algebra* $\mathfrak{A}$.

Proof. Let $e = \begin{pmatrix} 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{pmatrix}$. Put $V = e\mathbf{G}$ and $W = \mathbf{G}e$. It is clear that V is the closed right ideal ${}_1\mathbf{G}$ and W is the closed left ideal $\mathbf{G}_1$, and also $e\mathbf{G}e \cong \mathfrak{A}$.
Let $\beta : V \hat{\otimes}_{\mathbf{G}} W \to \mathfrak{A}$. We show that β is surjective. Define $\gamma : \mathfrak{A} \to V \hat{\otimes}_{\mathbf{G}} W$ by $\gamma(ege) = eg \otimes_{\mathbf{G}} e = e \otimes_{\mathbf{G}} ge \in e\mathbf{G} \otimes_{\mathbf{G}} \mathbf{G}e$. Note that if $ege = 0$, then $eg \otimes_{\mathbf{G}} e = eg \otimes_{\mathbf{G}} ee = ege \otimes_{\mathbf{G}} e = 0$, so γ is well-defined. We have $\beta\gamma(ege) = \beta(eg \otimes_{\mathbf{G}} e) = ege$. This shows that β is surjective. Now suppose that $\sum_n eg_n g'_n e = 0$ for $\sum_n \|eg_n\| \|g'_n e\| < \infty$, where $g_n, g'_n \in \mathbf{G}$. Then $\sum_n eg_n \otimes_{\mathbf{G}} g'_n e = \sum_n eg_n g'_n e \otimes_{\mathbf{G}} e = 0$ and β is injective, that proves $V \hat{\otimes}_{\mathbf{G}} W \cong \mathfrak{A}$. Now we illustrate $\alpha : W \hat{\otimes}_{\mathfrak{A}} V \to \mathbf{G}$ is an isomorphism. By Theorem 3.5, we have $\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} R \simeq R$, $S \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \simeq S$ and $\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \simeq \mathfrak{A}$, also by Proposition 3.3, $S \hat{\otimes}_{\mathfrak{A}} R \simeq \mathfrak{B}$. Therefore, by means of matrix multiplication, α is surjective. Consider $K = \ker \alpha$. First, we clarify $e\mathbf{G} \cdot K = 0$. Note that the domain α is a ($\mathbf{G}$-$\mathbf{G}$)-bimodule, so the equation $e\mathbf{G} \cdot K = 0$ makes sense. Let $x = \sum_i g_i e \otimes_{e\mathbf{G}e} eg'_i \in K$. Then $0 = \alpha(x) = \sum_i g_i e g'_i$, and so for any $g \in \mathbf{G}$

$$
\begin{aligned}
eg \cdot x &= eg \sum_i g_i e \otimes_{e\mathbf{G}e} eg'_i \\
&= \sum_i egg_i e \otimes_{e\mathbf{G}e} eg'_i \\
&= \sum_i e(egg_i e) \otimes_{e\mathbf{G}e} eg'_i \\
&= e \otimes_{e\mathbf{G}e} \sum_i egg_i eg'_i = 0.
\end{aligned}
$$

Also $K = \mathbf{G}K = (\mathbf{G}e\mathbf{G}) \cdot K = \mathbf{G}(e\mathbf{G} \cdot K) = 0$, so α is injective. Accordingly $W \hat{\otimes}_{\mathfrak{A}} V \cong \mathbf{G}$. Hence Banach algebras $\mathbf{G}$ and $\mathfrak{A}$ are Morita equivalent.

□

Consider the two implementing modules V and W in the previous theorem. Using Corollary 3.2, we have

Corollary 3.7. V *is right projective (flat) as a module in* ***unmod***-$\mathfrak{A}$ *and left projective (flat) as a module in* $\mathbf{G}$-***unmod*** *and* W *is right projective (flat) as a module in* $\mathbf{G}$-***unmod*** *and left projective (flat) as a module in* ***unmod***-$\mathfrak{A}$.

4. Morita Invariance of Bounded Hochschild (co)homology

In this section, we intend to demonstrate the bounded Hochschild (co)homology of $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ in terms of the (co)homology of the Banach algebra $\mathfrak{A}$.

Let $\mathfrak{A}$ be a Banach algebra and V be a Banach $\mathfrak{A}-$bimodule. For $n \in \mathbb{N}$, let $BL^n(\mathfrak{A}, V)$ denotes the space of all bounded $n-$linear maps from $\mathfrak{A}^n$ to V. The elements of $BL^n(\mathfrak{A}, V)$ are called continuous n-cochains on $\mathfrak{A}$ with coefficients in V. Consider the standard cohomological complex maps $\delta_0, ..., \delta_{n+1} : BL^n(\mathfrak{A}, V) \longrightarrow BL^{n+1}(\mathfrak{A}, V)$ where δ_n is defined by

$$\delta_0 T(a_1, ..., a_{n+1}) = a_1 T(a_2, ..., a_{n+1})$$
$$\delta_i T(a_1, ..., a_{n+1}) = T(a_1, ..., a_i a_{i+1}, ..., a_{n+1}) \ (1 \leq i \leq n)$$
$$\delta_{n+1} T(a_1, ..., a_{n+1}) = T(a_1, ..., a_n) a_{n+1}.$$

Conventionally $BL^0(\mathfrak{A}, V)$ is V and $\delta_0, \delta_1 : BL^0(\mathfrak{A}, V) \longrightarrow BL^1(\mathfrak{A}, V)$ are given by

$$(\delta_0 v)(a) = av, \ \ (\delta_1 v)(a) = va \quad (a \in \mathfrak{A}, v \in V).$$

Then $\delta : BL^n(\mathfrak{A}, V) \longrightarrow BL^{n+1}(\mathfrak{A}, V)$, the coboundary map, is given by $\delta = \sum_{i=0}^{n+1} (-1)^i \delta_i$.

For $n \in \mathbb{N}$, let $BL_n(\mathfrak{A}, V) = V \hat{\otimes} \mathfrak{A}^n$. The elements of $BL_n(\mathfrak{A}, V)$ are called continuous $n-$chains on $\mathfrak{A}$ with coefficients in V. The standard homological complex maps are $d_0, ..., d_{n+1} : BL_{n+1}(\mathfrak{A}, V) \longrightarrow BL_n(\mathfrak{A}, V)$ where

$$d_0(x \otimes a_1 \otimes ... \otimes a_{n+1}) = x \cdot a_1 \otimes ... \otimes a_{n+1}$$
$$d_i(x \otimes a_1 \otimes ... \otimes a_{n+1}) = x \otimes a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_{n+1} \ (1 \leq i \leq n)$$
$$d_{n+1}(x \otimes a_1 \otimes ... \otimes a_{n+1}) = a_{n+1} x \otimes a_1 \otimes ... \otimes a_n.$$

For convention; $BL_0(\mathfrak{A}, V)$ is V and $d_0, d_1 : BL_1(\mathfrak{A}, V) \longrightarrow BL_0(\mathfrak{A}, V)$ are given by

$$d_0(x \otimes a) = xa, \ \ d_1(x \otimes a) = ax \quad (x \in V, a \in \mathfrak{A}).$$

Then $d : BL_{n+1}(\mathfrak{A}, V) \longrightarrow BL_n(\mathfrak{A}, V)$, the boundary map, is given by $d = \sum_{i=0}^{n+1} (-1)^i d_i$.

The Hochschild homology group of $\mathfrak{A}$ with coefficients in V is $\mathcal{H}_n(\mathfrak{A}, V) = \frac{\ker d_{n-1}}{\text{im } d_n}$ $(\mathcal{H}_0(\mathfrak{A}, V) = \frac{V}{\text{im } d_0})$. Also, the Hochschild cohomology group of $\mathfrak{A}$ with coefficients in V is $\mathcal{H}^n(\mathfrak{A}, V) = \frac{\ker \delta_n}{\text{im } \delta_{n-1}}$ $\quad (\mathcal{H}^0(\mathfrak{A}, V) = \ker \delta_0 = \{x \in V : a \cdot x = x \cdot a \quad (a \in \mathfrak{A})\})$. For more information, see [9, 14].

In the remainder of this section, we will examine the connection between the Hochschild (co)homology groups $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ and $\mathfrak{A}$. The following theorem is a fundamental theorem for this purpose, and is a particular situation of what is stated in [6, Theorem 2.4].

Theorem 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital Morita equivalent Banach algebras with implementing modules $V \in \mathfrak{B}$-**mod**-$\mathfrak{A}$ and $W \in \mathfrak{A}$-**mod**-$\mathfrak{B}$. If V is right flat as a module in **mod**-$\mathfrak{A}$ and left flat as a module in $\mathfrak{B}$-**mod** , then there are natural isomorphisms*

$$\mathcal{H}_n(\mathfrak{A}, Y) = \mathcal{H}_n(\mathfrak{B}, V \underset{\mathfrak{A}}{\hat{\otimes}} Y \underset{\mathfrak{A}}{\hat{\otimes}} W),$$
$$\mathcal{H}^n(\mathfrak{A}, Y^*) = \mathcal{H}^n(\mathfrak{B}, (V \underset{\mathfrak{A}}{\hat{\otimes}} Y \underset{\mathfrak{A}}{\hat{\otimes}} W)^*),$$

*for all unit linked modules $Y \in \mathfrak{A}$-**mod**-$\mathfrak{A}$.*

For more information see [8]. By Theorem 4.1, Theorem 3.6 and Corollary 3.7, we have the following

Theorem 4.2. *Let $\mathbf{G} := \mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ be a generalized matrix Banach algebra. Then there are natural isomorphisms of (co)homology functors*

$$\mathcal{H}_n(\mathbf{G}, Y) = \mathcal{H}_n(\mathfrak{A}, V \underset{\mathbf{G}}{\hat{\otimes}} Y \underset{\mathbf{G}}{\hat{\otimes}} W),$$
$$\mathcal{H}^n(\mathbf{G}, Y^*) = \mathcal{H}^n(\mathfrak{A}, (V \underset{\mathbf{G}}{\hat{\otimes}} Y \underset{\mathbf{G}}{\hat{\otimes}} W)^*),$$

*for all unit linked modules $Y \in \mathbf{G}$-**mod**-$\mathbf{G}$. Therefore,*

$$\mathcal{H}_n(\mathbf{G}, \mathbf{G}) \cong \mathcal{H}_n(\mathfrak{A}, \mathfrak{A}),$$
$$\mathcal{H}^n(\mathbf{G}, \mathbf{G}^*) \cong \mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*).$$

To provide more results, we need the following proposition.

Proposition 4.3 ([2, Proposition 2.8.23])**.** *Let $\mathfrak{A}$ be a unital Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. Then $\mathcal{H}^n(\mathfrak{A}, E) \cong \mathcal{H}^n(\mathfrak{A}, e_{\mathfrak{A}} E e_{\mathfrak{A}})$ for all $n \in N$.*

We say that Banach algebra $\mathfrak{A}$ is n-amenable if $\mathcal{H}^n(\mathfrak{A}, Y^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule Y. Especially, $\mathfrak{A}$ is called amenable if and only if $\mathcal{H}^1(\mathfrak{A}, Y^*) = 0$, for all Banach $\mathfrak{A}$-bimodule Y.

A Banach algebra $\mathfrak{A}$ is said to be simplicially trivial ($n \geq 0$) if $\mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*) = 0$. It is called weakly amenable, if $n = 1$.

We apply Theorems 3.6, 4.2, and Proposition 4.3 to obtain the following corollaries.

Corollary 4.4. *A generalized matrix Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ is n-amenable if and only if $\mathfrak{A}$ is n-amenable. In particular, $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ is amenable if and only if $\mathfrak{A}$ is amenable.*

Corollary 4.5. *A generalized matrix Banach algebra $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ is simplicially trivial if and only if $\mathfrak{A}$ is simplicially trivial. In particular, $\mathfrak{G}(\mathfrak{A}, R, S, \mathfrak{B})$ is weakly amenable if and only if the Banach algebra $\mathfrak{A}$ is weakly amenable.*

Now, we describe the results with two examples.

Example 4.6. Let $\mathfrak{A}$ be a unital Banach algebra and let

$$M_k(\mathfrak{A}) = \begin{pmatrix} \mathfrak{A} & M_{1\times(k-1)}(\mathfrak{A}) \\ M_{(k-1)\times 1}(\mathfrak{A}) & M_{(k-1)}(\mathfrak{A}) \end{pmatrix}.$$

Then, by Theorem 3.6, $\mathfrak{A}$ and $M_k(\mathfrak{A})$ are Morita equivalent. Therefore $M_k(\mathfrak{A})$ is n-amenable (resp. simplicially trivial) if and only if $\mathfrak{A}$ is n-amenable (resp. simplicially trivial).

Example 4.7. Let $\mathfrak{A}$ be a unital Banach algebra and let I be a unital closed left ideal. Suppose that I in addition, has a left identity for $\mathfrak{A}$. By Theorem 3.5, $I\hat{\otimes}_I\mathfrak{A} \cong \mathfrak{A}$ and $\mathfrak{A}\hat{\otimes}_{\mathfrak{A}}I \cong I$, so $\mathfrak{A}$ and I are Morita equivalent. Consequently, by Theorem 3.6 $\mathbf{G} = \begin{pmatrix} \mathfrak{A} & I \\ \mathfrak{A} & I \end{pmatrix}$, and $\mathfrak{A}$ are Morita equivalent. So $\mathbf{G}$ is n-amenable (resp. simplicially trivial) if and only if $\mathfrak{A}$ and I are n-amenable (resp. simplicially trivial).

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