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## AN EXPONENTIALLY FITTED METHOD FOR TWO PARAMETER SINGULARLY PERTURBED PARABOLIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. This article devises an exponentially fitted method for the numerical solution of two parameter singularly perturbed parabolic boundary value problems. The proposed scheme is able to resolve the two lateral boundary layers of the solution. Error estimates show that the constructed scheme is parameter-uniformly convergent with a quadratic numerical rate of convergence. Some numerical test examples are taken from recently published articles to confirm the theoretical results and demonstrate a good performance of the current scheme.

## 1. Introduction

Many problems occurring in engineering and computational science are characterized by having solutions that vary unexpectedly in some narrow regions called boundary or interior layers. The boundary layer occurs when the highest order derivative term is multiplied by a small positive number termed singular perturbation parameter, and the interior layer arises when a discontinuity is there in the given data. The problems are generally named singular perturbation problems. Some of such problems are the flow field into two regions, electromagnetic field problem in moving media, chemical reactor theory, lubrication theory, fluid flow through unsaturated porous media, etc. [1, 12, 15, 31]. It is difficult to obtain an accurate solution for such problems by standard numerical methods [11, 13] specifically when the singular perturbation parameter goes to zero. Hence, searching for accurate numerical methods which do not require the introduction of very fine mesh discretization but still able to resolve the singularity of the problems, is the main concern of this study.

Numerical methods that are flexible enough to care of the singularity character of the problem (both time-dependent and steady-state version) have

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been widely studied [3,5–7,29]. By one or another, several authors have constructed these methods through the fitted operator [2, 10, 17, 18, 20], graded mesh [4,30,32], piecewise uniform mesh [11,25], and adaptively generated mesh techniques [7,8]. To ensure that the error in the approximate solution is independent of the value of the perturbation parameter, to develop a stable numerical method, and to obtain a numerical solution that is uniformly convergent, the scholars have used at least one of the aforementioned techniques and differ from each other only in the construction of the scheme.

However, the accuracy of the approximate solutions available in the literature is not more satisfactory, and still needs the development of further numerical methods which are able to yield a more accurate numerical solution. Thus, in this paper, a fitted operator method is devised in which we have replaced the standard finite difference operator with a difference operator which imitates the singularity character of the differential operator. Unlike the previous, to apply this method, we do not need a priori knowledge of the location and width of the boundary layers. Owing to this, a scheme is constructed through the Crank-Nicolson method discretization for the time variable, and finite difference approximation techniques accompanied by an exponentially fitted operator on equally spaced meshes for the spatial variable. It is shown that the scheme is a second-order accurate both in time and space, and the accuracy is analyzed on a uniform mesh by reducing it to a system of ordinary differential equations with respect to the spatial variable. It is proved that the numerical solution obtained by the proposed method converges uniformly to the solution of the continuous problem independent of the singular perturbation parameters.

#### 2. Description of the problem and method

The governing problem we considered for this study is

(1) 
$$L_{\varepsilon,\mu}u := \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu a(x,t) \frac{\partial u}{\partial x} + b(x,t)u = f(x,t), \ (x,t) \in D,$$

on the domain  $D = \Omega \times (0,T]$ ,  $\Omega = (0,1)$  subject to Dirichlet type boundary conditions

(2) 
$$\begin{cases} u(x,0) = s(x), \ x \in \overline{\Omega}, \\ u(0,t) = 0 = u(1,t), \ t \in [0,T], \end{cases}$$

where  $0 < \varepsilon \ll 1$  and  $0 \le \mu \ll 1$  are two small positive parameters. The functions a(x,t), b(x,t), f(x,t) and s(x) are assumed to be sufficiently continuously differentiable and satisfy  $a(x,t) \ge \alpha > 0$ ,  $b(x,t) \ge \beta > 0$ ,  $\forall (x,t) \in \overline{D}$ . Further, we consider the given boundary data are compatible and smooth so that the data match at the two corners (0,0) and (1,0). These restrictions ensure that there exists a unique solution u(x,t) displaying a boundary layer on both the left and right lateral boundaries of the spatial domain  $\overline{\Omega}$  with considerably different thickness depending on the relation between  $\mu^2$  and  $\varepsilon$  [11, 14, 22, 23, 26]. For chosen  $\gamma \approx \min_{(x,t)\in \overline{D}} a(x,t)/b(x,t)$ , the width of the layers can be coupled as follows. If  $\mu^2 \leq \varepsilon \gamma/\alpha$ , boundary layers of equal width are displayed. However, if  $\mu^2 \geq \varepsilon \gamma/\alpha$ , boundary layers of different width are expected.

#### 2.1. Bounds on the solution

In this section, we consider some properties of the continuous problem and bounds on the solution and on its derivatives through maximum principle. These properties are used in the error estimation that has occurred in our numerical approximations to the solution u(x, t) of Eqs. (1) and (2).

**Lemma 2.1.** Let  $u(x,t) \in C^{2,1}(\overline{D})$ . If  $u(x,t) \ge 0$  for all (x,t) on the boundary  $\partial D$  of the domain D, and  $L_{\varepsilon,\mu}u \ge 0$  for all  $(x,t) \in D$ , then  $u(x,t) \ge 0$  for all  $(x,t) \in \overline{D}$ .

*Proof.* It has shown in [11].

**Lemma 2.2.** Let u(x,t) be the solution of Eqs. (1) and (2). Then we have the estimate

$$||u|| \le \beta^{-1} ||f|| + \max |s(x)|,$$

where  $\|\cdot\|$  is the maximum norm.

*Proof.* See [18].

**Lemma 2.3.** The derivatives of the solution u(x,t) satisfy the following condition for all non-negative integers i, j such that  $i + 3j \leq 4$ .

$$\left\|\frac{\partial^{i+j}u}{\partial x^i\partial t^j}\right\|_{\overline{D}} \leq \begin{cases} C\varepsilon^{\frac{-i}{2}}, & \text{if } \mu^2 \leq \gamma\varepsilon/\alpha, \\ C\left(\frac{\mu}{\varepsilon}\right)^i, & \text{if } \mu^2 \geq \gamma\varepsilon/\alpha, \end{cases}$$

where the constant C is independent of  $\varepsilon$  and  $\mu$  and depends only on the bounded derivatives of the coefficients and the source term.

*Proof.* See [11].

# 2.2. Time variable discretization

Let us discretize the time interval [0, T] into M equal number of subintervals with mesh length k = T/M. This gives a time mesh  $D_t^M = \{t_m = (m-1)k, m = 1, 2, \ldots, M+1\}$ . Two level discretization of the time variable in Eqs. (1) and (2) by the Crank-Nicolson method and collecting the (m)th and (m-1)th time levels give a differential equation in the space variable x,

(3) 
$$\begin{cases} u(x,0) = s(x), \\ L^{M}_{\varepsilon,\mu}\tilde{u} \equiv -\varepsilon \tilde{u}_{xx}(x) - \mu a(x)\tilde{u}_{x}(x) + q(x)\tilde{u}(x) = g(x), \\ \tilde{u}(0) = 0 = \tilde{u}(1), \end{cases}$$

where  $q(x) = (b(x, t_m) + \frac{2}{k}), \tilde{u}(x) = u(x, t_m), a(x) = a(x, t_m), p(x) = (b(x, t_m) - \frac{2}{k}), H(x) = f(x, t_m) + f(x, t_{m-1}) \text{ and } g(x) = \varepsilon u_{xx}(x, t_{m-1}) + \mu a(x, t_{m-1}) u_x(x, t_{m-1}) - p(x)u(x, t_{m-1}) + H(x).$ 

Consider the characteristic equation corresponding to Eq. (3) for the purpose of describing the boundary layers in the solution of Eqs. (1) and (2).

(4) 
$$-\varepsilon r^2(x) - \mu a(x)r(x) + q(x) = 0.$$

This defines two continuous functions, which are given by

(5) 
$$\begin{cases} r_1(x) = (-\mu a(x) - \sqrt{(\mu a(x))^2 + 4\varepsilon q(x))}/(2\varepsilon), \\ r_2(x) = (-\mu a(x) + \sqrt{(\mu a(x))^2 + 4\varepsilon q(x))}/(2\varepsilon). \end{cases}$$

Let

$$\eta_1 = -\max_{x\in\overline{\Omega}} r_1(x) \text{ and } \eta_2 = \min_{x\in\overline{\Omega}} r_2(x).$$

Here, we can see that  $\eta_2 \leq \eta_1$  and as it becomes much less, the boundary layer on the left is stronger than on the right lateral boundary.

**Lemma 2.4.** For a fixed number  $0 < \rho < 1$ , a certain order  $\delta$  and the solution  $\tilde{u}(x)$  of Eq. (3), the following holds.

$$\left|\frac{\partial^{i}\tilde{u}}{\partial x^{i}}\right| \leq C\left(1 + \eta_{1}^{i}e^{-\rho\eta_{1}^{i}x} + \eta_{2}^{i}e^{-\rho\eta_{2}^{i}(1-x)}\right) \quad for \ \ 0 \leq i \leq \delta,$$

where C is a constant independent of  $\varepsilon$ ,  $\mu$ .

Proof. See [16,27].

**Lemma 2.5.** Let the error estimate in the temporal direction be denoted by  $TE_t$ . Then, its bound is given by

$$\|TE_t\| \le C(k^2)$$

*Proof.* The proof is given in [17].

## 2.3. Spatial variable discretization

Here, we discretize the domain  $\overline{\Omega} = [0, 1]$  into N equal number of subintervals with mesh length h = 1/N yielding a space mesh  $D_x^N = \{x_1, x_2, \ldots, x_{N+1}\}$ . Now denoting  $U(x_n, t_m) = U_n^m$  and the *i*th derivative  $d^i/dx^i U(x_n, t_m) = (U^{[i]})_n^m$ , at (m)th and (m - 1)th time level, we have the following Taylor's series expansions:

(7)  
$$U_{n+1}^{m} = U_{n}^{m} + h(U')_{n}^{m} + \frac{h^{2}}{2!}(U'')_{n}^{m} + \frac{h^{3}}{3!}(U''')_{n}^{m} + \frac{h^{4}}{4!}(U^{[4]})_{n}^{m} + \frac{h^{5}}{5!}(U^{[5]})_{n}^{m} + \frac{h^{6}}{6!}(U^{[6]})_{n}^{m} + \frac{h^{7}}{7!}(U^{[7]})_{n}^{m} + \frac{h^{8}}{8!}(U^{[8]})_{n}^{m} + O(h^{9}),$$

(8) 
$$U_{n-1}^{m} = U_{n}^{m} - h(U')_{n}^{m} + \frac{h^{2}}{2!}(U'')_{n}^{m} - \frac{h^{3}}{3!}(U''')_{n}^{m} + \frac{h^{4}}{4!}(U^{[4]})_{n}^{m} - \frac{h^{5}}{5!}(U^{[5]})_{n}^{m} + \frac{h^{6}}{6!}(U^{[6]})_{n}^{m} - \frac{h^{7}}{7!}(U^{[7]})_{n}^{m} + \frac{h^{8}}{8!}(U^{[8]})_{n}^{m} - O(h^{9}),$$

(9)  

$$U_{n+1}^{m-1} = U_n^{m-1} + h(U')_n^{m-1} + \frac{h^2}{2!} (U'')_n^{m-1} + \frac{h^3}{3!} (U''')_n^{m-1} + \frac{h^4}{4!} (U^{[4]})_n^{m-1} + \frac{h^5}{5!} (U^{[5]})_n^{m-1} + \frac{h^6}{6!} (U^{[6]})_n^{m-1} + \frac{h^7}{7!} (U^{[7]})_n^{m-1} + \frac{h^8}{8!} (U^{[8]})_n^{m-1} + O(h^9),$$

(10)  
$$U_{n-1}^{m-1} = U_n^{m-1} - h(U')_n^{m-1} + \frac{h^2}{2!} (U'')_n^{m-1} - \frac{h^3}{3!} (U''')_n^{m-1} + \frac{h^4}{4!} (U^{[4]})_n^{m-1} - \frac{h^5}{5!} (U^{[5]})_n^{m-1} + \frac{h^6}{6!} (U^{[6]})_n^{m-1} - \frac{h^7}{7!} (U^{[7]})_n^{m-1} + \frac{h^8}{8!} (U^{[8]})_n^{m-1} - O(h^9).$$

Using Eqs. (7)-(10) and following [28], we get

(11) 
$$U_{n-1}^m - 2U_n^m + U_{n+1}^m = \frac{h^2}{30} [(U'')_{n-1}^m + 28(U'')_n^m + (U'')_{n+1}^m] + \tau_1,$$

and

(12) 
$$U_{n-1}^{m-1} - 2U_n^{m-1} + U_{n+1}^{m-1} = \frac{h^2}{30} [(U'')_{n-1}^{m-1} + 28(U'')_n^{m-1} + (U'')_{n+1}^{m-1}] + \tau_2,$$

where  $\tau_1 = \frac{h^4}{20} (U^{[4]})_n^m - \frac{13h^8}{8!15} (U^{[8]})_n^m + O(h^{10}), \tau_2 = \frac{h^4}{20} (U^{[4]})_n^{m-1} - \frac{13h^8}{8!15} (U^{[8]})_n^{m-1} + O(h^{10}).$  Now adding Eq. (11) and Eq. (12), we arrive at

(13) 
$$U_{n-1}^{m} + U_{n-1}^{m-1} - 2(U_{n}^{m} + U_{n}^{m-1}) + U_{n+1}^{m} + U_{n+1}^{m-1} \\ = \frac{h^{2}}{30} [(U'')_{n-1}^{m} + 28(U'')_{n}^{m} + (U'')_{n+1}^{m}] \\ + \frac{h^{2}}{30} [(U'')_{n-1}^{m-1} + 28(U'')_{n}^{m-1} + (U'')_{n+1}^{m-1}].$$

From Eq. (3), solving for the highest order derivative gives

(14)  

$$(U'')_{n-1}^{m} + (U'')_{n-1}^{m-1}$$

$$= \frac{1}{\varepsilon} [-\mu a_{n-1}^{m} (U')_{n-1}^{m} + q_{n-1}^{m} U_{n-1}^{m} - f_{n-1}^{m}]$$

$$+ \frac{1}{\varepsilon} [-\mu a_{n-1}^{m-1} (U')_{n-1}^{m-1} + p_{n-1}^{m-1} U_{n-1}^{m-1} - f_{n-1}^{m-1}],$$

$$(U'')_{n-1}^{m} + (U'')_{n-1}^{m-1}$$

(15)  
$$(U'')_{n}^{m} + (U'')_{n}^{m-1}$$
$$= \frac{1}{\varepsilon} [-\mu a_{n}^{m} (U')_{n}^{m} + q_{n-1}^{m} U_{n}^{m} - f_{n}^{m}]$$
$$+ \frac{1}{\varepsilon} [-\mu a_{n}^{m-1} (U')_{n}^{m-1} + p_{n}^{m-1} U_{n}^{m-1} - f_{n}^{m-1}],$$

(16) 
$$(U'')_{n+1}^{m} + (U'')_{n+1}^{m-1} = \frac{1}{\varepsilon} [-\mu a_{n+1}^{m} (U')_{n+1}^{m} + q_{n+1}^{m} U_{n+1}^{m} - f_{n+1}^{m}] + \frac{1}{\varepsilon} [-\mu a_{n+1}^{m-1} (U')_{n+1}^{m-1} + p_{n+1}^{m-1} U_{n+1}^{m-1} - f_{n+1}^{m-1}].$$

Substituting Eqs. (14)-(16) into Eq. (13) gives

$$\varepsilon(U_{n-1}^{m} + U_{n-1}^{m-1} - 2(U_{n}^{m} + U_{n}^{m-1}) + U_{n+1}^{m} + U_{n+1}^{m-1})$$

$$= \frac{h^{2}}{30} [-\mu a_{n-1}^{m} (U')_{n-1}^{m} + q_{n-1}^{m} U_{n-1}^{m} - f_{n-1}^{m}]$$

$$+ \frac{28h^{2}}{30} [-\mu a_{n}^{m} (U')_{n}^{m} + q_{n}^{m} U_{n}^{m} - f_{n}^{m}]$$

$$+ \frac{h^{2}}{30} [-\mu a_{n+1}^{m} (U')_{n+1}^{m} + q_{n+1}^{m} U_{n+1}^{m} - f_{n+1}^{m}]$$

$$+ \frac{h^{2}}{30} [-\mu a_{n-1}^{m-1} (U')_{n-1}^{m-1} + p_{n-1}^{m-1} U_{n-1}^{m-1} - f_{n-1}^{m-1}]$$

$$+ \frac{28h^{2}}{30} [-\mu a_{n}^{m-1} (U')_{n}^{m-1} + p_{n}^{m-1} U_{n}^{m-1} - f_{n}^{m-1}]$$

$$+ \frac{h^{2}}{30} [-\mu a_{n+1}^{m-1} (U')_{n+1}^{m-1} + p_{n+1}^{m-1} U_{n+1}^{m-1} - f_{n+1}^{m-1}].$$

But from [28] we have

(18) 
$$\begin{cases} (U')_{n-1}^{m} = \frac{-3U_{n-1}^{m} + 4U_{n}^{m} - U_{n+1}^{m}}{2h} + h(U'')_{n}^{m} + O(h^{2}), \\ (U')_{n}^{m} = \frac{U_{n+1}^{m} - U_{n-1}^{m}}{2h} + O(h^{2}), \\ (U')_{n+1}^{m} = \frac{U_{n-1}^{m} - 4U_{n}^{m} + 3U_{n+1}^{m}}{2h} - h(U'')_{n}^{m} + O(h^{2}), \end{cases}$$

and

(19) 
$$\begin{cases} (U')_{n-1}^{m-1} = \frac{-3U_{n-1}^{m-1} + 4U_n^{m-1} - U_{n+1}^{m-1}}{2h} + h(U'')_n^{m-1} + O(h^2), \\ (U')_n^{m-1} = \frac{U_{n+1}^{m-1} - U_{n-1}^{m-1}}{2h} + O(h^2), \\ (U')_{n+1}^{m-1} = \frac{U_{n-1}^{m-1} - 4U_n^{m-1} + 3U_{n+1}^{m-1}}{2h} - h(U'')_n^{m-1} + O(h^2). \end{cases}$$

Plugging Eqs. (18) and (19) into Eq. (17) and introducing a fitting factor, we obtain

(20) 
$$L_{\varepsilon,\mu}^{N,M}U_n^m = \frac{1}{30}[f_n^m + 28f_n^m + f_n^m] + \frac{1}{30}[f_n^{m-1} + 28f_n^{m-1} + f_n^{m-1}],$$

where

$$L_{\varepsilon,\mu}^{N,M}U_{n}^{m} = -\left(\frac{\varepsilon\sigma_{j}^{m}}{h^{2}} + \frac{\mu a_{n-1}^{m}}{30h} - \frac{\mu a_{n+1}^{m}}{30h}\right)\left(U_{n-1}^{m} - 2U_{n}^{m} + U_{n+1}^{m}\right)$$

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$$\begin{split} &-\frac{\mu}{30h} \{a_{n-1}^m [-3U_{n-1}^m + 4U_n^m - U_{n+1}^m] + 28a_n^m [U_{n+1}^m - U_{n-1}^m]\}\\ &-\frac{\mu}{30h} a_{n+1}^m [U_{n-1}^m - 4U_n^m + 3U_{n+1}^m]\\ &+ \frac{1}{30} \left(q_{n-1}^m U_{n-1}^m + 28q_n^m U_n^m + q_{n+1}^m U_{n+1}^m\right)\\ &\left(\frac{\varepsilon \sigma_j^{m-1}}{h^2} + \frac{\mu a_{n-1}^{m-1}}{30h} - \frac{\mu a_{n+1}^{m-1}}{30h}\right) \left(U_{n-1}^{m-1} - 2U_n^{m-1} + U_{n+1}^{m-1}\right)\\ &- \frac{\mu}{30h} \{a_{n-1}^{m-1} [-3U_{n-1}^{m-1} + 4U_n^{m-1} - U_{n+1}^{m-1}]\\ &+ 28a_n^{m-1} [U_{n+1}^{m-1} - U_{n-1}^{m-1}]\}\\ &- \frac{\mu}{30h} a_{n+1}^{m-1} [U_{n-1}^{m-1} - 4U_n^{m-1} + 3U_{n+1}^{m-1}]\\ &+ \frac{1}{30} \left(p_{n-1}^{m-1} U_{n-1}^{m-1} + 28p_n^{m-1} U_n^{m-1} + p_{n+1}^{m-1} U_{n+1}^{m-1}\right). \end{split}$$

We can write Eq. (20) in the form of a system of equations as (21)  $A^{-}U^{m} + A^{0}U^{m} + A^{+}U^{m} + B^{-}U^{m-1} + B^{0}U^{m-1} + B^{+}U^{m}$ 

(21)  $A_n^- U_{n-1}^m + A_n^0 U_n^m + A_n^+ U_{n+1}^m + B_n^- U_{n-1}^{m-1} + B_n^0 U_n^{m-1} + B_n^+ U_{n+1}^{m-1} = F_n^m$ for  $n = 2, 3, \ldots, N$ ,  $m = 2, 3, \ldots, M$ , and together with the given conditions, it becomes  $(N + 1) \times (N + 1)$  system of equations and sufficient to be solved, where

$$\begin{split} A_n^- &= \frac{-\sigma_j^m \varepsilon}{h^2} + \frac{\mu}{60h} \left( a_{n-1}^m + 28a_n^m + a_{n+1}^m \right) + \frac{1}{30} \left( b_{n-1}^m + 2/k \right), \\ A_n^0 &= \frac{2\sigma_j^m \varepsilon}{h^2} + 14/15 \left( b_n^m + 2/k \right), \\ A_n^+ &= \frac{-\sigma_j^m \varepsilon}{h^2} - \frac{\mu}{60h} \left( a_{n-1}^m + 28a_n^m + a_{n+1}^m \right) + \frac{1}{30} \left( b_{n+1}^m + 2/k \right), \\ B_n^- &= \frac{-\sigma_j^{m-1} \varepsilon}{h^2} + \frac{\mu}{60h} \left( a_{n-1}^{m-1} + 28a_n^{m-1} + a_{n+1}^{m-1} \right) + \frac{1}{30} \left( b_{n-1}^{m-1} - 2/k \right), \\ B_n^0 &= \frac{2\sigma_j^{m-1} \varepsilon}{h^2} + 14/15 \left( b_n^{m-1} - 2/k \right), \\ B_n^+ &= \frac{-\sigma_j^{m-1} \varepsilon}{h^2} - \frac{\mu}{60h} \left( a_{n-1}^{m-1} + 28a_n^{m-1} + a_{n+1}^{m-1} \right) + \frac{1}{30} \left( b_{n+1}^{m-1} - 2/k \right), \\ F_n^m &= \frac{1}{30} \left( f_{n-1}^m + 28f_n^m + f_{n+1}^m + f_{n-1}^{m-1} + 28f_n^{m-1} + f_{n+1}^{m-1} \right), \end{split}$$

 $\sigma_j^m$  and  $\sigma_j^{m-1}$  are the fitting factors in the (m)th and (m-1)th time levels, respectively. Furthermore, multiplying Eq. (21) by  $h^2$ , it is easily observable that for sufficiently small h, the off-diagonal entries of the tridiagonal coefficient matrices are nonzero (the matrices are irreducible) and  $|A_n^0| \ge |A_n^-| + |A_n^+|$ ,  $|B_n^0| \ge |B_n^-| + |B_n^+|$  (the matrices are diagonally dominant). Hence, by [21], they are M-matrices and have an inverse. Therefore, the system of equations can be solved by matrix inverse method.

## 2.4. Determination of the fitting factor

From the theory given in [24], in the (m)th time level, Eq. (3) has asymptotically expanded solution of the form

(22) 
$$U^m(x) = U^m_r(x) + (U^m(0) - U^m_r(0)) \exp(-\mu a^m(0)x/\varepsilon),$$

and

(23) 
$$U^{m}(x) = U_{r}^{m}(x) + (U^{m}(1) - U_{r}^{m}(1)) \exp(\mu a^{m}(1)(1-x)/\varepsilon),$$

in the left and right layer respectively, where  $U_r^m(x)$  is the solution of the reduced problem (when  $\varepsilon = 0$ ):

$$-\mu a^{m}(x)\frac{dU^{m}(x)}{dx} + q^{m}(x)U^{m}(x) = g^{m}(x).$$

Left layer fitting factor. Here, we assign j = 0 and from (22), we can have

(24) 
$$\begin{cases} U_n^m = (U_r)_1^m + (U_1^m - (U_r)_1^m) \exp(-\mu\rho a_1^m n), \\ U_{n-1}^m = (U_r)_1^m + (U_1^m - (U_r)_1^m) \exp(-\mu\rho a_1^m (n-1)), \\ U_{n+1}^m = (U_r)_1^m + (U_1^m - (U_r)_1^m) \exp(-\mu\rho a_1^m (n+1)), \end{cases}$$

and

(25) 
$$\begin{cases} U_n^{m-1} = (U_r)_1^{m-1} + (U_1^{m-1} - (U_r)_1^{m-1}) \exp(-\mu\rho a_1^{m-1}n), \\ U_{n-1}^{m-1} = (U_r)_1^{m-1} + (U_1^{m-1} - (U_r)_1^{m-1}) \exp(-\mu\rho a_1^{m-1}(n-1)), \\ U_{n+1}^{m-1} = (U_r)_1^{m-1} + (U_1^{m-1} - (U_r)_1^{m-1}) \exp(-\mu\rho a_1^{m-1}(n+1)), \end{cases}$$

where  $\rho = h/\varepsilon$ .

Substituting Eqs. (24) and (25) into Eq. (20), multiplying both sides of Eq. (20) by h, restricting the Taylor series expansion of each coefficient to its first term, taking the limit as  $h \to 0$ , and then simplifying successively, we obtain

(26) 
$$\begin{cases} \sigma_0^m = \frac{\rho \mu a_1^m}{2} \coth(\frac{\rho \mu a_1^m}{2}), \\ \sigma_0^{m-1} = \frac{\rho \mu a_1^{m-1}}{2} \coth(\frac{\rho \mu a_1^{m-1}}{2}). \end{cases}$$

Right layer fitting factor. Here, we assign j = 1 and from (23), we arrive at

(27) 
$$\begin{cases} U_n^m = (U_r)_{N+1}^m + (U_{N+1}^m - (U_r)_{N+1}^m) \exp(\mu a_{N+1}^m (1/\varepsilon - \rho n)), \\ U_{n-1}^m = (U_r)_{N+1}^m + (U_{N+1}^m - (U_r)_{N+1}^m) \exp(\mu a_{N+1}^m (1/\varepsilon - \rho (n-1))), \\ U_{n+1}^m = (U_r)_{N+1}^m + (U_{N+1}^m - (U_r)_{N+1}^m) \exp(\mu a_{N+1}^m (1/\varepsilon - \rho (n+1))), \end{cases}$$

and

(28) 
$$\begin{cases} U_n^{m-1} = (U_r)_{N+1}^{m-1} + (U_{N+1}^{m-1} - (U_r)_{N+1}^{m-1}) \exp(-\mu\rho a_{N+1}^{m-1}n), \\ U_{n-1}^{m-1} = (U_r)_{N+1}^{m-1} + (U_{N+1}^{m-1} - (U_r)_{N+1}^{m-1}) \exp(-\mu\rho a_{N+1}^{m-1}(n-1)), \\ U_{n+1}^{m-1} = (U_r)_{N+1}^{m-1} + (U_{N+1}^{m-1} - (U_r)_{N+1}^{m-1}) \exp(-\mu\rho a_{N+1}^{m-1}(n+1)). \end{cases}$$

Following the same fashion we did in the left layer, we get

(29) 
$$\begin{cases} \sigma_1^m = \frac{\rho \mu a_{N+1}^m}{2} \coth(\frac{\rho \mu a_{N+1}^m}{2}), \\ \sigma_1^{m-1} = \frac{\rho \mu a_{N+1}^{m-1}}{2} \coth(\frac{\rho \mu a_{N+1}^{m-1}}{2}). \end{cases}$$

#### 3. Convergence and stability analysis

In this section, the stability and convergence analysis is shown for the scheme (20).

**Lemma 3.1** (Discrete maximum principle). Assume the discrete function  $\Phi_n^m$  satisfies  $\Phi_1^m \ge 0$ ,  $\Phi_{N+1}^m \ge 0$ , and  $L_{\varepsilon,\mu}^{N,M} \Phi_n^m \ge 0$  on  $D_x^N \times D_t^M$ . Then  $\Phi_n^m \ge 0$  at each point of  $\overline{D}_x^N \times \overline{D}_t^M$ .

*Proof.* To follow the proof by contradiction, let there exist a point  $(\iota, m)$  where  $\iota \in \{1, 2, ..., N + 1\}$  such that

$$\Phi^m_\iota = \min_{1 \le n \le N+1} \Phi^m_n,$$

and suppose that  $\Phi_{\iota}^m < 0$ . Then we have  $\iota \neq 1, N + 1$ . But by using the assumptions  $a(x,t) \geq \alpha$  and  $b(x,t) \geq \beta$ , and the series representation  $x \coth x = 1 + x^2/3 + O(x^4)$  for the expressions in Eqs. (26) and (29) into Eq. (21) we arrive at

$$L_{\varepsilon,\mu}^{N,M}\Phi_{\iota}^{m} < 0.$$

This contradicts the assumption  $L_{\varepsilon,\mu}^{N,M}\Phi_n^m \ge 0$  on  $D_x^N \times D_t^M$ , and then it follows that  $\Phi_n^m \ge 0$  at each point of  $\overline{D}_x^N \times \overline{D}_t^M$ .

This guarantees for the existence of unique discrete solution. The uniform stability of the discrete solution is discussed as follows.

**Lemma 3.2.** The solution  $U_n^m$  of the discrete scheme (20) satisfies the following bound

(30) 
$$\|U_n^m\| \le \frac{\|L_{\varepsilon,\mu}^{N,M} U_n^m\|}{q^*} + \max\left\{\|U_1^m\|, \|U_{N+1}^m\|\right\},$$

where  $Q(x,t) = q(x,t) + p(x,t) \ge Q^* > 0.$ 

*Proof.* Let  $\Theta = (\|L_{\varepsilon,\mu}^{N,M}U_n^m\|)/Q^* + \max\{\|U_1^m\|, \|U_{N+1}^m\|\}$  and define barrier functions as

$$\left(\vartheta^{\pm}\right)_{n}^{m} = \Theta \pm U_{n}^{m}.$$

The values of these barrier functions on the boundary points are

$$\left(\vartheta^{\pm}\right)_{1}^{m} = \Theta \pm U_{1}^{m} = \frac{\|L_{\varepsilon,\mu}^{N,M}U_{n}^{m}\|}{Q^{*}} + \max\left\{\|U_{1}^{m}\|, \|U_{N+1}^{m}\|\right\} \pm U_{1}^{m} \ge 0,$$
$$\left(\vartheta^{\pm}\right)_{N+1}^{m} = \Theta \pm U_{N+1}^{m} = \frac{\|L_{\varepsilon,\mu}^{N,M}U_{n}^{m}\|}{Q^{*}} + \max\left\{\|U_{1}^{m}\|, \|U_{N+1}^{m}\|\right\} \pm U_{N+1}^{m} \ge 0,$$

and on the discretized domain is

$$\begin{split} &L_{\varepsilon,\mu}^{N,M}\left(\boldsymbol{\vartheta}^{\pm}\right)_{n}^{m} \\ = &- \left(\frac{\varepsilon\sigma_{j}^{m}}{h^{2}} + \frac{\mu a_{n-1}^{m}}{30h} - \frac{\mu a_{n+1}^{m}}{30h}\right) \left(\boldsymbol{\Theta} \pm U_{n-1}^{m} - 2(\boldsymbol{\Theta} \pm U_{n}^{m}) + \boldsymbol{\Theta} \pm U_{n+1}^{m}\right) \\ &- \frac{\mu}{30h} \{a_{n-1}^{m}[-3(\boldsymbol{\Theta} \pm U_{n-1}^{m}) + 4(\boldsymbol{\Theta} \pm U_{n}^{m}) - (\boldsymbol{\Theta} \pm U_{n+1}^{m})] \\ &+ 28a_{n}^{m}[\boldsymbol{\Theta} \pm U_{n+1}^{m} - (\boldsymbol{\Theta} \pm U_{n-1}^{m})]\} \\ &- \frac{\mu}{30h}a_{n+1}^{m}[\boldsymbol{\Theta} \pm U_{n-1}^{m} - 4(\boldsymbol{\Theta} \pm U_{n}^{m}) + 3(\boldsymbol{\Theta} \pm U_{n+1}^{m})] \\ &+ \frac{1}{30}\left(q_{n-1}^{m}(\boldsymbol{\Theta} \pm U_{n-1}^{m}) + 28q_{n}^{m}(\boldsymbol{\Theta} \pm U_{n}^{m}) + q_{n+1}^{m}(\boldsymbol{\Theta} \pm U_{n+1}^{m})\right) \\ &- \left(\frac{\varepsilon\sigma_{j}^{m-1}}{h^{2}} + \frac{\mu a_{n-1}^{m-1}}{30h} - \frac{\mu a_{n+1}^{m-1}}{30h}\right) \left(\boldsymbol{\Theta} \pm U_{n-1}^{m-1} - 2(\boldsymbol{\Theta} \pm U_{n}^{m-1}) + \boldsymbol{\Theta} \pm U_{n+1}^{m-1}\right) \\ &- \frac{\mu}{30h} \{a_{n-1}^{m-1}[-3(\boldsymbol{\Theta} \pm U_{n-1}^{m-1}) + 4(\boldsymbol{\Theta} \pm U_{n}^{m-1}) - (\boldsymbol{\Theta} \pm U_{n+1}^{m-1})] \\ &+ 28a_{n}^{m-1}[\boldsymbol{\Theta} \pm U_{n+1}^{m-1} - (\boldsymbol{\Theta} \pm U_{n-1}^{m-1})]\} \\ &- \frac{\mu}{30h}a_{n+1}^{m-1}[\boldsymbol{\Theta} \pm U_{n-1}^{m-1} - 4(\boldsymbol{\Theta} \pm U_{n}^{m-1}) + 3(\boldsymbol{\Theta} \pm U_{n+1}^{m-1})] \\ &+ \frac{1}{30}\left(p_{n-1}^{m-1}(\boldsymbol{\Theta} \pm U_{n-1}^{m-1}) + 28p_{n}^{m-1}(\boldsymbol{\Theta} \pm U_{n-1}^{m-1}) + p_{n+1}^{m-1}(\boldsymbol{\Theta} \pm U_{n+1}^{m-1})\right) \\ &= \frac{1}{30}\left(q_{n-1}^{m} + 28q_{n}^{m} + q_{n+1}^{m} + p_{n-1}^{m-1} + 28f_{n}^{m-1} + f_{n+1}^{m-1}\right) \geq 0. \end{split}$$

Then by applying the discrete maximum principle given in Lemma 3.1, the required bound can easily be achieved.  $\hfill \Box$ 

As a result, the method is uniformly stable in the maximum norm. To establish a parameter-uniform convergence of the discrete scheme (21) let the truncation error in the spatial variable, fixing time, be given by

$$TE_{x} = A_{n}^{-}U_{n-1}^{m} + A_{n}^{0}U_{n}^{m} + A_{n}^{+}U_{n+1}^{m} + B_{n}^{-}U_{n-1}^{m-1} + B_{n}^{0}U_{n}^{m-1} + B_{n}^{+}U_{n+1}^{m-1} - F_{n}^{m} = \left(A_{n}^{-}U_{n-1}^{m} + A_{n}^{0}U_{n}^{m} + A_{n}^{+}U_{n+1}^{m}\right) - \frac{1}{30}\left(f_{n-1}^{m} + 28f_{n}^{m} + f_{n+1}^{m}\right) + \left(B_{n}^{-}U_{n-1}^{m-1} + B_{n}^{0}U_{n}^{m-1} + B_{n}^{+}U_{n+1}^{m-1}\right) - \frac{1}{30}\left(f_{n-1}^{m-1} + 28f_{n}^{m-1} + f_{n+1}^{m-1}\right).$$

The Taylor series expansion of each term in space up to third order derivative and collecting terms with  $U_n^m$ ,  $(U')_n^m$ ,  $(U'')_n^m$ ,  $(U''')_n^m$ ,  $U_n^{m-1}$ ,  $(U')_n^{m-1}$ ,

$$(U'')_{n}^{m-1} \text{ and } (U''')_{n}^{m-1} \text{ give}$$

$$TE_{x} = \zeta_{0,n}^{m} U_{n}^{m} + \zeta_{1,n}^{m} (U')_{n}^{m} + \zeta_{2,n}^{m} (U'')_{n}^{m} + \zeta_{3,n}^{m} (U''')_{n}^{m}$$

$$(32) + \zeta_{0,n}^{m-1} U_{n}^{m-1} + \zeta_{1,n}^{m-1} (U')_{n}^{m-1} + \zeta_{2,n}^{m-1} (U'')_{n}^{m-1}$$

$$+ \zeta_{3,n}^{m-1} (U''')_{n}^{m-1},$$

where

$$\begin{split} \zeta_{0,n}^{m} &= A_{n}^{-} + A_{n}^{0} + A_{n}^{+} - \frac{1}{30} \left( q_{n-1}^{m} + 28q_{n}^{m} + q_{n+1}^{m} \right), \\ \zeta_{1,n}^{m} &= -hA_{n}^{-} + hA_{n}^{+} + \frac{\mu}{30} \left( a_{n-1}^{m} + 28a_{n}^{m} + a_{n+1}^{m} \right) + \frac{h}{30} \left( q_{n-1}^{m} - q_{n+1}^{m} \right), \\ \zeta_{2,n}^{m} &= \frac{h^{2}}{2} \left( A_{n}^{-} + A_{n}^{+} \right) - \frac{h\mu}{30} \left( a_{n-1}^{m} - a_{n+1}^{m} \right) - \frac{h^{2}}{60} \left( q_{n-1}^{m} - q_{n+1}^{m} \right) + \sigma_{j}^{m} \varepsilon, \\ \zeta_{3,n}^{m} &= -\frac{h^{3}}{2} \left( A_{n}^{-} - A_{n}^{+} \right) + \frac{h^{2}\mu}{60} \left( a_{n-1}^{m} + a_{n+1}^{m} \right) + \frac{h^{3}}{180} \left( q_{n-1}^{m} - q_{n+1}^{m} \right), \\ \zeta_{0,n}^{m-1} &= B_{n}^{-} + B_{n}^{0} + B_{n}^{+} - \frac{1}{30} \left( p_{n-1}^{m-1} + 28p_{n}^{m-1} + p_{n+1}^{m-1} \right), \\ \zeta_{1,n}^{m-1} &= -hB_{n}^{-} + hB_{n}^{+} + \frac{\mu}{30} \left( a_{n-1}^{m-1} + 28a_{n}^{m-1} + a_{n+1}^{m-1} \right) + \frac{h}{30} \left( p_{n-1}^{m-1} - p_{n+1}^{m-1} \right), \\ \zeta_{2,n}^{m-1} &= \frac{h^{2}}{2} \left( B_{n}^{-} + B_{n}^{+} \right) - \frac{h\mu}{30} \left( a_{n-1}^{m-1} - a_{n+1}^{m-1} \right) - \frac{h^{2}}{60} \left( p_{n-1}^{m-1} - p_{n+1}^{m-1} \right) + \sigma_{j}^{m-1} \varepsilon, \\ \zeta_{3,n}^{m-1} &= -\frac{h^{3}}{2} \left( B_{n}^{-} - B_{n}^{+} \right) + \frac{h^{2}\mu}{60} \left( a_{n-1}^{m-1} + a_{n+1}^{m-1} \right) + \frac{h^{3}}{180} \left( p_{n-1}^{m-1} - p_{n+1}^{m-1} \right). \end{split}$$

Using the coefficients in Eq. (21) and restricting the expansion of each term to its first term yield

(33) 
$$\begin{cases} \zeta_{0,n}^{m} = \zeta_{1,n}^{m} = \zeta_{2,n}^{m} = \zeta_{0,n}^{m-1} = \zeta_{1,n}^{m-1} = \zeta_{2,n}^{m-1} = 0, \\ \zeta_{3,n}^{m} = \frac{h^{2}\mu}{90} \left( a_{n-1}^{m} - 14a_{n}^{m} + a_{n+1}^{m} \right), \\ \zeta_{3,n}^{m-1} = \frac{h^{2}\mu}{90} \left( a_{n-1}^{m-1} - 14a_{n}^{m-1} + a_{n+1}^{m-1} \right). \end{cases}$$

Using the derivative bound given in Lemma 2.3, this results

$$||TE_x|| \le C(h^2)$$

Therefore, the presented method is with second-order accuracy in the spatial direction. We can also realize that, as the step sizes tend to zero, the errors in Eqs. (6) and (34) tend to zero. This shows the constructed scheme is consistent. Moreover, it is also stable as the solution and its derivatives are bounded, and the error is estimated. Hence, the proposed scheme is convergent. But the parameter uniform convergence of the proposed scheme is shown by the following theorem:

**Theorem 3.3.** Let  $u_n^m$  and  $U_n^m$  be solutions of Eqs. (1) and (20), respectively, at the node  $(x_n, t_m)$ . Then the proposed scheme satisfies the following error

*estimate*:

(35) 
$$\sup_{0 < \varepsilon, \mu} \left\| u_n^m - U_n^m \right\| \le C \left( h^2 + k^2 \right).$$

*Proof.* The proof immediate from Lemma 2.5 and Eq. (34).

TABLE 1. Comparison of the maximum point-wise error and rate of convergence for Example 1 for  $\mu = 10^{-6}$  and different values of  $\varepsilon$ .

$\varepsilon\downarrow$	N = 32	N = 64	N = 128	N = 256	N = 512	
	M = 10	M = 20	M = 40	M = 80	M = 160	
Current result						
$10^{-2}$	2.9752e-04	7.4609e-05	1.8668e-05	4.6682e-06	1.1672e-06	
	1.9956	1.9988	1.9996	1.9998		
$10^{-4}$	2.2974e-04	5.7610e-05	1.4413e-05	3.6039e-06	9.0101e-07	
	1.9956	1.9989	1.9997	1.9999		
$10^{-6}$	2.2902e-04	5.7428e-05	1.4367 e-05	3.5925e-06	8.9816e-07	
	1.9956	1.9990	1.9997	1.9999		
$10^{-8}$	2.2896e-04	5.7406e-05	1.4361e-05	3.5916e-06	8.9802e-07	
	1.9958	1.9990	1.9995	1.9998		
$10^{-10}$	2.2896e-04	5.7395e-05	1.4353e-05	3.5882e-06	$8.9621 e{-} 07$	
	1.9961	1.9996	2.0000	2.0014		
$10^{-12}$	2.2896e-04	5.7395e-05	1.4353e-05	3.5882e-06	$8.9621 e{-} 07$	
	1.9961	1.9996	2.0000	2.0014		
Result in [30]						
$10^{-2}$	8.6053e-03	4.4951e-03	2.2857e-03	1.1438e-03	5.7093 e- 04	
	0.937	0.976	0.999	1.002		
$10^{-4}$	8.6095e-03	4.4529e-03	2.2631e-03	1.1406e-03	5.7256e-04	
	0.951	0.976	0.988	0.994		
$10^{-6}$	8.6014e-03	4.4513e-03	2.2628e-03	1.1406e-03	5.7356e-04	
	0.951	0.976	0.988	0.994		
$10^{-8}$	8.6006e-03	4.4512e-03	2.2628e-03	1.1406e-03	5.7356e-04	
	0.951	0.976	0.988	0.994		
$10^{-10}$	8.6006e-03	4.4512e-03	2.2628e-03	1.1406e-03	5.7356e-04	
	0.951	0.976	0.988	0.994		
$10^{-12}$	8.6006e-03	4.4512e-03	2.2628e-03	1.1406e-03	5.7356e-04	
	0.951	0.976	0.988	0.994		

## 4. Numerical examples and discussion

To illustrate the accuracy and efficiency of the proposed method and verify the theoretical outcomes experimentally, we have solved the following commonly used examples.

TABLE 2. Comparison of the maximum point-wise error and rate of convergence for Example 2 for  $\mu = 10^{-6}$  and different values of  $\varepsilon$ .

$\varepsilon\downarrow$	N = 32	N = 64	N = 128	N = 256	N = 512	
	M = 10	M = 20	M = 40	M = 80	M = 160	
Current result						
$10^{-2}$	3.8115e-03	1.0712e-03	2.8461e-04	7.3391e-05	1.8637e-05	
	1.8311	1.9122	1.9553	1.9774		
$10^{-4}$	3.9321e-03	1.1112e-03	2.9595e-04	7.6409e-05	1.9414e-05	
	1.8232	1.9087	1.9535	1.9766		
$10^{-6}$	3.9336e-03	1.1117e-03	2.9608e-04	7.6445 e-05	1.9423e-05	
	1.8231	1.9087	1.9535	1.9767		
$10^{-8}$	3.9334e-03	1.1116e-03	2.9603e-04	7.6430e-05	1.9419e-05	
	1.8231	1.9088	1.9535	1.9767		
$10^{-10}$	3.9334e-03	1.1115e-03	2.9600e-04	7.6401e-05	1.9400e-05	
	1.8233	1.9088	1.9539	1.9775		
$10^{-12}$	3.9334e-03	1.1115e-03	2.9600e-04	7.6401 e- 05	1.9400e-05	
	1.8233	1.9088	1.9539	1.9775		
Result	in [30]					
$10^{-2}$	3.6825e-02	1.8188e-02	8.5040e-03	4.2227e-03	2.1179e-03	
	1.018	1.097	1.010	0.995		
$10^{-4}$	3.9442e-02	1.9359e-02	9.5692 e- 03	4.7539e-03	2.3691e-03	
	1.027	1.016	1.009	1.005		
$10^{-6}$	3.9402e-02	1.9391e-02	9.5773e-03	4.7594 e- 03	2.3717e-03	
	1.023	1.018	1.009	1.005		
$10^{-8}$	3.9418e-02	1.9392e-02	9.5791e-03	4.7594 e- 03	2.3718e-03	
	1.023	1.017	1.009	1.005		
$10^{-10}$	3.9418e-02	1.9392e-02	9.5791e-03	4.7594 e- 03	2.3718e-03	
	1.023	1.017	1.009	1.005		
$10^{-12}$	3.9418e-02	1.9392e-02	9.5791e-03	4.7594 e- 03	2.3718e-03	
	1.023	1.017	1.009	1.005		

**Example 1.** The first example we have considered is given in [30]

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu (1+x) \frac{\partial u}{\partial x} + u = -16x^2 (1-x)^2,$$

subject to u(x, 0) = 0, u(0, t) = 0 = u(1, t).

**Example 2.** Consider the following singular perturbation initial boundary value problem in [30]

$$\begin{split} &\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu (1 + \exp(x)) \frac{\partial u}{\partial x} + (1 + x^5) u = -10 \exp(t^2) x^2 (1 - x^2),\\ &\text{subject to } u(x, 0) = 0, \, u(0, t) = 0 = u(1, t). \end{split}$$

$\varepsilon\downarrow$	N = 64	N = 128	N = 256	N = 512		
·	M = 16	M = 32	M = 64	M = 128		
Current result						
$10^{-6}$	2.6377e-05	6.5877 e-06	1.6474 e-06	4.1182e-07		
	2.0014	1.9996	2.0001	1.9999		
$10^{-7}$	2.6377e-05	6.5877 e-06	1.6474e-06	4.1182e-07		
	2.0014	1.9996	2.0001	1.9999		
$10^{-8}$	2.6377e-05	6.5877 e-06	1.6474e-06	4.1182e-07		
	2.0014	1.9996	2.0001	1.9999		
$10^{-9}$	2.6377e-05	6.5877 e-06	1.6474e-06	4.1182e-07		
	2.0014	1.9996	2.0001	1.9999		
Result in [10]						
$10^{-6}$	4.6780E-4	1.1982E-4	3.0257 E-5	7.5982E-6		
$10^{-7}$	4.6780 E-4	1.1982E-4	3.0257 E-5	7.5982E-6		
$10^{-8}$	4.6780 E-4	1.1982E-4	3.0257 E-5	7.5982E-6		
$10^{-9}$	4.6780 E-4	1.1982E-4	3.0257 E-5	7.5982E-6		
Result in [8]						
$10^{-6}$	0.096949e-2	0.049906e-2	0.025231e-2	0.012824e-2		
	0.95802	0.98400	0.97638	0.99233		
$10^{-7}$	0.098712e-2	0.050049e-2	0.025485e-2	0.012853e-2		
	0.97987	0.97368	0.98758	0.99493		
$10^{-8}$	0.0951284e-2	0.050026e-2	0.025237e-2	0.012781e-2		
	0.92720	0.98713	0.98156	0.98839		
$10^{-9}$	0.096746e-2	0.050012e-2	0.025461e-2	0.0128036e-2		
	0.95193	0.97394	0.99186	0.99188		

TABLE 3. Comparison of the maximum point-wise error and rate of convergence for Example 3 for  $\mu = 10^{-7}$  and different values of  $\varepsilon$ .

**Example 3.** Consider the following singular perturbation initial boundary value problem in [8]

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu (1 + x - x^2 + t^2) \frac{\partial u}{\partial x} + (1 + 5xt)u = (x^2 - x)(e^t - 1),$$
  
subject to  $u(x, 0) = 0, u(0, t) = 0 = u(1, t).$ 

As the considered examples have no exact solution, we calculate the absolute maximum errors using the double mesh principle [9] as follows:

$$E_{\varepsilon,\mu}^{N,M} = \max_{1 \le n \le N+1, 1 \le m \le M+1} |U_N^M - U_{2N}^{2M}|,$$

where  $U_N^M$  is an approximate solution obtained using M and N subintervals in the t and x directions respectively, and  $U_{2N}^{2M}$  is an approximate solution obtained by bisecting each subinterval. As well, the corresponding numerical



FIGURE 1. The numerical solution profile when  $N=M=2^7, \ \varepsilon=10^{-2}$  and  $\mu=10^{-10}$  for Example 1.



FIGURE 2. Log-log plot for Example 1 when  $\mu = 10^{-6}$ .

rate of convergence is defined by

$$R = \frac{\log(E_{\varepsilon,\mu}^{N,M}) - \log(E_{\varepsilon,\mu}^{2N,2M})}{\log(2)}$$

For  $\mu = 10^{-6}$  and different values of  $\varepsilon$ , the maximum absolute error and numerical rate of convergence for Examples 1 and 2 are given in Tables 1 and 2, respectively. The numerical solutions of these examples are plotted in Figs. 1 and 3. From the results in these tables, one can observe that the current method converges independently of the perturbation parameters and gives more accurate numerical results than that of the article [30]. Furthermore, comparison of our numerical results with the results in [8, 10] is presented in Table 3 and



FIGURE 3. The numerical solution profile when  $N=M=2^7,$  $\varepsilon=10^{-2}$  and  $\mu=10^{-10}$  for Example 2.



FIGURE 4. Log-log plot for Example 2 when  $\mu = 10^{-6}$ .

verify the hypothesized result. The solution profile illustrated in Figs. 1, 3 and 5 indicate that the boundary layers occur at the two endpoints of the spatial domain. The steepness of the boundary layers is equal in Fig. 1, but more strong on the right and left lateral corner of the spatial domain in Figs. 3 and 5, respectively. From the Log-Log plots, we can conclude that our method is parameter uniform. In general, the numerical results obtained confirm the theoretical findings very well.



FIGURE 5. The numerical solution profile when N = 64, M = 64,  $\varepsilon = 10^{-2}$  and  $\mu = 10^{-10}$  for Example 3.



FIGURE 6. Log-log plot for Example 3 when  $\mu = 10^{-7}$ .

## 5. Conclusion

A parameter uniform numerical scheme is constructed for solving singularly perturbed parabolic problems whose diffusion and convection terms are multiplied by small perturbation parameters. Due to the presence of these two parameters, the solution of the problem exhibits twin boundary layers at both the lateral surfaces of the rectangular domain. This is demonstrated by taking numerical examples. To handle the singularity character of the solution in the inner layers, a fitted operator method is devised which requires neither a priori information about the location and width of the boundary layers nor the introduction of extremely fine meshes in the layer regions. Moreover, the analysis clearly shows the developed method is convergent regardless of the perturbation parameters. The obtained results are more accurate and converge faster than the results available in the literature.

*Remark* 5.1. The notable differences of the current manuscript and our previous paper [19] are given as follows. In the previous paper, the singularity character of the problem under consideration is controlled by fitting mesh (piecewise Shishkin mesh, i.e., unequal mesh sizes are used), whereas in this work, it is controlled by fitting operator (equal mesh sizes are used throughout the domain and the perturbation parameter is multiplied by the fitting operator). That is why it is exponentially fitted. Furthermore, the numerical result obtained in this work is relatively better than the said paper.

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