Commun. Korean Math. Soc. **38** (2023), No. 1, pp. 179–193
https://doi.org/10.4134/CKMS.c210026
pISSN: 1225-1763 / eISSN: 2234-3024

GEOMETRIC INEQUALITIES FOR WARPED PRODUCTS SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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Abstract. In this article, we derived Chen's inequality for warped product bi-slant submanifolds in generalized complex space forms using semi-symmetric metric connections and discuss the equality case of the inequality. Further, we discuss non-existence of such minimal immersion. We also provide various applications of the obtained inequalities.

1. Introduction

Since the celebrated theory of J. F. Nash of isometric immersion of a Riemannian manifold into a suitable Euclidean space gives very important and effective motivation to view each Riemannian manifold as a submanifold in a Euclidean space, the problem of discovering simple basic relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifold theory. The main extrinsic invariant is the squared mean curvature, and the main intrinsic invariants include the classical curvature invariants: the Ricci curvature and the scalar curvature.

Apart from Hermitian geometry, the theory of product manifolds has important physical and geometrical aspects. In physics, the spacetime of Einstein's general relativity could be considered as a product of 3-dimensional space and 1-dimensional time, both having their metrics, thus its topology is generated by the metrics of these spaces. There are also nice applications of product manifolds in Kaluza-Klein theory, brane theory and gauge theory. In 1969, R. L. Bishop et al. [5] introduced a generalized case of Riemannian product manifolds to study manifolds of negative sectional curvature called warped product manifold. They defined warped products as follows:

Let us consider a Riemannian manifold M_1 of dimension n_1 with Riemannian metric g_1, M_2 of dimension n_2 with Riemannian metric g_2 and σ be positive

Received January 16, 2021; Accepted June 28, 2022.
2020 *Mathematics Subject Classification.* Primary 53B40, 53B42, 53C15.
Key words and phrases. Chen's inequality, warped product, bi-slant submanifolds, generalized complex space forms, semi-symmetric metric connection.

differentiable functions on M_1. Consider the warped product manifold $M_1 \times M_2$ with its projections $\iota_1 : M_1 \times M_2 \to M_1$ and $\iota_2 : M_1 \times M_2 \to M_2$. Then, their warped product manifold $M = M_1 \times_\sigma M_2$ is the product manifold equipped with the structure

$$g(X,Y) = g_1(\iota_{1*}X, \iota_{1*}Y) + (\sigma \circ \iota_1)^2 g_2(\iota_{2*}X, \iota_{2*}Y)$$

for any vector fields X, Y on M, where $*$ denotes the symbol for tangent maps.

Due to its usefulness, many research article has been published in this area [6, 7, 9–12, 14, 16, 19].

On the other hand, as a generalization of the complex space form Tricerri and Vanhecke [18] introduced the notion of generalized complex space form.

Afterwards, many interesting results have been proved in this ambient space [1, 3, 4, 15, 17].

In this article, our main aim is to obtain Chen's inequality for warped product bi-slant submanifolds of generalized complex space forms endowed with semi-symmetric metric connections. We also obtained various application of the derived results.

2. Preliminaries

Let J and g be an almost complex structure and a Riemannian metric on an almost Hermitian manifold $\overline{M}$, respectively. Then, $\overline{M}$ is said to be:

- a nearly Kaehler manifold if $(\overline{\nabla}_X J)X = 0$.
- a Kaehler manifold if $\overline{\nabla} J = 0$ for all $X \in T\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of the Riemannian metric g.
- a generalized complex space form, denoted by $\overline{M}(f_1, f_2)$, if the Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}\end{aligned}$$

for all $X, Y, Z \in T\overline{M}$, where f_1 and f_2 are smooth functions on $\overline{M}(f_1, f_2)$.

Let $\overline{M}^{2m}$ be an almost Hermitian manifold and M^n be a submanifold $\overline{M}^{2m}$ with induced metric g. Let ∇ be an induced connection on the tangent bundle TM and $\nabla^\perp$ be an induced connection on the normal bundle $T^\perp M$ of M. Then, the Gauss and Weingarten formulas are given by

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\ \overline{\nabla}_X N &= -\mathcal{A}_N X + \nabla^\perp_X N,\end{aligned}$$

where $X, Y \in TM$, $N \in T^\perp M$ and h, $\mathcal{A}_N$ are the second fundamental form and the shape operator, respectively.

The relation between the shape operator and the second fundamental form is given by

$$g(h(X,Y), N) = g(\mathcal{A}_N X, Y)$$

for vector fields $X, Y \in TM$ and $N \in T^{\perp}M$.

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}(c)$ and M, respectively. Then, the Gauss equation is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W)\\ &\quad + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)),\end{aligned} \tag{1}$$

$X, Y, Z, W \in TM$.

The notion of the semi-symmetric linear connection was introduced by Friedmann and Schouten [13]. If ω is the 1-form given by $\omega(X) = g(X, U)$ for any vector fields $X, Y, U \in TM$ and the torsion tensor T of a linear connection satisfies the relation

$$T(X,Y) = \omega(y)X - \omega(X)y,$$

then such a linear connection is said to be semi-symmetric. If a semi-symmetric connection satisfies

$$\tilde{\nabla} g = 0,$$

then it is said to be a semi-symmetric metric connection $\tilde{\nabla}$.

Further, with respect to semi-symmetric metric connection $\tilde{\nabla}$ on $\overline{M}(c)$ the curvature tensor $\tilde{R}$ can be written as

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) - \alpha(Y,Z)g(X,W)\\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,W) + \alpha(Y,W)g(X,Z)\end{aligned} \tag{2}$$

for any $X, Y, Z, W \in TM$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X, Y*) = (\overline{\nabla}_X \omega)Y - \omega(X)\omega Y + \frac{1}{2}\omega(P)g(X,Y)$$

for all $X, Y \in TM$.

Let M be an n-dimensional submanifold of a generalized complex space form $\overline{M}(f_1, f_2)$ of complex dimension m. Then, we know that

$$JX = PX + QX,$$

where P and Q are the tangential and normal components of JX, respectively and $X \in TM$.

It should be noted that:

- The submanifold is said to be an anti-invariant submanifold if $P = 0$.
- The submanifold is said to be an invariant submanifold if $Q = 0$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space TM of M. Then, the squared norm of P at $p \in M$ is defined by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

Let $\pi \subset T_xM$ at a point $x \in M$ be a plane section. If $\{e_1, \ldots, e_n\}$ is the orthonormal basis of T_xM and $\{e_{n+1}, \ldots, e_{2m}\}$ is the orthonormal basis of

$T_x^{\perp}M$ at any $x \in M$, then the sectional curvature $K(\pi)$ of a Riemannian manifold M is given by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where τ is the scalar curvature. On the other hand, the mean curvature vector field $\mathcal{H}$ on M is given by

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^{n} g(h(e_i, e_i)).$$

Definition. A submanifold is said to be a minimal submanifold if the mean curvature vector $\mathcal{H}$ vanishes identically, that is $\mathcal{H} = 0$.

We also recall the definition of slant submanifolds.

Definition ([11])**.** Let $\tilde{M}^{2m}$ be an almost hermitian manifold. Then, a submanifold M^n of $\tilde{M}^{2m}$ is said to be slant if for each given point $x \in M^n$ and for any non-zero vector $X \in T_xM$, the angle $\theta(X)$ between JX and T_xM is free from the choice of X.

Definition ([19])**.** Let $\tilde{M}^{2m}$ be an almost hermitian manifold. Then, a submanifold M^n of $\tilde{M}^{2m}$ is said a bi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

(i) $TM^n = D_1 \oplus D_1$,
(ii) $JD_1 \perp D_2$ and $JD_2 \perp D_1$;
(iii) For $i = 1, 2$, each distribution D_i is slant with a slant angle θ_i.

Indeed, bi-slant submanifolds englobe not only slant submanifolds but semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds also. The first author assembled it in the following table [2]:

Table 1. Definition

S.N.	$\tilde{M}$	M	D_1	D_2	θ_1	θ_2
(1)	$\tilde{M}$	bi-slant	slant	slant	slant angle	slant angle
(2)	$\tilde{M}$	semi-slant	invariant	slant	0	slant angle
(3)	$\tilde{M}$	hemi-slant	slant	anti-invariant	slant angle	$\frac{\pi}{2}$
(4)	$\tilde{M}$	CR	invariant	anti-invariant	0	$\frac{\pi}{2}$
(5)	$\tilde{M}$	slant	either $D_1 = 0$ or $D_2 = 0$		either $\theta_1 = \theta_2 = \theta$ or $\theta_1 = \theta_2 \neq \theta$	

Further, a slant submanifold is said to be

- invariant if the slant angle $\theta = 0$.
- anti-invariant if the slant angle $\theta = \frac{\pi}{2}$.
- proper slant if the slant angle $\theta \in (0, \frac{\pi}{2})$.

Furthermore, since M^n is a bi-slant submanifold, we define an adapted orthonormal frame as:

$$\begin{aligned} &n = 2d_1 + 2d_2 \text{ follows} \\ &\{e_1, e_2 = \sec\theta_1 P_1, \dots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 P_{e_{2d_1-1}}, \dots, e_{2d_1+1}, \\ &\quad e_{2d_1+2} = \sec\theta_2 P e_{2d_1+1}, \dots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 P_{2d_1+2d_2-1}\}. \end{aligned}$$

Then, by setting $g(e_1, Je_2) = -g(Je_1, e_2) = -g(Je_1, \sec\theta_1 Pe_1)$, one can obtain $g(e_1, Je_2) = -\sec\theta_1 g(Pe_1, Pe_2)$. Following ((2.8) in [12]), we get $g(e_1, Je_2) = \cos\theta_1 g(e_1, e_2)$. This implies

$$g^2(e_i, Je_j) = \begin{cases} \cos^2\theta_1 & \text{for each } i = 1, \dots, 2d_1 - 1, \\ \cos^2\theta_2 & \text{for each } j = 2d_1 + 1, \dots, 2d_1 + 2d_2 - 1. \end{cases}$$

Hence, we have

$$\sum_{i,j=1}^{n} g^2(Je_i, e_j) = (n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2).$$

Now, we recall the following well-known algebraic lemma for later use.

Lemma 2.1 ([10]). *For $n \geq 2$, let $a_1, \dots, a_n, b$ be real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Finally, we conclude the section with the following relation between sectional curvature and Laplacian of warping function for warped product by B. Y. Chen [8]. According to him, we have

$$\sum_{1\leq i\leq n_1} \sum_{n_1+1\leq j\leq n} K(e_i \wedge e_j) = n_2 \frac{\Delta\sigma}{\sigma} = n_2\Big(\Delta(\ln\sigma) - ||\nabla\sigma||^2\Big), \tag{3}$$

where Δ is the Laplacian operator.

3. Chen's inequality for warped product submanifolds in generalized complex space forms

In this section we state and prove the main result of the article. More precisely we have the following result.

Theorem 3.1. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be an isometric immersion of warped product bi-slant submanifold into $\tilde{M}^{2m}(c)$. Then,*

$$n_2 \frac{\Delta\sigma}{\sigma} \leq \frac{n^2}{4}||\mathcal{H}||^2 + n_1 n_2 f_1 + \lambda - \frac{f_2}{2}\big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\big), \tag{4}$$

where λ denotes the trace of α and θ_1 and θ_2 are slant functions along M_1 and M_2, respectively. The equality case holds in (4) if and only if φ is a mixed totally geodesic isometric immersion and the following satisfies

$$\frac{\mathcal{H}_1}{\mathcal{H}_2} = \frac{n_1}{n_2},$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the mean curvature vectors along $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

Proof. From (1), (2) and the Gauss equation with respect to the semi-symmetric metric connection, we have

$$\begin{aligned} &R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \\ &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ &\quad - g(Y,JZ)JX + 2g(X,JY)JZ\} - \alpha(Y,Z)g(X,W) \\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,W) + \alpha(Y,W)g(X,Z). \end{aligned}$$

Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ and by summing after $1 \leq i,j \leq n$, it follows from the previous relation that

$$\begin{aligned} 2\tau = {} & f_1 n\big(n-1\big) + f_2\big(3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2\big) \\ & + 2(n-1)\lambda + n^2||\mathcal{H}||^2 - ||h||^2. \end{aligned} \tag{5}$$

Let us assume that

$$\begin{aligned} \delta = {} & 2\tau - f_1 n\big(n-1\big) - f_2\big(3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2\big) \\ & - 2(n-1)\lambda - \frac{n^2}{2}||\mathcal{H}||^2. \end{aligned} \tag{6}$$

Then, we derive from (5) and (6) that

$$n^2||\mathcal{H}||^2 = 2(\delta + ||h||^2). \tag{7}$$

Thus, with respect to the chosen orthonormal frame, (7) takes the form

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}.$$

If we substitute $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation reduces to

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right.$$

$$- \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\Bigg\}.$$

Thus a_1, a_2, a_3 satisfy the lemma of Chen (for $n = 3$), that is,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

Then $2a_1a_2 \ge b$ with equality holding if and only if $a_1 + a_2 = a_3$. In the case under considering, this means

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}$$

$$\ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} (h_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^{r})^2. \tag{8}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{9}$$

Again using the Gauss equation, we have

$$n_2\frac{\Delta\sigma}{\sigma} = \tau - \sum_{1\le j<k\le n_1} \kappa(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} \kappa(e_s\wedge e_t). \tag{10}$$

Combining (3), (8) and (10) yields the following relation

$$\begin{aligned} n_2\frac{\Delta\sigma}{\sigma} &= \tau - \frac{f_1}{2}n_1\big(n_1-1\big) - 3n_1f_2\cos^2\theta_1 \\ &\quad - (n_1-1)\lambda - \sum_{r=n+1}^{2m}\sum_{1\le j<k\le n_1}\big(h_{jj}^r h_{kk}^r - (h_{jk}^r)^2\big) \\ &\quad - \frac{f_1}{2}n_2\big(n_2-1\big) - 3n_2f_2\cos^2\theta_2 \\ &\quad - (n_2-1)\lambda - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\big(h_{ss}^r h_{tt}^r - (h_{st}^r)^2\big). \end{aligned} \tag{11}$$

Taking into account (8) and (11), we find

$$\begin{aligned} n_2\frac{\Delta\sigma}{\sigma} &\le \tau - \frac{f_1}{2}n\big(n-1\big) + n_1n_2f_1 - (n_1-1)\lambda \\ &\quad - (n_2-1)\lambda - 3n_1f_2\cos^2\theta_1 - 3n_2f_2\cos^2\theta_2 - \frac{\delta}{2}. \end{aligned}$$

Using (6) in the above equation, we obtain

$$n_2\frac{\Delta\sigma}{\sigma} \le \frac{n^2}{4}||\mathcal{H}||^2 + n_1n_2f_1 - (2n-3)\lambda$$

$$-\frac{f_2}{2}\big(3n_1\cos^2\theta_1+3n_2\cos^2\theta_2\big),$$

which is the required inequality.

From (8) and (9), we conclude that the equality in (4) holds if and only if

$$\sum_{r=n+1}^{2m}\sum_{i=1}^{n_1}h_{ii}^r=\sum_{r=n+1}^{2m}\sum_{t=n_1+1}^{n}h_{tt}^r=0,$$

and $n_1\mathcal{H}_1=n_2\mathcal{H}_2$ are partially mean curvature vectors on $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

Moreover from (8), we obtain

$$\begin{aligned} h_{\alpha\beta}^r=0,\ \forall\ 1\le\alpha\le n_1,\\ n+1\le\beta\le n,\\ n+1\le r\le 2m \end{aligned}$$

and the converse is also true in case of warped product bi-slant submanifolds into the generalized complex space form. This concludes the proof of the result. □

An immediate consequence of Theorem 3.1 is the following.

Corollary 3.2. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi: M^n=M_1^{n_1}\times_\sigma M_2^{n_2}\to\tilde{M}^{2m}(c)$ be an isometric immersion from warped product submanifold into $\tilde{M}^{2m}(c)$. Then we have the following inequalities*:

Table 2.

S.N.	$\overline{M}(f_1,f_2)$	*M*	*Inequality*
(1)	$\overline{M}(f_1,f_2)$	*semi-slant*	$n_2\frac{\Delta\sigma}{\sigma}\le\frac{n^2}{4}\|\|\mathcal{H}\|\|^2+n_1n_2f_1+\lambda-\frac{3f_2}{2}\big(n_1+n_2\cos^2\theta_2\big)$,
(2)	$\overline{M}(f_1,f_2)$	*hemi-slant*	$n_2\frac{\Delta\sigma}{\sigma}\le\frac{n^2}{4}\|\|\mathcal{H}\|\|^2+n_1n_2f_1+\lambda-\frac{3f_2}{2}\big(n_1\cos^2\theta_1\big)$,
(3)	$\overline{M}(f_1,f_2)$	*CR*	$n_2\frac{\Delta\sigma}{\sigma}\le\frac{n^2}{4}\|\|\mathcal{H}\|\|^2+n_1n_2f_1+\lambda-n_1\frac{3f_2}{2}$,
(4)	$\overline{M}(f_1,f_2)$	*slant*	$n_2\frac{\Delta\sigma}{\sigma}\le\frac{n^2}{4}\|\|\mathcal{H}\|\|^2+n_1n_2f_1+\lambda-\frac{3f_2}{4}\big(n\cos^2\theta\big)$,
(5)	$\overline{M}(f_1,f_2)$	*invariant*	$n_2\frac{\Delta\sigma}{\sigma}\le\frac{n^2}{4}\|\|\mathcal{H}\|\|^2+n_1n_2f_1+\lambda-3n_1\frac{f_2}{2}$,

S.N.	$\overline{M}(f_1, f_2)$	M	*Inequality*
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$n_2 \frac{\Delta\sigma}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - 3n_2 \frac{f_2}{2}$,

Since we know that if M^n is a compact oriented Riemannian submanifold without boundary, then we have the formula with respect to the volume element dV:

$$\int_{M^n} \Delta\sigma dV = 0. \tag{12}$$

Hence, as a consequence we obtain the following result.

Theorem 3.3. *Let $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be a compact oriented warped product bi-slant submanifold in the generalized complex space form. Then, M^n is simply a Riemannian product if and only if*

$$||\mathcal{H}||^2 \geq \frac{2f_2}{n^2}\big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\big) - \frac{4}{n^2} n_1 n_2 f_1 + \lambda. \tag{13}$$

Proof. We consider the warped product bi-slant submanifolds as a compact oriented Riemannian manifold without boundary. If the inequality (4) holds,

$$\begin{aligned} & n_2\big(\Delta(\ln\sigma) - ||\nabla\sigma||^2\big) \\ \leq\ & \frac{n^2}{4}||\mathcal{H}||^2 + n_1 n_2 f_1 + \lambda - \frac{f_2}{2}\big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\big). \end{aligned}$$

Since M^n is a compact warped product submanifold, then from (12), we obtain

$$\begin{aligned} \int_{M^n} \Big(- n_2 ||\nabla^1\sigma||^2\Big) dV \leq \int_{M^n} \Big[& \frac{1}{4}\sum_{i=1}^{n} (h_{ii}^{n+1})^2 + n_1 n_2 f_1 + \lambda \\ & - \frac{f_2}{2}\big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\big)\Big] dV. \end{aligned} \tag{14}$$

Now, let us assume that M^n is a Riemannian product and the warping function σ and σ_2 must be constant of M^n. Then, from (14), we obtain the inequality (13).

Conversely, suppose that the inequality (13) holds. Then from (14), we obtain

$$0 \leq \int_{M^n} (n_2 ||\nabla\sigma||^2) dV \leq 0.$$

The above condition implies that $||\nabla\sigma||^2 = 0$. This means that σ is a constant function on M^n. Hence, M^n is simply a Riemannian product of $M_1^{n_1}$ and $M_2^{n_2}$, respectively. Thus, Theorem 3.3 is proved. □

From the above theorem we have the following result.

Corollary 3.4. *Let $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be a compact oriented warped product bi-slant submanifold in the generalized complex space form. Then, M^n is simply a Riemannian product if and only if*

Table 3.

S.N.	$\overline{M}(f_1, f_2)$	M	*Inequality*
(1)	$\overline{M}(f_1, f_2)$	*semi-slant*	$\|\|\mathcal{H}\|\|^2 \geq 6f_2\big(n_1+n_2\cos^2\theta_2\big)+\lambda-\frac{4}{n^2}n_1n_2f_1$
(2)	$\overline{M}(f_1, f_2)$	*hemi-slant*	$\|\|\mathcal{H}\|\|^2 \geq 6f_2n_1\cos^2\theta_1+\lambda-\frac{4}{n^2}n_1n_2f_1$
(3)	$\overline{M}(f_1, f_2)$	*CR*	$\|\|\mathcal{H}\|\|^2 \geq 6f_2n_1+\lambda-\frac{4}{n^2}n_1n_2f_1$
(4)	$\overline{M}(f_1, f_2)$	*slant*	$\|\|\mathcal{H}\|\|^2 \geq 3f_2n\cos^2\theta+\lambda-\frac{4}{n^2}n_1n_2f_1$
(5)	$\overline{M}(f_1, f_2)$	*invariant*	$\|\|\mathcal{H}\|\|^2 \geq 3f_2n+\lambda-\frac{4}{n^2}n_1n_2f_1$
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$\|\|\mathcal{H}\|\|^2 \geq \lambda-\frac{4}{n^2}n_1n_2f_1$

Let ϕ be a positive differentiable C^∞-differentiable function. Then the Hessian tensor of function ϕ is a symmetric 2-covariant tensor field on M^n defined by

$$\mathbb{H}^\phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$$

such that

$$\mathbb{H}^\phi(X, Y) = \mathbb{H}^\phi_{ij} X^i Y^j$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathbb{H}^\phi_{ij}$ can be expressed by

$$\mathbb{H}^\phi_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial\phi}{\partial x_k}.$$

Let us assume that $\phi = \ln\sigma$. Then as a consequence of Theorem 3.1 and the above relation, we conclude the following result.

Theorem 3.5. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be an isometric immersion of warped product bi-slant submanifold into $\tilde{M}^{2m}(c)$. Then,*

$$\begin{aligned} & n_2\frac{trace\mathbb{H}^\phi}{\sigma} \\ & \leq \frac{n^2}{4}||\mathcal{H}||^2 + n_1n_2f_1 + \lambda - \frac{f_2}{2}\big(3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2\big). \end{aligned}$$

The following corollary follows from Theorem 3.5.

Corollary 3.6. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be an isometric immersion from warped product submanifold into $\tilde{M}^{2m}(c)$. Then we have the following inequalities*:

Table 4.

S.N.	$\overline{M}(f_1, f_2)$	*M*	*Inequality*
(1)	$\overline{M}(f_1, f_2)$	*semi-slant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - \frac{3f_2}{2}\big(n_1 + n_2 \cos^2\theta_2\big)$,
(2)	$\overline{M}(f_1, f_2)$	*hemi-slant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - \frac{3f_2}{2}\big(n_1 \cos^2\theta_1\big)$,
(3)	$\overline{M}(f_1, f_2)$	*CR*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - n_1 \frac{3f_2}{2}$,
(4)	$\overline{M}(f_1, f_2)$	*slant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - \frac{3f_2}{4}\big(n \cos^2\theta\big)$,
(5)	$\overline{M}(f_1, f_2)$	*invariant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - 3n_1 \frac{f_2}{2}$,
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} \leq \frac{n^2}{4}\|\|\mathcal{H}\|\|^2 + n_1 n_2 f_1 + \lambda - 3n_2 \frac{f_2}{2}$,

4. Non-existence of warped product submanifolds in generalized complex space forms

In this section we obtain the obstruction to the minimal immersion of warped product submanifolds in the generalized complex space forms.

Theorem 4.1. *For $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$, if*

$$n_2 \frac{\Delta\sigma}{\sigma} > n_1 n_2 f_1 + \lambda - \frac{f_2}{2}\big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\big),$$

then M^n cannot be minimally immersed in $\tilde{M}^{2m}(c)$.

Proof. From Theorem 3.1 and the definition of minimality, we have the required non-existence result. □

Theorem 4.1 gives the following result.

Corollary 4.2. *For $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$, if*

Table 5.

S.N.	$\overline{M}(f_1, f_2)$	*M*	*Inequality*
(1)	$\overline{M}(f_1, f_2)$	*semi-slant*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{2}\big(3n_1 + 3n_2\cos^2\theta_2\big)$
(2)	$\overline{M}(f_1, f_2)$	*hemi-slant*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{2}3n_1$
(3)	$\overline{M}(f_1, f_2)$	*CR*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{2}3n_1$
(4)	$\overline{M}(f_1, f_2)$	*slant*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{4}3n\cos^2\theta$
(5)	$\overline{M}(f_1, f_2)$	*invariant*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{4}3n$
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$n_2\frac{\Delta\sigma}{\sigma} > n_1n_2f_1 + \lambda$

then M^n cannot be minimally immersed in $\tilde{M}^{2m}(c)$.

Proof. From, Table 1 and the definition of minimality, we have the required non-existence result. □

Further, from Theorem 3.3 and the definition of the minimality we have the following non-existence result.

Theorem 4.3. *For $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$, if*

$$\frac{f_1}{f_2} < \frac{3}{2}\Big(\frac{\cos^2\theta_1}{n_2} + \frac{\cos^2\theta_2}{n_1}\Big) + \Big(\frac{3n^2}{2n_1n_2}\Big)\frac{\lambda}{f_2},$$

then M^n cannot be minimally immersed in $\tilde{M}^{2m}(c)$.

Proof. From, Theorem 3.3 and the definition of minimality, we have the required non-existence result. □

Theorem 4.3 yields the following.

Corollary 4.4. *Let $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be a compact oriented warped product submanifold in the generalized complex space form. If*

Table 6.

S.N.	$\overline{M}(f_1, f_2)$	*M*	*Inequality*
(1)	$\overline{M}(f_1, f_2)$	*semi-slant*	$\frac{f_1}{f_2} < \frac{3}{2}\Big(\frac{1}{n_2} + \frac{\cos^2\theta_2}{n_1}\Big) + \Big(\frac{3n^2}{2n_1n_2}\Big)\frac{\lambda}{f_2}$
(2)	$\overline{M}(f_1, f_2)$	*hemi-slant*	$\frac{f_1}{f_2} < \frac{3}{2}\frac{\cos^2\theta_1}{n_2} + \Big(\frac{3n^2}{2n_1n_2}\Big)\frac{\lambda}{f_2}$

S.N.	$\overline{M}(f_1, f_2)$	*M*	*Inequality*
(3)	$\overline{M}(f_1, f_2)$	*CR*	$\frac{f_1}{f_2} < \frac{3}{2n_2} + \left(\frac{3n^2}{2n_1n_2}\right)\frac{\lambda}{f_2}$
(4)	$\overline{M}(f_1, f_2)$	*slant*	$\frac{f_1}{f_2} < \frac{3n}{4n_1n_2}\cos^2\theta + \left(\frac{3n^2}{2n_1n_2}\right)\frac{\lambda}{f_2}$
(5)	$\overline{M}(f_1, f_2)$	*invariant*	$\frac{f_1}{f_2} < \frac{3n}{4n_1n_2} + \left(\frac{3n^2}{2n_1n_2}\right)\frac{\lambda}{f_2}$
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$f_1 < \left(\frac{3n^2}{2n_1n_2}\right)\lambda$

then M^n cannot be minimally immersed in $\tilde{M}^{2m}(c)$.

Proof. From Theorem 4.3 with Table 1, we obtain the required result. □

Theorem 3.5 yields the following obstruction result.

Theorem 4.5. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be an isometric immersion of warped product bi-slant submanifold into $\tilde{M}^{2m}(c)$. If*

$$n_2\frac{trace\mathbb{H}^\phi}{\sigma} > n_1n_2f_1 + \lambda - \frac{f_2}{2}\big(3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2\big),$$

then M^n cannot be minimally immersed in $\tilde{M}^{2m}(c)$.

Proof. The application of the definition of minimality to Theorem 3.5 yields the required result. □

The following corollary follows from Theorem 3.5.

Corollary 4.6. *Let $\tilde{M}^{2m}(c)$ be the generalized complex space form and $\varphi : M^n = M_1^{n_1} \times_\sigma M_2^{n_2} \to \tilde{M}^{2m}(c)$ be an isometric immersion from warped product submanifold into $\tilde{M}^{2m}(c)$. If*

Table 7.

S.N.	$\overline{M}(f_1, f_2)$	*M*	*Inequality*
(1)	$\overline{M}(f_1, f_2)$	*semi-slant*	$n_2\frac{trace\mathbb{H}^\phi}{\sigma} > n_1n_2f_1 + \lambda - \frac{3f_2}{2}\big(n_1 + n_2\cos^2\theta_2\big)$,
(2)	$\overline{M}(f_1, f_2)$	*hemi-slant*	$n_2\frac{trace\mathbb{H}^\phi}{\sigma} > n_1n_2f_1 + \lambda - \frac{3f_2}{2}\big(n_1\cos^2\theta_1\big)$,
(3)	$\overline{M}(f_1, f_2)$	*CR*	$n_2\frac{trace\mathbb{H}^\phi}{\sigma} > n_1n_2f_1 + \lambda - n_1\frac{3f_2}{2}$,

S.N.	$\overline{M}(f_1, f_2)$	M	*Inequality*
(4)	$\overline{M}(f_1, f_2)$	*slant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} > n_1 n_2 f_1 + \lambda - \frac{3f_2}{4}\big(n\cos^2\theta\big)$,
(5)	$\overline{M}(f_1, f_2)$	*invariant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} > n_1 n_2 f_1 + \lambda - 3n_1 \frac{f_2}{2}$,
(6)	$\overline{M}(f_1, f_2)$	*anti-invariant*	$n_2 \frac{trace\mathbb{H}^\phi}{\sigma} > n_1 n_2 f_1 + \lambda - 3n_2 \frac{f_2}{2}$,

Proof. Making use of Table 1 in Theorem 4.5, we have the required result. □

References

[1] P. Alegre, A. Carriazo, Y. H. Kim, and D. W. Yoon, *B. Y. Chen's inequality for submanifolds of generalized space forms*, Indian J. Pure Appl. Math. **38** (2007), no. 3, 185–201.

[2] M. Aquib, *Bounds for generalized normalized δ-Casorati curvatures for bi-slant submanifolds in T-space forms*, Filomat **32** (2018), no. 1, 329–340. https://doi.org/10.2298/fil1801329a

[3] M. Aquib, M. N. Boyom, M. H. Shahid, and G. E. Vilcu, *The first fundamental equation and generalized wintgen-type inequalities for submanifolds in generalized space forms*, Mathematics **7** (2019), 1151.

[4] M. Aquib, M. H. Shahid, and M. Jamali, *Lower extremities for generalized normalized δ-Casorati curvatures of bi-slant submanifolds in generalized complex space forms*, Kragujevac J. Math. **42** (2018), no. 4, 591–605. https://doi.org/10.5937/kgjmath1804591a

[5] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49. https://doi.org/10.2307/1995057

[6] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133** (2001), no. 3, 177–195. https://doi.org/10.1007/s006050170019

[7] B.-Y. Chen, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinb. Math. Soc. **45** (2002), 579–587.

[8] B.-Y. Chen, *Another general inequality for CR-warped products in complex space forms*, Hokkaido Math. J. **32** (2003), no. 2, 415–444. https://doi.org/10.14492/hokmj/1350657533

[9] B.-Y. Chen, *Geometry of warped product submanifolds: a survey*, J. Adv. Math. Stud. **6** (2013), no. 2, 1–43.

[10] B.-Y. Chen, *Differential Geometry of Warped Manifolds and Submanifolds*, World Scientific, Hackensack, NJ, USA, 2017.

[11] B.-Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turkish J. Math. **36** (2012), no. 4, 630–640.

[12] B.-Y. Chen and S. Uddin, *Warped product pointwise bi-slant submanifolds of Kaehler manifolds*, Publ. Math. Debrecen **92** (2018), no. 1-2, 183–199.

[13] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Z. **21** (1924), no. 1, 211–223. https://doi.org/10.1007/BF01187468

[14] M. A. Khan and A. A. Ishan, *CR-warped product submanifolds of a generalized complex space form*, Cogent Math. **4** (2017), Art. ID 1306153, 13 pp. https://doi.org/10.1080/23311835.2017.1306153

[15] C. W. Lee, J. W. Lee, G. Vîlcu, and D. W. Yoon, *Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, Bull. Korean Math. Soc. **52** (2015), no. 5, 1631–1647. https://doi.org/10.4134/BKMS.2015.52.5.1631

[16] A. Mihai, *Warped product submanifolds in generalized complex space forms*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **21** (2005), no. 1, 79–87.

[17] Z. Olszak, *On the existence of generalized complex space forms*, Israel J. Math. **65** (1989), no. 2, 214–218. https://doi.org/10.1007/BF02764861

[18] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trnas. Amer. Math. J. **26** (2003), 85–94.

[19] S. Uddin, B.-Y. Chen, and F. R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math. **14** (2017), no. 2, Paper No. 95, 11 pp. https://doi.org/10.1007/s00009-017-0896-8

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