Commun. Korean Math. Soc. **38** (2023), No. 1, pp. 97–112
https://doi.org/10.4134/CKMS.c220016
pISSN: 1225-1763 / eISSN: 2234-3024

THE u-S-WEAK GLOBAL DIMENSIONS OF COMMUTATIVE RINGS

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Abstract. In this paper, we introduce and study the u-S-weak global dimension u-S-w.gl.dim(R) of a commutative ring R for some multiplicative subset S of R. Moreover, the u-S-weak global dimensions of factor rings and polynomial rings are investigated.

Throughout this article, R is always a commutative ring with identity 1 and S is always a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S, s_2 \in S$. We denote by U(R) the set of all units in R. In 2002, Anderson and Dumitrescu [1] defined an S-Noetherian ring R for which any ideal of R is S-finite. Recall from [1] that an R-module M is called S-finite provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M. An R-module T is called u-S-torsion if $sT = 0$ for some $s \in S$ (see [7]). So an R-module M is S-finite if and only if M/F is u-S-torsion for some finitely generated submodule F of M. The idea derived from u-S-torsion modules is deserved to be further investigated. In [7], the author of this paper introduced the class of u-S-flat modules F for which the functor $F \otimes_R -$ preserves u-S-exact sequences. The class of u-S-flat modules can be seen as a "uniform" generalization of that of flat modules, since an R-module F is u-S-flat if and only if $\mathrm{Tor}_1^R(F, M)$ is u-S-torsion for any R-module M (see [7, Theorem 3.2]). The class of u-S-flat modules owns the following u-S-hereditary property: let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence, if B and C are u-S-flat so is A (see [7, Proposition 3.4]). So it is worth to study the u-S-analogue of flat dimensions of R-modules and the u-S-analogue of a weak global dimension of commutative rings.

In this article, we define the u-S-flat dimension u-S-$fd_R(M)$ of an R-module M to be the length of the shortest u-S-flat u-S-resolution of M. We characterize u-S-flat dimensions of R-modules using the uniform torsion property of the "Tor" functors in Proposition 3.2. Besides, we obtain a new local characterization of flat dimensions of R-modules (see Corollary 3.7). The u-S-weak

Received January 25, 2022; Accepted May 25, 2022.
2020 *Mathematics Subject Classification.* 13D05, 13D07, 13B25.
Key words and phrases. u-S-flat module, u-S-flat dimension, u-S-weak global dimension, polynomial ring.

global dimension u-S-w.gl.dim(R) of a commutative ring R is defined to be the supremum of u-S-flat dimensions of all R-modules. A characterization of u-S-weak global dimensions is given in Proposition 3.2. Examples of rings R for which u-S-w.gl.dim$(R) \neq$ w.gl.dim(R_S) can be found in Example 3.11. U-S-von Neumann regular rings are firstly introduced in [7] for which there exist $s \in S$ and $r \in R$ such that $sa = ra^2$ for any $a \in R$. By [7, Theorem 3.11], a ring R is u-S-von Neumann regular if and only if all R-modules are u-S-flat. So u-S-von Neumann regular rings are exactly commutative rings with u-S-weak global dimensions equal to 0 (see Corollary 3.8). We also study commutative rings R with u-S-w.gl.dim(R) at most 1. The nontrivial example of a commutative ring R with u-S-w.gl.dim$(R) \leq 1$ but an infinite weak global dimension is given in Example 3.11. In the final section, we investigate the u-S-weak global dimensions of factor rings and polynomial rings and show that u-S-w.gl.dim$(R[x]) = u$-S-w.gl.dim$(R) + 1$ for a ring R under some condition (see Theorem 4.7).

1. Preliminaries

Recall from [7] that an R-module T is called a u-S-torsion module provided that there exists an element $s \in S$ such that $sT = 0$. An R-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called *u-S-exact* (at N) provided that there is an element $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. We say a long R-sequence $\cdots \to A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \to \cdots$ is u-S-exact, if for any n there is an element $s \in S$ such that $s\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n)$ and $s\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1})$. A u-S-exact sequence $0 \to A \to B \to C \to 0$ is called a short u-S-exact sequence. An R-homomorphism $f : M \to N$ is a *u-S-monomorphism* (resp., *u-S-epimorphism*, *u-S-isomorphism*) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. It is easy to verify an R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$, both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$) is a u-S-torsion module.

Proposition 1.1 ([6, Lemma 2.1]). *Let R be a ring and S a multiplicative subset of R. Suppose there is a u-S-isomorphism $f : M \to N$ for R-modules M and N. Then there is a u-S-isomorphism $g : N \to M$. Moreover, there is $t \in S$ such that $f \circ g = t\mathrm{Id}_N$ and $g \circ f = t\mathrm{Id}_M$.*

Let R be a ring and S a multiplicative subset of R. Let M and N be R-modules. We say M is u-S-isomorphic to N if there exists a u-S-isomorphism $f : M \to N$. A family $\mathcal{C}$ of R-modules is said to be closed under u-S-isomorphisms if M is u-S-isomorphic to N and M is in $\mathcal{C}$, then N is also in $\mathcal{C}$. It follows from Proposition 1.1 that the existence of u-S-isomorphisms of two R-modules is an equivalence relation. Next, we give a u-S-analogue of the Five Lemma.

Theorem 1.2 (u-S-analogue of Five Lemma)**.** *Let R be a ring and S a multiplicative subset of R. Consider the following commutative diagram with u-S-exact rows*:

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f_A"'] & B \arrow[r, "g_2"] \arrow[d, "f_B"'] & C \arrow[r, "g_3"] \arrow[d, "f_C"] & D \arrow[r, "g_4"] \arrow[d, "f_D"] & E \arrow[d, "f_E"] \\ A' \arrow[r, "h_1"] & B' \arrow[r, "h_2"] & C' \arrow[r, "h_3"] & D' \arrow[r, "h_4"] & E'. \end{tikzcd}$$

(1) *If f_B and f_D are u-S-monomorphisms and f_A is a u-S-epimorphism, then f_C is a u-S-monomorphism.*
(2) *If f_B and f_D are u-S-epimorphisms and f_E is a u-S-monomorphism, then f_C is a u-S-epimorphism.*
(3) *If f_A is a u-S-epimorphism, f_E is a u-S-monomorphism, and f_B and f_D are u-S-isomorphisms, then f_C is a u-S-isomorphism.*
(4) *If f_A, f_B, f_D and f_E are all u-S-isomorphisms, then f_C is a u-S-isomorphism.*

Proof. (1) Let $x \in \mathrm{Ker}(f_C)$. Then $f_D g_3(x) = h_3 f_C(x) = 0$. Since f_D is a u-S-monomorphism, $s_1\mathrm{Ker}(f_D) = 0$ for some $s_1 \in S$. So $s_1 g_3(x) = g_3(s_1 x) = 0$. Since the top row is u-S-exact, there exists $s_2 \in S$ such that $s_2\mathrm{Ker}(g_3) \subseteq \mathrm{Im}(g_2)$. Thus there exists $b \in B$ such that $g_2(b) = s_2 s_1 x$. Hence $h_2 f_B(b) = f_C g_2(b) = f_C(s_2 s_1 x) = 0$. Thus there exists $s_3 \in S$ such that $s_3\mathrm{Ker}(h_2) \subseteq \mathrm{Im}(h_1)$. So there exists $a' \in A'$ such that $h_1(a') = s_2 f_B(b)$. Since f_A is a u-S-epimorphism, there exists $s_4 \in S$ such that $s_4 A' \subseteq \mathrm{Im}(f_A)$. So there exists $a \in A$ such that $s_4 a' = f_A(a)$. Hence $s_4 s_2 f_B(b) = s_4 h_1(a') = h_1(f_A(a)) = f_B(g_1(a))$. So $s_4 s_2 b - g_1(a) \in \mathrm{Ker}(f_B)$. Since f_B is a u-S-monomorphism, there exists $s_5 \in S$ such that $s_5\mathrm{Ker}(f_B) = 0$. Thus $s_5(s_4 s_2 b - g_1(a)) = 0$. So $s_5 s_4 s_2 s_2 s_1 x = s_5(g_2(s_4 s_2 b)) = s_5 g_2(g_1(a))$. Since the top row is u-S-exact at B, there exists $s_6 \in S$ such that $s_6\mathrm{Im}(g_1) \subseteq \mathrm{Ker}(g_2)$. So $s_6 s_5 s_4 s_2 s_2 s_1 x = s_5 g_2(s_6 g_1(a)) = 0$. Consequently, if we set $s = s_6 s_5 s_4 s_2 s_2 s_1$, then $s\mathrm{Ker}(f_C) = 0$. It follows that f_C is a u-S-monomorphism.

(2) Let $x \in C'$. Since f_D is a u-S-epimorphism, there exists $s_1 \in S$ such that $s_1 D' \subseteq \mathrm{Im}(f_D)$. Thus there exists $d \in D$ such that $f_D(d) = s_1 h_3(x)$. By the commutativity of the right square, we have $f_E g_4(d) = h_4 f_D(d) = s_1 h_4(h_3(x))$. Since the bottom row is u-S-exact at D', there exists $s_2 \in S$ such that $s_4\mathrm{Im}(h_3) \subseteq \mathrm{Ker}(h_4)$. So $s_4 f_E(g_4(d)) = s_1 h_4(s_4 h_3(x)) = 0$. Since f_E is a u-S-monomorphism, there exists $s_3 \in S$ such that $s_3\mathrm{Ker}(f_E) = 0$. Thus $s_3 s_4 g_4(d) = 0$. Since the top row is u-S-exact at D, there there exists $s_5 \in S$ such that $s_5\mathrm{Ker}(g_4) \subseteq \mathrm{Im}(g_3)$. So there exists $c \in C$ such that $s_5 s_3 s_4 d = g_3(c)$. Hence $s_5 s_3 s_4 f_D(d) = f_D(g_3(c)) = h_3(f_C(c))$. Since $s_5 s_3 s_4 f_D(d) = h_3(s_1 s_5 s_3 s_4 x)$, we have $f_C(c) - s_1 s_5 s_3 s_4 x \in \mathrm{Ker}(h_3)$. Since the bottom row is u-S-exact at C', there exists $s_6 \in S$ such that $s_6\mathrm{Ker}(h_3) \subseteq \mathrm{Im}(h_2)$. Thus there exists $b' \in B'$ such that $s_6(f_C(c) - s_1 s_5 s_3 s_4 x) = h_2(b')$. Since f_B is a u-S-epimorphism, there exists $s_7 \in S$ such that $s_7 B' \subseteq \mathrm{Im}(f_B)$. So $s_7 b' = f_B(b)$ for some $b \in B$. Thus $f_C(g_2(b)) = h_2(f_B(b)) = s_7 h_2(b') = s_7(s_6(f_C(c) -$

$s_1s_5s_3s_4x)$). So $s_7s_6s_1s_5s_3s_4x = s_7s_6f_C(c) - f_C(g_2(b)) = f_C(s_7s_6c - g_2(b)) \in \mathrm{Im}(f_C)$. Consequently, if we set $s = s_7s_6s_1s_5s_3s_4$, then $sC' \subseteq \mathrm{Im}(f_C)$. It follows that f_C is a u-S-epimorphism.

It is easy to see (3) follows from (1) and (2), while (4) follows from (3). $\square$

Recall from [7, Definition 3.1] that an R-module F is called u-S-flat provided that for any u-S-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact. It is easy to verify that the class of u-S-flat modules is closed under u-S-isomorphisms by the following result.

Lemma 1.3 ([7, Theorem 3.2]). *Let R be a ring, S a multiplicative subset of R and F an R-module. The following assertions are equivalent*:

(1) *F is u-S-flat*;
(2) *for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to 0$ is u-S-exact*;
(3) *$\mathrm{Tor}_1^R(M, F)$ is u-S-torsion for any R-module M*;
(4) *$\mathrm{Tor}_n^R(M, F)$ is u-S-torsion for any R-module M and $n \geq 1$.*

The following result says that a short u-S-exact sequence induces a long u-S-exact sequence by the functor "Tor" as the classical case.

Theorem 1.4. *Let R be a ring, S a multiplicative subset of R and N an R-module. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence of R-modules. Then for any $n \geq 1$ there is an R-homomorphism $\delta_n : \mathrm{Tor}_n^R(C, N) \to \mathrm{Tor}_{n-1}^R(A, N)$ such that the induced sequence*

$$\cdots \to \mathrm{Tor}_n^R(A, N) \to \mathrm{Tor}_n^R(B, N) \to \mathrm{Tor}_n^R(C, N) \xrightarrow{\delta_n} \mathrm{Tor}_{n-1}^R(A, N) \to$$

$$\mathrm{Tor}_{n-1}^R(B, N) \to \cdots \to \mathrm{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

is u-S-exact.

Proof. Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is u-S-exact at B, there are three exact sequences $0 \to \mathrm{Ker}(f) \xrightarrow{i_{\mathrm{Ker}(f)}} A \xrightarrow{\pi_{\mathrm{Im}(f)}} \mathrm{Im}(f) \to 0$, $0 \to \mathrm{Ker}(g) \xrightarrow{i_{\mathrm{Ker}(g)}} B \xrightarrow{\pi_{\mathrm{Im}(g)}} \mathrm{Im}(g) \to 0$ and $0 \to \mathrm{Im}(g) \xrightarrow{i_{\mathrm{Im}(g)}} C \xrightarrow{\pi_{\mathrm{Coker}(g)}} \mathrm{Coker}(g) \to 0$ with $\mathrm{Ker}(f)$ and $\mathrm{Coker}(g)$ u-S-torsion. There also exists $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. Denote $T = \mathrm{Ker}(f)$ and $T' = \mathrm{Coker}(g)$.

Firstly, consider the exact sequence

$$\mathrm{Tor}_{n+1}^R(T', N) \to \mathrm{Tor}_n^R(\mathrm{Im}(g), N) \xrightarrow{\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)}, N)} \mathrm{Tor}_n^R(C, N) \to \mathrm{Tor}_n^R(T', N).$$

Since T' is u-S-torsion, $\mathrm{Tor}_{n+1}^R(T', N)$ and $\mathrm{Tor}_n^R(T', N)$ are u-S-torsion. Thus $\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)}, N)$ is a u-S-isomorphism. So there is also a u-S-isomorphism

$h^n_{\mathrm{Im}(g)}:\mathrm{Tor}^R_n(C,N)\to\mathrm{Tor}^R_n(\mathrm{Im}(g),N)$ by Proposition 1.1. Consider the exact sequence:

$$\mathrm{Tor}^R_{n-1}(T,N)\to\mathrm{Tor}^R_{n-1}(A,N)\xrightarrow{\mathrm{Tor}^R_{n-1}(\pi_{\mathrm{Im}(f)},N)}\mathrm{Tor}^R_{n-1}(\mathrm{Im}(f),N)\to\mathrm{Tor}^R_{n-2}(T,N).$$

Since T is u-S-torsion, we have $\mathrm{Tor}^R_{n-1}(\pi_{\mathrm{Im}(f)},N)$ is a u-S-isomorphism. So there is also a u-S-isomorphism $h^{n-1}_{\mathrm{Im}(f)}:\mathrm{Tor}^R_{n-1}(\mathrm{Im}(f),N)\to\mathrm{Tor}^R_{n-1}(A,N)$ by Proposition 1.1. We have two exact sequences

$$\mathrm{Tor}^R_{n+1}(T_1,N)\to\mathrm{Tor}^R_n(s\mathrm{Ker}(g),N)\xrightarrow{\mathrm{Tor}^R_n(i^1_{s\mathrm{Ker}(g)},N)}\mathrm{Tor}^R_n(\mathrm{Im}(f),N)\to\mathrm{Tor}^R_{n+1}(T_1,N)$$

and

$$\mathrm{Tor}^R_{n+1}(T_2,N)\to\mathrm{Tor}^R_n(s\mathrm{Ker}(g),N)\xrightarrow{\mathrm{Tor}^R_n(i^2_{s\mathrm{Ker}(g)},N)}\mathrm{Tor}^R_n(\mathrm{Ker}(g),N)\to\mathrm{Tor}^R_{n+1}(T_2,N),$$

where $T_1=\mathrm{Im}(f)/s\mathrm{Ker}(g)$ and $T_2=\mathrm{Im}(f)/s\mathrm{Im}(f)$ are u-S-torsion. So $\mathrm{Tor}^R_n(i^1_{s\mathrm{Ker}(g)},N)$ and $\mathrm{Tor}^R_n(i^2_{s\mathrm{Ker}(g)},N)$ are u-S-isomorphisms. Thus there is a u-S-isomorphism $h^n_{s\mathrm{Ker}(g)}:\mathrm{Tor}^R_n(\mathrm{Ker}(g),N)\to\mathrm{Tor}^R_n(s\mathrm{Ker}(g),N)$. Note that there is an exact sequence $\mathrm{Tor}^R_n(B,N)\xrightarrow{\mathrm{Tor}^R_n(\pi_{\mathrm{Im}(g)},N)}\mathrm{Tor}^R_n(\mathrm{Im}(g),N)\xrightarrow{\delta^n_{\mathrm{Im}(g)}}\mathrm{Tor}^R_{n-1}(\mathrm{Ker}(g),N)\xrightarrow{\mathrm{Tor}^R_{n-1}(i_{\mathrm{Ker}(g)},N)}\mathrm{Tor}^R_{n-1}(B,N)$. Set $\delta_n=h^n_{\mathrm{Im}(g)}\circ\delta^n_{\mathrm{Im}(g)}\circ h^n_{s\mathrm{Ker}(g)}\circ\mathrm{Tor}^R_n(i^1_{s\mathrm{Ker}(g)},N)\circ h^{n-1}_{\mathrm{Im}(f)}:\mathrm{Tor}^R_n(C,N)\to\mathrm{Tor}^R_{n-1}(A,N)$. Since $h^n_{\mathrm{Im}(g)},\delta^n_{\mathrm{Im}(g)},h^n_{s\mathrm{Ker}(g)}$ and $h^{n-1}_{\mathrm{Im}(f)}$ are u-S-isomorphisms, we have the sequence $\mathrm{Tor}^R_n(B,N)\to\mathrm{Tor}^R_n(C,N)\xrightarrow{\delta^n}\mathrm{Tor}^R_{n-1}(A,N)\to\mathrm{Tor}^R_{n-1}(B,N)$ is u-S-exact.

Secondly, consider the exact sequence:

$$\mathrm{Tor}^R_{n+1}(T,N)\to\mathrm{Tor}^R_n(A,N)\xrightarrow{\mathrm{Tor}^R_n(i_{\mathrm{Im}(f)},N)}\mathrm{Tor}^R_n(\mathrm{Im}(f),N)\to\mathrm{Tor}^R_n(T,N).$$

Since T is u-S-torsion, $\mathrm{Tor}^R_n(i_{\mathrm{Im}(f)},N)$ is a u-S-isomorphism. Consider the exact sequences:

$$\mathrm{Tor}^R_{n+1}(\mathrm{Im}(g),N)\to\mathrm{Tor}^R_n(\mathrm{Ker}(g),N)\xrightarrow{\mathrm{Tor}^R_n(i_{\mathrm{Ker}(g)},N)}\mathrm{Tor}^R_n(B,N)\to\mathrm{Tor}^R_n(\mathrm{Im}(g),N)$$

and

$$\mathrm{Tor}^R_{n+1}(T',N)\to\mathrm{Tor}^R_n(\mathrm{Im}(g),N)\xrightarrow{\mathrm{Tor}^R_n(i_{\mathrm{Im}(g)},N)}\mathrm{Tor}^R_n(C,N)\to\mathrm{Tor}^R_n(T',N).$$

Since T' is u-S-torsion, we have $\mathrm{Tor}^R_n(i_{\mathrm{Im}(g)},N)$ is a u-S-isomorphism. Since $\mathrm{Tor}^R_n(i^1_{s\mathrm{Ker}(g)},N)$ and $\mathrm{Tor}^R_n(i^2_{s\mathrm{Ker}(g)},N)$ are u-S-isomorphisms as above,

$$\mathrm{Tor}^R_n(A,N)\to\mathrm{Tor}^R_n(B,N)\to\mathrm{Tor}^R_n(C,N)$$

is u-S-exact at $\mathrm{Tor}^R_n(B,N)$.

Continue by the above method, we have a u-S-exact sequence:

$$\cdots\to\mathrm{Tor}^R_n(A,N)\to\mathrm{Tor}^R_n(B,N)\to\mathrm{Tor}^R_n(C,N)\xrightarrow{\delta_n}\mathrm{Tor}^R_{n-1}(A,N)\to$$

$$\mathrm{Tor}^R_{n-1}(B,N)\to\cdots\to\mathrm{Tor}^R_1(C,N)\xrightarrow{\delta_1}A\otimes_R N\to B\otimes_R N\to C\otimes_R N\to 0.$$

□

Corollary 1.5. *Let R be a ring, S a multiplicative subset of R and N an R-module. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-exact sequence of R-modules where B is u-S-flat. Then $\mathrm{Tor}^R_{n+1}(C, N)$ is u-S-isomorphic to $\mathrm{Tor}^R_n(A, N)$ for any $n \geq 0$. Consequently, $\mathrm{Tor}^R_{n+1}(C, N)$ is u-S-torsion if and only if $\mathrm{Tor}^R_n(A, N)$ is u-S-torsion for any $n \geq 0$.*

Proof. It follows from Lemma 1.3 and Theorem 1.4. □

2. On the u-S-flat dimensions of modules

Let R be a ring. The flat dimension of an R-module M is defined as the shortest flat resolution of M. We now introduce the notion of a u-S-flat dimension of an R-module as follows.

Definition 2.1. Let R be a ring, S a multiplicative subset of R and M an R-module. We write u-S-$fd_R(M) \leq n$ (u-S-fd abbreviates a *uniformly S-flat dimension*) if there exists a u-S-exact sequence of R-modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{$\diamondsuit$}$$

where each F_i is u-S-flat for $i = 0, \ldots, n$. The u-S-exact sequence ($\diamondsuit$) is said to be a u-S-flat u-S-resolution of length n of M. If such a finite u-S-flat u-S-resolution does not exist, then we say u-S-$fd_R(M) = \infty$; otherwise, define u-S-$fd_R(M) = n$ if n is the length of the shortest u-S-flat u-S-resolution of M.

Trivially, the u-S-flat dimension of an R-module M cannot exceed its flat dimension for any multiplicative subset S of R. And if S is composed of units, then u-S-$fd_R(M) = fd_R(M)$. It is also obvious that an R-module M is u-S-flat if and only if u-S-$fd_R(M) = 0$.

Lemma 2.2. *Let R be a ring and S a multiplicative subset of R. If A is u-S-isomorphic to B, then u-S-$fd_R(A) = $ u-S-$fd_R(B)$.*

Proof. Let $f : A \to B$ be a u-S-isomorphism. If $\cdots \to F_n \to \cdots \to F_1 \to F_0 \xrightarrow{g} A \to 0$ is a u-S-resolution of A, then $\cdots \to F_n \to \cdots \to F_1 \to F_0 \xrightarrow{f \circ g} B \to 0$ is a u-S-resolution of B. So u-S-$fd_R(A) \geq u$-S-$fd_R(B)$. Note that there is a u-S-isomorphism $g : B \to A$ by Proposition 1.1. Similarly we have u-S-$fd_R(B) \geq u$-S-$fd_R(A)$. □

Proposition 2.3. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent for an R-module M:*

1. *u-S-$fd_R(M) \leq n$;*
2. *$\mathrm{Tor}^R_{n+k}(M, N)$ is u-S-torsion for all R-modules N and all $k > 0$;*
3. *$\mathrm{Tor}^R_{n+1}(M, N)$ is u-S-torsion for all R-modules N;*
4. *there exists $s \in S$ such that $s\mathrm{Tor}^R_{n+1}(M, R/I) = 0$ for all ideals I of R;*
5. *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is a u-S-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are u-S-flat R-modules, then F_n is u-S-flat;*

(6) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is a* u-S*-exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules, then* F_n *is* u-S*-flat*;
(7) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are* u-S*-flat* R*-modules, then* F_n *is* u-S*-flat*;
(8) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules, then* F_n *is* u-S*-flat*;
(9) *there exists a* u-S*-exact sequence* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, *where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules and* F_n *is* u-S*-flat*;
(10) *there exists an exact sequence* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, *where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules and* F_n *is* u-S*-flat*;
(11) *there exists an exact sequence* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, *where* $F_0, F_1, \ldots, F_n$ *are* u-S*-flat* R*-modules.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. For the case $n = 0$, we have M is u-S-flat, and then (2) holds by [7, Theorem 3.2]. If $n > 0$, then there is a u-S-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is u-S-flat for $i = 0, \ldots, n$. Set $K_0 = \mathrm{Ker}(F_0 \to M)$ and $L_0 = \mathrm{Im}(F_1 \to F_0)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to L_0 \to 0$ are u-S-exact. Since u-S-$fd_R(L_0) \le n-1$ and L_0 is u-S-isomorphic to K_0, u-S-$fd_R(K_0) \le n-1$ by Lemma 2.2. By induction, $\mathrm{Tor}^R_{n-1+k}(K_0, N)$ is u-S-torsion for all u-S-torsion R-modules N and all $k > 0$. It follows from Corollary 1.5 that $\mathrm{Tor}^R_{n+k}(M, N)$ is u-S-torsion.

(2) $\Rightarrow$ (3), (5) $\Rightarrow$ (6) $\Rightarrow$ (8) and (5) $\Rightarrow$ (7) $\Rightarrow$ (8): Trivial.

(3) $\Rightarrow$ (4): Let $N = \bigoplus_{I \trianglelefteq R} R/I$. Then there exists an element $s \in S$ such that $s\mathrm{Tor}^R_{n+1}(M, N) = 0$. So $s \bigoplus_{I \trianglelefteq R} \mathrm{Tor}^R_{n+1}(M, R/I) = 0$. It follows that $s\mathrm{Tor}^R_{n+1}(M, R/I) = 0$ for all ideals I of R.

(4) $\Rightarrow$ (3): Let N be generated by $\{n_i : i \in \Gamma\}$. Set $N_0 = 0$ and $N_\alpha = \langle n_i : i < \alpha \rangle$ for each $\alpha \le \Gamma$. Then N has a continuous filtration $\{N_\alpha : \alpha \le \Gamma\}$ with $N_{\alpha+1}/N_\alpha \cong R/I_{\alpha+1}$ and $I_\alpha = \mathrm{Ann}_R(n_\alpha + N_\alpha \cap Rn_\alpha)$. Since $s\mathrm{Tor}^R_{n+1}(M, R/I_\alpha) = 0$ for each $\alpha \le \Gamma$, it is easy to verify $s\mathrm{Tor}^R_{n+1}(M, N_\alpha) = 0$ by transfinite induction on α. So $s\mathrm{Tor}^R_{n+1}(M, N) = 0$.

(3) $\Rightarrow$ (5): Let $0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \to \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ be a u-S-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are u-S-flat. Then F_n is u-S-flat if and only if $\mathrm{Tor}^R_1(F_n, N)$ is u-S-torsion for all R-modules N, if and only if $\mathrm{Tor}^R_2(\mathrm{Im}(d_{n-1}), N)$ is u-S-torsion for all R-modules N. Following these steps, we can show F_n is u-S-flat if and only if $\mathrm{Tor}^R_{n+1}(M, N)$ is u-S-torsion for all R-modules N.

(10) $\Rightarrow$ (11) $\Rightarrow$ (1) and (10) $\Rightarrow$ (9) $\Rightarrow$ (1): Trivial.

(8) $\Rightarrow$ (10): Let $\cdots \to P_n \to P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M. Set $F_n = \mathrm{Ker}(d_{n-1})$. Then we have an exact sequence $0 \to F_n \to P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \cdots \to P_0 \to M \to 0$. By (8), F_n is u-S-flat. So (10) holds. $\square$

Corollary 2.4. *Let R be a ring and $S' \subseteq S$ multiplicative subsets of R. Let M be an R-module. Then u-S-$fd_R(M) \leq S'$-$fd_R(M)$.*

Proof. Let N be an R-module. If $\mathrm{Tor}^R_{n+1}(M, N)$ is uniformly S'-torsion, then $\mathrm{Tor}^R_{n+1}(M, N)$ is u-S-torsion. The result follows by Proposition 2.3. $\square$

Let R be a ring, S a multiplicative subset of R and M an R-module. For any $s \in S$, we denote by R_s the localization of R at $\{s^n : n \geq 0\}$ and denote $M_s = M \otimes_R R_s$ as an R_s-module.

Corollary 2.5. *Let R be a ring, S a multiplicative subset of R and M an R-module. If u-S-$fd_R(M) \leq n$, then there exists an element $s \in S$ such that $fd_{R_s}(M_s) \leq n$.*

Proof. Let M be an R-module with u-S-$fd_R(M) \leq n$. Then there is an element $s \in S$ such that $s\mathrm{Tor}^R_{n+1}(R/I, M) = 0$ for any ideal I of R by Proposition 2.3. Let I_s be an ideal of R_s with I an ideal of R. Then $\mathrm{Tor}^{R_s}_{n+1}(R_s/I_s, M_s) \cong \mathrm{Tor}^R_{n+1}(R/I, M) \otimes_R R_s = 0$ since $s\mathrm{Tor}^R_{n+1}(R/I, M) = 0$. Hence $fd_{R_s}(M_s) \leq n$. $\square$

Corollary 2.6. *Let R be a ring and S a multiplicative subset of R. Let M be an R-module. Then u-S-$fd_R(M) \geq fd_{R_S} M_S$. Moreover, if S is composed of finite elements, then u-S-$fd_R(M) = fd_{R_S} M_S$.*

Proof. Let $\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence with each F_i u-S-flat. By localizing at S, we can obtain a flat resolution of M_S over R_S as follows:

$$\to (F_n)_S \to \cdots \to (F_1)_S \to (F_0)_S \to (M)_S \to 0.$$

So u-S-$fd_R(M) \geq fd_{R_S} M_S$ by Proposition 2.3. Suppose S is composed of finite elements and $fd_{R_S} M_S = n$. Let $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence, where F_i is flat over R for any $i = 0, \ldots, n-1$. Localizing at S, we have $(F_n)_S$ is flat over R_S. By [7, Proposition 3.8], F is u-S-flat. So u-S-$fd_R(M) \leq n$ by Proposition 2.3. $\square$

Proposition 2.7. *Let R be a ring and S a multiplicative subset of R. Let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence of R-modules. Then the following assertions hold.*

1. *u-S-$fd_R(C) \leq 1 + \max\{u\text{-}S\text{-}fd_R(A), u\text{-}S\text{-}fd_R(B)\}$.*
2. *If u-S-$fd_R(B) < u$-S-$fd_R(C)$, then u-S-$fd_R(A) = u$-S-$fd_R(C) - 1 > u$-S-$fd_R(B)$.*

Proof. The proof is similar with that of the classical case (see [5, Theorem 3.6.7]). So we omit it. $\square$

Let $\mathfrak{p}$ be a prime ideal of R and M an R-module. We denote u-$\mathfrak{p}$-$fd_R(M)$ to be u-$(R-\mathfrak{p})$-$fd_R(M)$ briefly. The next result gives a new local characterization of flat dimension of an R-module.

Proposition 2.8. *Let R be a ring and M an R-module. Then*

$$\begin{aligned} fd_R(M) &= \sup\{u\text{-}\mathfrak{p}\text{-}fd_R(M) : \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &= \sup\{u\text{-}\mathfrak{m}\text{-}fd_R(M) : \mathfrak{m} \in \mathrm{Max}(R)\}. \end{aligned}$$

Proof. Trivially, $\sup\{u\text{-}\mathfrak{m}\text{-}fd_R(M) : \mathfrak{m} \in \mathrm{Max}(R)\} \leq \sup\{u\text{-}\mathfrak{p}\text{-}fd_R(M) : \mathfrak{p} \in \mathrm{Spec}(R)\} \leq fd_R(M)$. Suppose $\sup\{u\text{-}\mathfrak{m}\text{-}fd_R(M) : \mathfrak{m} \in \mathrm{Max}(R)\} = n$. For any R-module N, there exists an element $s^{\mathfrak{m}} \in R - \mathfrak{m}$ such that $s^{\mathfrak{m}}\mathrm{Tor}_{n+1}^R(M,N) = 0$ by Proposition 2.3. Since the ideal generated by all $s^{\mathfrak{m}}$ is R, we have $\mathrm{Tor}_{n+1}^R(M,N) = 0$ for all R-modules N. So $fd_R(M) \leq n$. Suppose $\sup\{u\text{-}\mathfrak{m}\text{-}fd_R(M) : \mathfrak{m} \in \mathrm{Max}(R)\} = \infty$. Then for any $n \geq 0$, there exists a maximal ideal $\mathfrak{m}$ and an element $s^{\mathfrak{m}} \in R - \mathfrak{m}$ such that $s^{\mathfrak{m}}\mathrm{Tor}_{n+1}^R(M,N) \neq 0$ for some R-module N. So for any $n \geq 0$, we have $\mathrm{Tor}_{n+1}^R(M,N) \neq 0$ for some R-module N. Thus $fd_R(M) = \infty$. So the equalities hold. □

3. On the u-S-weak global dimensions of rings

Recall that the weak global dimension w.gl.dim(R) of a ring R is the supremum of flat dimensions of all R-modules. Now, we introduce the u-S-analogue of weak global dimensions of rings R for a multiplicative subset S of R.

Definition 3.1. The u-S-weak global dimension of a ring R is defined by

$$u\text{-}S\text{-w.gl.dim}(R) = \sup\{u\text{-}S\text{-}fd_R(M) : M \text{ is an } R\text{-module}\}.$$

Obviously, u-S-w.gl.dim$(R) \leq$w.gl.dim(R) for any multiplicative subset S of R. And if S is composed of units, then u-S-w.gl.dim$(R) =$ w.gl.dim(R). The next result characterizes the u-S-weak global dimension of a ring R.

Proposition 3.2. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent for R:*

(1) *u-S-w.gl.dim$(R) \leq n$;*
(2) *u-S-$fd_R(M) \leq n$ for all R-modules M;*
(3) *$\mathrm{Tor}_{n+k}^R(M,N)$ is u-S-torsion for all R-modules M, N and all $k > 0$;*
(4) *$\mathrm{Tor}_{n+1}^R(M,N)$ is u-S-torsion for all R-modules M, N;*
(5) *there exists an element $s \in S$ such that $s\mathrm{Tor}_{n+1}^R(R/I, R/J)$ for any ideals I and J of R.*

Proof. (1) ⇒ (2) and (3) ⇒ (4): These are trivial.

(2) ⇒ (3): This follows from Proposition 2.3.

(4) ⇒ (1): Let M be an R-module and $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules. To complete the proof, it suffices, by Proposition 2.3, to prove that F_n is u-S-flat. Let N be an R-module. Thus u-S-$fd_R(N) \leq n$ by (4). It follows from Corollary 1.5 that $\mathrm{Tor}_1^R(N, F_n) \cong \mathrm{Tor}_{n+1}^R(N, M)$ is u-S-torsion. Thus F_n is u-S-flat.

(4) $\Rightarrow$ (5): Let $M = \bigoplus_{I \trianglelefteq R} R/I$ and $N = \bigoplus_{J \trianglelefteq R} R/J$. Then there exists $s \in S$ such that

$$s\mathrm{Tor}_{n+1}^R(M,N) = s \bigoplus_{I \trianglelefteq R, J \trianglelefteq R} \mathrm{Tor}_{n+1}^R(R/I, R/J) = 0.$$

Thus $s\mathrm{Tor}_{n+1}^R(R/I, R/J) = 0$ for any ideals I, J of R.

(5) $\Rightarrow$ (4): Suppose M is generated by $\{m_i : i \in \Gamma\}$ and N is generated by $\{n_i : i \in \Lambda\}$. Set $M_0 = 0$ and $M_\alpha = \langle m_i : i < \alpha \rangle$ for each $\alpha \leq \Gamma$. Then M has a continuous filtration $\{M_\alpha : \alpha \leq \Gamma\}$ with $M_{\alpha+1}/M_\alpha \cong R/I_{\alpha+1}$ and $I_\alpha = \mathrm{Ann}_R(m_\alpha + M_\alpha \cap Rm_\alpha)$. Similarly, N has a continuous filtration $\{N_\beta : \beta \leq \Lambda\}$ with $N_{\beta+1}/N_\beta \cong R/J_{\beta+1}$ and $J_\beta = \mathrm{Ann}_R(n_\beta + N_\beta \cap Rn_\beta)$. Since $s\mathrm{Tor}_{n+1}^R(R/I_\alpha, R/J_\beta) = 0$ for each $\alpha \leq \Gamma$ and $\beta \leq \Lambda$, it is easy to verify $s\mathrm{Tor}_{n+1}^R(M,N) = 0$ by transfinite induction on both positions of M and N. $\square$

The following Corollaries 3.4, 3.5, 3.3 and 3.7 can be deduced by Corollaries 2.5, 2.6, 2.4 and Proposition 2.8.

Corollary 3.3. *Let R be a ring and $S' \subseteq S$ multiplicative subsets of R. Then u-S-$w.gl.dim(R) \leq S'$-$w.gl.dim(R)$.*

Corollary 3.4. *Let R be a ring and S a multiplicative subset of R. If u-S-$w.gl.dim(R) \leq n$, then there exists an element $s \in S$ such that $w.gl.dim(R_s) \leq n$.*

Corollary 3.5. *Let R be a ring and S a multiplicative subset of R. Then u-S-$w.gl.dim(R) \leq w.gl.dim(R_S)$. Moreover, if S is composed of finite elements, then u-S-$w.gl.dim(R) =w.gl.dim(R_S)$.*

The following example shows that the reverse inequality of Corollary 3.5 does not hold in general.

Example 3.6. Let $R = k[x_1, x_2, \ldots, x_{n+1}]$ be a polynomial ring with $n+1$ indeterminates over a field k ($n \geq 0$). Set $S = k[x_1] - \{0\}$. Then S is a multiplicative subset of R and $R_S = k(x_1)[x_2, \ldots, x_{n+1}]$ is a polynomial ring with n indeterminates over the field $k(x_1)$. So $\mathrm{w.gl.dim}(R_S) = n$ by [5, Theorem 3.8.23]. Let $s \in S$. Then we have $R_s = k[x_1]_s[x_2, \ldots, x_{n+1}]$. Since $k[x_1]$ is not a G-domain, $k[x_1]_s$ is not a field (see [4, Theorem 21]). Thus $\mathrm{w.gl.dim}(k[x_1]_s) = 1$. So $\mathrm{w.gl.dim}(R_s) = n+1$ for any $s \in S$ by [5, Theorem 3.8.23] again. Consequently u-S-$\mathrm{w.gl.dim}(R) \geq n+1$ by Corollary 3.4.

Let $\mathfrak{p}$ be a prime ideal of a ring R and u-$\mathfrak{p}$-$w.gl.dim(R)$ denote u-$(R-\mathfrak{p})$-$w.gl.dim(R)$ briefly. We have a new local characterization of weak global dimensions of commutative rings.

Corollary 3.7. *Let R be a ring. Then*

$$\begin{aligned} w.gl.dim(R) &= \sup\{u\text{-}\mathfrak{p}\text{-}w.gl.dim(R) : \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &= \sup\{u\text{-}\mathfrak{m}\text{-}w.gl.dim(R) : \mathfrak{m} \in \mathrm{Max}(R)\}. \end{aligned}$$

The rest of this section mainly consider rings with u-S-weak global dimensions at most one. Recall from [7] that a ring R is called u-S-von Neumann regular provided that there exist $s \in S$ and $r \in R$ such that $sa = ra^2$ for any $a \in R$. Thus by [7, Theorem 3.11], the following result holds.

Corollary 3.8. *Let R be a ring and S a multiplicative subset of R. The following assertions are equivalent*:

(1) *R is a u-S-von Neumann regular ring*;
(2) *for any R-module M and N, there exists $s \in S$ such that $s\mathrm{Tor}_1^R(M, N) = 0$*;
(3) *there exists $s \in S$ such that $s\mathrm{Tor}_1^R(R/I, R/J) = 0$ for any ideals I and J of R*;
(4) *any R-module is u-S-flat*;
(5) *u-S-w.gl.dim$(R) = 0$.*

Trivially, von Neumann regular rings are u-S-von Neumann regular, and if a ring R is a u-S-von Neumann regular ring, then R_S is von Neumann regular. It was proved in [7, Proposition 3.17] that if the multiplicative subset S of R is composed of non-zero-divisors, then R is u-S-von Neumann regular if and only if R is von Neumann regular. Examples of u-S-von Neumann regular rings that are not von Neumann regular, and a ring R for which R_S is von Neumann regular but R is not u-S-von Neumann regular are given in [7].

Proposition 3.9. *Let R be a ring and S a multiplicative subset of R. The following assertions are equivalent*:

(1) *u-S-w.gl.dim$(R) \leq 1$*;
(2) *any submodule of u-S-flat modules is u-S-flat*;
(3) *any submodule of flat modules is u-S-flat*;
(4) *$\mathrm{Tor}_2^R(M, N)$ is u-S-torsion for all R-modules M, N*;
(5) *there exists an element $s \in S$ such that $s\mathrm{Tor}_2^R(R/I, R/J) = 0$ for any ideals I, J of R.*

Proof. The equivalences follow from Proposition 3.2. □

The following lemma can be found in [2, Chapter 1 Exercise 6.3] for integral domains. However it is also true for any commutative ring and we give a proof for completeness.

Lemma 3.10. *Let R be a ring and I, J ideals of R. Then $\mathrm{Tor}_2^R(R/I, R/J) \cong \mathrm{Ker}(\phi)$, where $\phi : I \otimes J \to IJ$ is an R-homomorphism defined by $\phi(a \otimes b) = ab$.*

Proof. Let I and J be ideals of R. Then $\mathrm{Tor}_2^R(R/I, R/J) \cong \mathrm{Tor}_1^R(R/I, J)$. Consider the following exact sequence: $0 \to \mathrm{Tor}_1^R(R/I, J) \to I \otimes_R J \xrightarrow{\phi} R \otimes_R J$, where ϕ is an R-homomorphism such that $\phi(a \otimes b) = ab$. We have $\mathrm{Tor}_2^R(R/I, R/J) \cong \mathrm{Ker}(\phi)$. □

Trivially, a ring R with w.gl.dim$(R) \leq 1$ has a u-S-weak global dimension at most one. The following example shows the converse does not hold generally.

Example 3.11. Let A be a ring with w.gl.dim$(A) = 1$, $T = A \times A$ the direct product of A. Set $s = (1, 0) \in T$. Then $s^2 = s$. Let $R = T[x]/\langle sx, x^2\rangle$ with x an indeterminate and $S = \{1, s\}$ be a multiplicative subset of R. Then u-S-w.gl.dim$(R) = 1$ but w.gl.dim$(R) = \infty$.

Proof. Since $x^2 = 0$ and $sx = 0$ in R, every element in R can uniquely be written as $r = (a, b) + (0, c)x$ where $a, b, c \in A$. Let $f : R \to A$ be a ring homomorphism defined by $f((a, b) + (0, c)x) = a$. Then f makes A a module retract of R. Let I and J be ideals of R. Let $r_1 = (a_1, b_1) + (0, c_1)x$ and $r_2 = (a_2, b_2) + (0, c_2)$ be elements in I and J, respectively, such that $r_1 \otimes r_2 \in \mathrm{Ker}(\phi)$, where $\phi : I \otimes_R J \to IJ$ is the multiplicative homomorphism. Then $r_1r_2 = (a_1a_2, b_1b_2) + (0, b_1c_2 + b_2c_1)x = 0$, and so $a_1a_2 = 0$ in A. By Lemma 3.10, $a_1 \otimes_A a_2 = 0$ in $f(I) \otimes_A f(J)$ since w.gl.dim$(A) = 1$. Consequently $s^2r_1 \otimes_R r_2 = sr_1 \otimes_R sr_2 = (a_1, 0) \otimes_R (a_2, 0) = 0$ in $I \otimes J$. So $s^2\mathrm{Tor}^R_2(R/I, R/J) = 0$ by Lemma 3.10. It follows that u-S-w.gl.dim$(R) \leq 1$ by Proposition 3.9. Since $R_S \cong A$ has a weak global dimension 1, u-S-w.gl.dim$(R) = 1$ by Corollary 3.8 and [7, Corollary 3.14]. Since R is a non-reduced coherent ring, it follows from [3, Corollary 4.2.4] that w.gl.dim$(R) = \infty$. $\square$

4. *U*-*S*-weak global dimensions of factor rings and polynomial rings

In this section, we mainly consider the u-S-weak global dimensions of factor rings and polynomial rings. Firstly, we give an inequality of u-S-weak global dimensions for ring homomorphisms. Let $\theta : R \to T$ be a ring homomorphism. Let S be a multiplicative subset of R. Then $\theta(S) = \{\theta(s) : s \in S\}$ is a multiplicative subset of T.

Lemma 4.1. *Let $\theta : R \to T$ be a ring homomorphism and S a multiplicative subset of R. Suppose L is a u-$\theta(S)$-flat T-module. Then for any R-module X and any $n \geq 0$, $\mathrm{Tor}^R_n(X, L)$ is u-S-isomorphic to $\mathrm{Tor}^R_n(X, T) \otimes_T L$. Consequently, u-S-$fd_R(L) \leq u$-S-$fd_R(T)$.*

Proof. If $n = 0$, then $X \otimes_R L \cong X \otimes_R (T \otimes_T L) \cong (X \otimes_R T) \otimes_T L$.

If $n = 1$, let $0 \to A \to P \to X \to 0$ be an exact sequence of R-modules where P is free. Thus we have two exact sequences of T-modules: $0 \to \mathrm{Tor}^R_1(X, T) \to A \otimes_R T \to P \otimes_R T \to X \otimes_R T \to 0$ and $0 \to \mathrm{Tor}^R_1(X, L) \to A \otimes_R L \to P \otimes_R L \to X \otimes_R L \to 0$. Consider the following commutative diagram with exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Tor}^R_1(X,L) \arrow[r] \arrow[d, "h"'] & A \otimes_R L \arrow[r] \arrow[d, "\cong"] & P \otimes_R L \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Ker}(\delta) \arrow[r] & \mathrm{Tor}^R_1(X,T) \otimes_T L \arrow[r, "\delta"] & (A \otimes_R T) \otimes_T L \arrow[r] & (P \otimes_R T) \otimes_T L.
\end{tikzcd}$$

Since L is a u-$\theta(S)$-flat T-module, δ is a u-$\theta(S)$-monomorphism. By Theorem 1.2, h is a u-$\theta(S)$-isomorphism over T. So h is a u-S-isomorphism over R since T-modules are viewed as R-modules through θ. By dimension-shifting, we can obtain that $\mathrm{Tor}_n^R(X,L)$ is u-S-isomorphic to $\mathrm{Tor}_n^R(X,T)\otimes_T L$ for any R-module X and any $n \geq 0$.

Thus for any R-module X, if $\mathrm{Tor}_n^R(X,T)$ is u-S-torsion, then $\mathrm{Tor}_n^R(X,L)$ is also u-S-torsion. Consequently, $u\text{-}S\text{-}fd_R(L) \leq u\text{-}S\text{-}fd_R(T)$. □

Proposition 4.2. *Let $\theta : R \to T$ be a ring homomorphism and S a multiplicative subset of R. Let M be an T-module. Then*

$$u\text{-}S\text{-}fd_R(M) \leq u\text{-}\theta(S)\text{-}fd_T(M) + u\text{-}S\text{-}fd_R(T).$$

Proof. Assume $u\text{-}\theta(S)\text{-}fd_T(M) = n < \infty$. If $n = 0$, then M is u-$\theta(S)$-flat over T. By Lemma 4.1, $u\text{-}S\text{-}fd_R(M) \leq n + u\text{-}S\text{-}fd_R(T)$.

Now we assume $n > 0$. Let $0 \to A \to F \to M \to 0$ be an exact sequence of T-modules, where F is a free T-module. Then $u\text{-}\theta(S)\text{-}fd_T(A) = n-1$ by Corollary 1.5 and Proposition 2.3. By induction, $u\text{-}S\text{-}fd_R(A) \leq n - 1 + u\text{-}S\text{-}fd_R(T)$. Note that $u\text{-}S\text{-}fd_R(T) = u\text{-}S\text{-}fd_R(F)$. By Proposition 2.7, we have

$$\begin{aligned} u\text{-}S\text{-}fd_R(M) &\leq 1 + \max\{u\text{-}S\text{-}fd_R(F), u\text{-}S\text{-}fd_R(A)\} \\ &\leq 1 + n - 1 + u\text{-}S\text{-}fd_R(T) \\ &= u\text{-}\theta(S)\text{-}fd_T(M) + u\text{-}S\text{-}fd_R(T). \end{aligned}$$

□

Let R be a ring, I an ideal of R and S a multiplicative subset of R. Then $\pi : R \to R/I$ is a ring epimorphism and $\pi(S) := \overline{S} = \{s + I \in R/I : s \in S\}$ is naturally a multiplicative subset of R/I.

Proposition 4.3. *Let R be a ring and S a multiplicative subset of R. Let $a \in R$ be an element such that $u\text{-}S\text{-}fd_R(R/aR) = 1$. Written $\overline{R} = R/aR$ and $\overline{S} = \{s + aR \in \overline{R} : s \in S\}$. Then the following assertions hold.*

(1) *Let M be a nonzero $\overline{R}$-module. If $u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) < \infty$, then*
$$u\text{-}S\text{-}fd_R(M) = u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) + 1.$$

(2) *If $u\text{-}\overline{S}\text{-}w.gl.dim(\overline{R}) < \infty$, then*
$$u\text{-}S\text{-}w.gl.dim(R) \geq u\text{-}\overline{S}\text{-}w.gl.dim(\overline{R}) + 1.$$

Proof. (1) Set $u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) = n$. By Proposition 4.2, we have $u\text{-}S\text{-}fd_R(M) \leq u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) + 1 = n + 1$. Since $u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) = n$, then there is an injective $\overline{R}$-module C such that $\mathrm{Tor}_n^{\overline{R}}(M,C)$ is not u-$\overline{S}$-torsion. By [5, Theorem 2.4.22], there is an injective R-module E such that $0 \to C \to E \to E \to 0$ is exact. By [5, Proposition 3.8.12(4)], $\mathrm{Tor}_{n+1}^R(M,E) \cong \mathrm{Tor}_n^{\overline{R}}(M,C)$. Thus $\mathrm{Tor}_{n+1}^R(M,E)$ is not u-S-torsion. So $u\text{-}S\text{-}fd_R(M) = u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) + 1$.

(2) Let $n = u\text{-}\overline{S}\text{-}\mathrm{w.gl.dim}(\overline{R})$. Then there is a nonzero $\overline{R}$-module M such that $u\text{-}\overline{S}\text{-}fd_{\overline{R}}(M) = n$. Thus $u\text{-}S\text{-}fd_R(M) = n+1$ by (1). So $u\text{-}S\text{-}\mathrm{w.gl.dim}(R)$ $\geq u\text{-}\overline{S}\text{-}\mathrm{w.gl.dim}(\overline{R}) + 1$. □

Let R be a ring and M an R-module. Denote by $R[x]$ the polynomial ring with one indeterminate, where all coefficients are in R. Set $M[x] = M \otimes_R R[x]$. Then $M[x]$ can be seen as an $R[x]$-module naturally. It is well-known w.gl.dim($R[x]$) =w.gl.dim(R) (see [5, Theorem 3.8.23]). In this section, we give a u-S-analogue of this result. Let S be a multiplicative subset of R. Then S is a multiplicative subset of $R[x]$ naturally.

Lemma 4.4. *Let R be a ring and S a multiplicative subset of R. Let T be an R-module and F an $R[x]$-module. Then the following assertions hold.*

(1) *T is u-S-torsion over R if and only if $T[x]$ is a u-S-torsion $R[x]$-module.*
(2) *If F is u-S-flat over $R[x]$, then F is u-S-flat over R.*

Proof. (1) If $sT[x] = 0$ for some $s \in S$, then trivially $sT = 0$. So T is u-S-torsion over R. Suppose $sT = 0$ for some $s \in S$. Then $sT[x] \cong (sT)[x] = 0$. Thus $T[x]$ is a u-S-torsion $R[x]$-module.

(2) Suppose F is a u-S-flat $R[x]$-module. By [3, Theorem 1.3.11],

$$\mathrm{Tor}_1^R(F, L)[x] \cong \mathrm{Tor}_1^{R[x]}(F[x], L[x])$$

is u-S-torsion. Thus there exists an element $s \in S$ such that $s\mathrm{Tor}_1^R(F, L)[x] = 0$. So $s\mathrm{Tor}_1^R(F, L) = 0$. It follows that F is a u-S-flat R-module. □

Proposition 4.5. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then u-S-$fd_{R[x]}(M[x]) = u$-S-$fd_R(M)$.*

Proof. Assume that u-S-$fd_R(M) \leq n$. Then $\mathrm{Tor}_{n+1}^R(M, N)$ is u-S-torsion for any R-module N by Proposition 2.3. Thus for any $R[x]$-module L,

$$\mathrm{Tor}_{n+1}^{R[x]}(M[x], L) \cong \mathrm{Tor}_{n+1}^R(M, L)$$

is u-S-torsion for any $R[x]$-module L by [3, Theorem 1.3.11]. Consequently, u-S-$fd_{R[x]}(M[x]) \leq n$ by Proposition 2.3.

Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M[x] \to 0$ be an exact sequence with each F_i u-S-flat over $R[x]$ $(1 \leq i \leq n)$. Then it is also a u-S-flat resolution of $M[x]$ over R by Lemma 4.4. Thus $\mathrm{Tor}_{n+1}^R(M[x], N)$ is u-S-torsion for any R-module N by Proposition 2.3. It follows that $s\mathrm{Tor}_{n+1}^R(M[x], N) = s\bigoplus_{i=1}^{\infty} \mathrm{Tor}_{n+1}^R(M, N) = 0$. Thus $\mathrm{Tor}_{n+1}^R(M, N)$ is u-S-torsion. Consequently, u-S-$fd_R(M) \leq u$-S-$fd_{R[x]}(M[x])$ by Proposition 2.3 again. □

Let M be an $R[x]$-module. Then M can be naturally viewed as an R-module. Define $\psi : M[x] \to M$ by

$$\psi(\sum_{i=0}^{n} x^i \otimes m_i) = \sum_{i=0}^{n} x^i m_i, \qquad m_i \in M.$$

And define $\varphi : M[x] \to M[x]$ by

$$\varphi(\sum_{i=0}^{n} x^i \otimes m_i) = \sum_{i=0}^{n} x^{i+1} \otimes m_i - \sum_{i=0}^{n} x^i \otimes x m_i, \qquad m_i \in M.$$

Lemma 4.6 ([5, Theorem 3.8.22]). *Let R be a ring. For any $R[x]$-module M,*

$$0 \to M[x] \xrightarrow{\varphi} M[x] \xrightarrow{\psi} M \to 0$$

is exact.

Theorem 4.7. *Let R be a ring and S a multiplicative subset of R satisfying u-S-$fd_{R[x]}(R) = 1$. Then u-S-w.gl.dim$(R[x]) = u$-S-w.gl.dim$(R) + 1$.*

Proof. Let M be an $R[x]$-module. Then, by Lemma 4.6, there is an exact sequence over $R[x]$:

$$0 \to M[x] \to M[x] \to M \to 0.$$

By Proposition 2.7 and Proposition 4.5,

$$(*) \quad u\text{-}S\text{-}fd_R(M) \leq u\text{-}S\text{-}fd_{R[x]}(M) \leq 1 + u\text{-}S\text{-}fd_{R[x]}(M[x]) = 1 + u\text{-}S\text{-}fd_R(M).$$

Thus if u-S-w.gl.dim$(R) < \infty$, then u-S-w.gl.dim$(R[x]) < \infty$.

Conversely, if u-S-w.gl.dim$(R[x]) < \infty$, then for any R-module M, u-S-$fd_R(M) = u$-S-$fd_{R[x]}(M[x]) < \infty$ by Proposition 4.5. Therefore we have u-S-w.gl.dim$(R) < \infty$ if and only if u-S-w.gl.dim$(R[x]) < \infty$. Now we assume that both of these are finite. Then u-S-w.gl.dim$(R[x]) \leq u$-S-w.gl.dim$(R) + 1$ by $(*)$. Since $R \cong R[x]/xR[x]$ and u-S-$fd_{R[x]}(R[x]/xR[x]) = u$-$S$-$fd_{R[x]}(R) = 1$, u-S-w.gl.dim$(R[x]) \geq u$-S-w.gl.dim$(R) + 1$ by Proposition 4.3. Consequently, we have u-S-w.gl.dim$(R[x]) = u$-S-w.gl.dim$(R) + 1$. □

Remark 4.8. Remark that u-S-$fd_{R[x]}(R) = 1$ in the following cases.

(1) $S = \mathrm{U(R)}$ is consist of units.
(2) $R = R_1 \times R_2$ and $S = \mathrm{U(R_1)} \times 0$.

We also remark that u-S-w.gl.dim$(R[x])$ may not be equal to u-S-w.gl.dim$(R)+$ 1 in general. For example, let S be a multiplicative subset of R that contains 0. Then u-S-w.gl.dim$(R) = u$-S-w.gl.dim$(R[x]) = 0$.

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