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GEOMETRY OF BILINEAR FORMS ON A NORMED SPACE $\mathbb{R}^n$

Sung Guen Kim

Abstract. For every $n \geq 2$, let $\mathbb{R}^n_{\|\cdot\|}$ be $\mathbb{R}^n$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points more than $2n$. We devote to the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ and $\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$, where $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ is the space of bilinear forms on $\mathbb{R}^n_{\|\cdot\|}$, and $\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$ is the subspace of $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ consisting of symmetric bilinear forms. Let $\mathcal{F} = \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ or $\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$. First we classify the extreme and exposed points of the closed unit ball of $\mathcal{F}$. We also show that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. It is shown that $\operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$ and $\exp B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})} = \exp B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$, which expand some results of [18, 23, 28, 29, 35, 38, 40, 41, 43].

1. Introduction

Throughout the paper, we let $n, m \in \mathbb{N}$, $n, m \geq 2$. We write B_E and S_E for the closed unit ball and sphere of a real Banach space E. The dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{ext} B_E$, $\exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm

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$\|T\| = \sup_{\|x_k\|=1} |T(x_1,\ldots,x_n)|$. $\mathcal{L}_s({}^nE)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}({}^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} x^{(1),i} \otimes \cdots \otimes x^{(n),i},\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i},\ldots,x^{(n),i}\Big).$$

Notice also that $\mathcal{L}_s({}^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!}\Big(\sum_{\sigma} x^{\sigma(1),i} \otimes \cdots \otimes x^{\sigma(n),i}\Big),\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i},\ldots,x^{(n),i}\Big),$$

where σ goes over all permutations on $\{1,\ldots,n\}$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [8].

Let us sketch the history of classification problems of the extreme points, the exposed points and smooth points of the unit ball of continuous n-homogeneous polynomials on a Banach space.

We let $l_p^n = \mathbb{R}^n$ for every $1 \le p \le \infty$ equipped with the l_p-norm. Choi and Kim [3] initiated and classified $\operatorname{ext} B_{\mathcal{P}({}^2l_2^2)}$ and $\operatorname{sm} B_{\mathcal{P}({}^2l_2^2)}$. Choi, Ki and Kim [7] classified $\operatorname{ext} B_{\mathcal{P}({}^2l_1^2)}$. Choi and Kim [5, 6] classified $\operatorname{sm} B_{\mathcal{P}({}^2l_1^2)}$ and $\exp B_{\mathcal{P}({}^2l_p^2)}$ for $p = 1, 2, \infty$. Grecu [12] classified $\operatorname{ext} B_{\mathcal{P}({}^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim and Lee [45] showed that if E is a separable real Hilbert space with $\dim(E) \ge 2$, then, $\operatorname{ext} B_{\mathcal{P}({}^2E)} = \exp B_{\mathcal{P}({}^2E)}$. Kim [17] classified $\exp B_{\mathcal{P}({}^2l_p^2)}$ for $1 \le p \le \infty$. Kim [19, 21] characterized $\operatorname{ext} B_{\mathcal{P}({}^2d_*(1,w)^2)}$ and $\operatorname{sm} B_{\mathcal{P}({}^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_w = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. Kim [26] classified $\exp B_{\mathcal{P}({}^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}({}^2d_*(1,w)^2)} \ne \operatorname{ext} B_{\mathcal{P}({}^2d_*(1,w)^2)}$. Recently, Kim [31, 34] classified $\operatorname{ext} B_{\mathcal{P}({}^2\mathbb{R}^2_{h(1/2)})}$ and $\exp B_{\mathcal{P}({}^2\mathbb{R}^2_{h(1/2)})}$, where $\mathbb{R}^2_{h(1/2)} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(1/2)} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}({}^nE)}, \exp B_{\mathcal{P}({}^nE)}$ and $\operatorname{sm} B_{\mathcal{P}({}^nE)}$, it seems to be very natural to study the classification problems of the extreme points, the exposed points and smooth points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

Kim [18] initiated and classified $\operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$, $\exp B_{\mathcal{L}_s({}^2l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^2)} = \exp B_{\mathcal{L}_s({}^2l_\infty^2)}$. Kim [20, 22, 23, 25] classified $\operatorname{ext} B_{\mathcal{L}_s({}^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}({}^2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s({}^2d_*(1,w)^2)}$, and $\exp B_{\mathcal{L}({}^2d_*(1,w)^2)}$.

Kim [29, 30] also classified $\operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^3)}$ and $\exp B_{\mathcal{L}_s({}^3l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^3)} = \exp B_{\mathcal{L}_s({}^2l_\infty^3)}$ and $\operatorname{ext} B_{\mathcal{L}_s({}^3l_\infty^2)} = \exp B_{\mathcal{L}_s({}^3l_\infty^2)}$. Kim [33] characterized $\operatorname{ext} B_{\mathcal{L}({}^2l_\infty^n)}$ and $\operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^n)}$, and showed that $\exp B_{\mathcal{L}({}^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}({}^2l_\infty^n)}$ and $\exp B_{\mathcal{L}_s({}^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^n)}$. Kim [35] characterized $\operatorname{ext} B_{\mathcal{L}({}^2l_\infty^3)}$ and $\exp B_{\mathcal{L}({}^2l_\infty^3)}$. Kim [36] characterized $\operatorname{sm} B_{\mathcal{L}_s({}^nl_\infty^2)}$. Kim [37] studied $\operatorname{ext} B_{\mathcal{L}({}^2l_\infty)}$. Cavalcante et al. [2] characterized $\operatorname{ext} B_{\mathcal{L}({}^nl_\infty^m)}$. Kim [40] classified $\operatorname{ext} B_{\mathcal{L}({}^nl_\infty^2)}$ and $\operatorname{ext} B_{\mathcal{L}_s({}^nl_\infty^2)}$. It was shown that $|\operatorname{ext} B_{\mathcal{L}({}^nl_\infty^2)}| = 2^{(2^n)}$ and $|\operatorname{ext} B_{\mathcal{L}_s({}^nl_\infty^2)}| = 2^{n+1}$, and that $\exp B_{\mathcal{L}({}^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}({}^nl_\infty^2)}$ and $\exp B_{\mathcal{L}_s({}^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s({}^nl_\infty^2)}$. Kim [39, 42] characterized $\operatorname{ext} B_{\mathcal{L}_s({}^nl_\infty^m)}, \operatorname{ext} B_{\mathcal{L}({}^nl_\infty^m)}, \exp B_{\mathcal{L}_s({}^ml_\infty^m)}, \exp B_{\mathcal{L}({}^nl_\infty^m)}$, $\operatorname{sm} B_{\mathcal{L}_s({}^nl_\infty^m)}$ and $\operatorname{sm} B_{\mathcal{L}({}^nl_\infty^m)}$ for every $n, m \geq 2$. Kim [44] characterized $\operatorname{ext} B_{\mathcal{L}_s({}^ml_1)}, \operatorname{ext} B_{\mathcal{L}({}^ml_1)}, \exp B_{\mathcal{L}_s({}^ml_1)}, \exp B_{\mathcal{L}({}^ml_1)}, \operatorname{sm} B_{\mathcal{L}_s({}^ml_1^n)}$ and $\operatorname{sm} B_{\mathcal{L}({}^ml_1^n)}$ for $n, m \geq 2$. Recently, Kim [43] characterized $\operatorname{ext} B_{\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})}$, $\operatorname{ext} B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})}$, $\exp B_{\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})}$, and $\exp B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})}$ if $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = 2m$ for $m \geq 2$. It was shown that every extreme point is exposed in this case.

We refer to ([1–7, 9–15, 17–54] and references therein) for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

For every $n \geq 2$, let $\mathbb{R}^n_{\|\cdot\|}$ be $\mathbb{R}^n$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points more than $2n$. We devote to the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}({}^2\mathbb{R}^n_{\|\cdot\|})$ and $\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$. Let $\mathcal{F} = \mathcal{L}({}^2\mathbb{R}^n_{\|\cdot\|})$ or $\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$. First we classify the extreme and exposed points of the closed unit ball of $\mathcal{F}$. We also show that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. It is shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}({}^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$ and $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})} = \exp B_{\mathcal{L}({}^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$. We expand some results of [18, 23, 28, 29, 35, 38, 40, 41, 43].

2. Extreme and exposed points of $\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$

Throughout the paper, we let $n \geq 2$ and $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm $\|\cdot\|$ such that $B_{\mathbb{R}^n_{\|\cdot\|}}$ has finitely many extreme points more than $2n$. Let $\operatorname{ext} B_{\mathbb{R}^n_{\|\cdot\|}} = \{\pm U_1, \ldots, \pm U_m\}$ for some $m \geq n$ and $U_i \neq U_j$ for $1 \leq i \neq j \leq m$. Let

$$F_{ls} := \frac{x_l y_s + x_s y_l}{2} \text{ for } 1 \leq l \leq s \leq n.$$

Notice that $\{F_{ls} : 1 \leq l \leq s \leq n\}$ is a basis for $\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$. Hence, $\dim(\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})) = \frac{n(n+1)}{2}$. By Mazur's theorem, $B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$ is compact and convex. By the Krein-Milman theorem, $\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$ is nonempty.

Let $T \in \mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$. Then

$$T\Big((x_1, \ldots, x_n),\ (y_1, \ldots, y_n)\Big) = \sum_{1 \leq l, s \leq n} a_{ls} x_l y_s = \sum_{1 \leq l \leq n} a_{ll} F_{ll} + \sum_{1 \leq l < s \leq n} 2a_{ls} F_{ls}$$

for some $a_{ls} \in \mathbb{R}$.

For simplicity, we denote

$$T = \Big(a_{11}, 2a_{12}, \dots, 2a_{1n}, a_{22}, 2a_{23}, \dots, 2a_{2n}, \dots, a_{n-1n-1}, 2a_{n-1n}, a_{nn}\Big)^t \in \mathbb{R}^{\frac{n(n+1)}{2}}.$$

For $j = 1, \dots, m$, we let $U_j = \sum_{1\le k\le n} \lambda_k^{(j)} e_k$ for some $\lambda_k^{(j)} \in \mathbb{R}$.

It follows that for $1 \le i \le j \le m$,

$$\begin{aligned} T(U_i, U_j) &= T\Big(\sum_{1\le k\le n} \lambda_k^{(i)} e_k, \sum_{1\le k\le n} \lambda_k^{(j)} e_k\Big) = \sum_{1\le k_1,k_2\le n} \lambda_{k_1}^{(i)}\lambda_{k_2}^{(j)}\, T(e_{k_1}, e_{k_2}) \\ &= \sum_{1\le k_1,k_2\le n} \lambda_{k_1}^{(i)}\lambda_{k_2}^{(j)} a_{k_1k_2} = X_{(i,j)} \cdot T, \end{aligned}$$

where

$$\begin{aligned} X_{(i,j)} = \Big(&\lambda_1^{(i)}\lambda_1^{(j)}, \frac{\lambda_1^{(i)}\lambda_2^{(j)} + \lambda_2^{(i)}\lambda_1^{(j)}}{2}, \dots, \frac{\lambda_1^{(i)}\lambda_n^{(j)} + \lambda_n^{(i)}\lambda_1^{(j)}}{2}, \lambda_2^{(i)}\lambda_2^{(j)}, \\ &\frac{\lambda_2^{(i)}\lambda_3^{(j)} + \lambda_3^{(i)}\lambda_2^{(j)}}{2}, \dots, \frac{\lambda_2^{(i)}\lambda_n^{(j)} + \lambda_n^{(i)}\lambda_2^{(j)}}{2}, \dots, \lambda_{n-1}^{(i)}\lambda_{n-1}^{(j)}, \\ &\frac{\lambda_{n-1}^{(i)}\lambda_n^{(j)} + \lambda_n^{(i)}\lambda_{n-1}^{(j)}}{2}, \lambda_n^{(i)}\lambda_n^{(j)}\Big) \in \mathbb{R}^{\frac{n(n+1)}{2}} \end{aligned}$$

and $X_{(i,j)} \cdot T$ denotes the dot product of $X_{(i,j)}$ and T on $\mathbb{R}^{\frac{n(n+1)}{2}}$.

Let $\Gamma := \{(i,j) : 1 \le i \le j \le m\}$. Then $|\Gamma| = \frac{m(m+1)}{2} \ge \frac{n(n+1)}{2}$. Notice that there are at most $\frac{n(n+1)}{2}$ linearly independent vectors in $\{X_{(i,j)} : (i,j) \in \Gamma\}$ since $\{X_{(i,j)} : (i,j) \in \Gamma\} \subseteq \mathbb{R}^{\frac{n(n+1)}{2}}$.

In this section we characterize $\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$ and $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$, which expand some results of [18, 23, 28, 29, 35, 38, 40, 41, 43]. First, we present some examples.

Examples. (a) Let $n \ge 2$ and $\mathbb{R}^n_{\|\cdot\|} = l^n_\infty$. Then

$$\operatorname{ext} B_{l^n_\infty} = \Big\{ \pm (1, t_2, \dots, t_n) : t_j = \pm 1, j = 2, \dots, n\Big\}.$$

Hence, $2n \le |\operatorname{ext} B_{l^n_\infty}| = 2^n$.

(b) Let $0 < w < 1$ and $\mathbb{R}^2_{\|\cdot\|} = \mathbb{R}^2_{*(w)}$ with the octagonal norm $\|(x,y)\|_{*(w)} = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$. Then $2 \cdot 2 < \Big|\operatorname{ext} B_{\mathbb{R}^2_{*(w)}}\Big| = 8$.

(c) Let $0 < w < 1$ and $\mathbb{R}^2_{\|\cdot\|} = \mathbb{R}^2_{h(w)}$ with the hexagonal norm $\|(x,y)\|_{h(w)} = \max\Big\{|y|, |x| + w|y|\Big\}$. Then $2 \cdot 2 < \Big|\operatorname{ext} B_{\mathbb{R}^2_{h(w)}}\Big| = 6$.

(d) Let $\mathbb{R}^6_{\|\cdot\|} = \mathbb{R}^6$ with the $\mathcal{L}({}^2 l^2_\infty)$-norm

$$\begin{aligned} \Big\|(a,b,c,d,e,f)\Big\|_{\mathcal{L}_s({}^2 l^2_\infty)} := \max\Big\{&|a|,\ |b|,\ |d|,\ \frac{1}{2}\Big(|a-d| + |e|\Big), \\ &\frac{1}{2}\Big(|b-d| + |f|\Big), \frac{1}{4}\Big(|a+b-2d| + |c|\Big), \end{aligned}$$

$$\frac{1}{4}\Big|\ |a+b-2d|-|c|\ \Big|+\frac{1}{2}|e-f|\Big\}.$$

Kim [41, Theorem 2] showed that $2\cdot 6 < \Big|\operatorname{ext} B_{\mathbb{R}^6_{\|\cdot\|}}\Big| = 26$.

We present an explicit formulae for the norm of $T\in\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$.

Theorem 2.1. *Let $n\geq 2$ and let $\mathbb{R}^n_{\|\cdot\|}=\mathbb{R}^n$ with the norm $\|\cdot\|$ be such that*

$$\operatorname{ext} B_{\mathbb{R}^n_{\|\cdot\|}} = \{\pm U_1,\dots,\pm U_m\}$$

for some $m\geq n$ and $U_i\neq U_j$ for $1\leq i\neq j\leq m$.

(a) *If $T\in\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$, then*

$$\|T\| = \sup_{1\leq i\leq j\leq m}|T(U_i,U_j)| = \sup_{1\leq k\leq \frac{m(m+1)}{2}}|X_{(i_k,j_k)}\cdot T|.$$

(b) *If $c_{(i,j)}\in\mathbb{R}$ for $(i,j)\in\Gamma$ with $c_{(i,j)}=c_{(j,i)}$, then there is a unique $S\in\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$ such that $S(U_i,U_j)=c_{(i,j)}$ for all $(i,j)\in\Gamma$.*

Proof. It follows from the Krein-Milman theorem and bilinearity of T. □

We are in position to prove the main result in this section.

Theorem 2.2. *Let $n\geq 2$ and let $\mathbb{R}^n_{\|\cdot\|}=\mathbb{R}^n$ with the norm $\|\cdot\|$ be such that*

$$\operatorname{ext} B_{\mathbb{R}^n_{\|\cdot\|}} = \{\pm U_1,\dots,\pm U_m\}$$

for some $m\geq n$ and $U_i\neq U_j$ for $1\leq i\neq j\leq m$. Let $T\in\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$ with $\|T\|=1$. Then $T\in\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$ if and only if there are $\frac{n(n+1)}{2}$ linearly independent vectors $X_{(i_1,j_1)},\dots,X_{(i_{n(n+1)/2},\ j_{n(n+1)/2})}$ in $\mathbb{R}^{\frac{n(n+1)}{2}}$ for some $(i_1,j_1),\dots,(i_{n(n+1)/2},j_{n(n+1)/2})\in\Gamma$ such that $|X_{(i_k,j_k)}\cdot T|=1$ for all $k=1,\dots,\frac{n(n+1)}{2}$.

Proof. ($\Rightarrow$) Suppose that T is extreme.

Claim: There are $\frac{n(n+1)}{2}$ linearly independent vectors $X_{(i_1,j_1)},\dots,X_{(i_{n(n+1)/2},\ j_{n(n+1)/2})}$ in $\mathbb{R}^{\frac{n(n+1)}{2}}$ for some $(i_1,j_1),\dots,(i_{n(n+1)/2},\ j_{n(n+1)/2})\in\Gamma$.

Assume the contrary. Let $N\in\mathbb{N}$ be the largest number of linearly independent vectors among $\{X_{(i,j)}:(i,j)\in\Gamma\}$. Then $N<\frac{n(n+1)}{2}$ and so there are $\epsilon_{(i_k,j_k)}\in\mathbb{R}$ for some $(i_k,j_k)\in\Gamma$ and $k=1,\dots,\frac{n(n+1)}{2}$ such that

$$\mathcal{E} = (\epsilon_{(i_k,j_k)})^t_{1\leq k\leq\frac{n(n+1)}{2}}\neq 0 \text{ and } X_{(i,j)}\cdot\mathcal{E}=0 \text{ all } (i,j)\in\Gamma.$$

Let $T^\pm = T\pm\mathcal{E}$. We will show that $\|T^\pm\|\leq 1$. It follows that for $(i,j)\in\Gamma$,

$$\begin{aligned}|X_{(i,j)}\cdot T^\pm| &\leq \max\Big\{|X_{(i,j)}\cdot T + X_{(i,j)}\cdot\mathcal{E}|,\ |X_{(i,j)}\cdot T - X_{(i,j)}\cdot\mathcal{E}|\Big\}\\ &= |X_{(i,j)}\cdot T|\leq\|T\|=1.\end{aligned}$$

By Theorem 2.1(a), $\|T^{\pm}\| \leq 1$. Since $T^{\pm} \neq T$ and $T = \frac{1}{2}(T^{+} + T^{-})$, T is not extreme. This is a contradiction.

Claim: $|X_{(i_k,j_k)} \cdot T| = 1$ for all $k = 1, \ldots, n(n+1)/2$.

Assume the contrary. There is $k_0 \in \{1, \ldots, n(n+1)/2\}$ such that $|X_{(i_{k_0},j_{k_0})} \cdot T| < 1$. Let $t_0 \in \mathbb{R}$ such that $0 < t_0 < 1 - |X_{(i_{k_0},j_{k_0})} \cdot T|$.

By Theorem 2.1(b), there are $L^{\pm} \in \mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$ such that

$$L^{\pm}(U_i, U_j) := T(U_i, U_j) \text{ for } (i,j) \in \Gamma \backslash \{(i_{k_0}, j_{k_0}), (j_{k_0}, i_{k_0})\} \text{ and}$$
$$L^{\pm}(U_{i_{k_0}}, U_{j_{k_0}}) := T(U_{i_{k_0}}, U_{j_{k_0}}) \pm t_0.$$

By Theorem 2.1(a), $\|L^{\pm}\| \leq 1$ for $l = 1, 2$. Since $L^{\pm} \neq T$ and $T = \frac{1}{2}(L^{+} + L^{-})$, T is not extreme. This is a contradiction.

($\Leftarrow$) Let $S_1, S_2 \in \mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})$ be such that $\|S_l\| = 1$ for $l = 1, 2$ and $T = \frac{1}{2}(S_1 + S_2)$.

Claim: $T = S_l$ for $l = 1, 2$.

Since $\|S_l\| = 1$ for $l = 1, 2$, by Theorem 2.1(a),

$$|X_{(i_k,j_k)} \cdot S_l| \leq 1 \text{ for all } k = 1, \ldots, n(n+1)/2.$$

Let M be the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$-matrix such that the k-th row of M equals to $X_{(i_k,j_k)}$ for $k = 1, \ldots, n(n+1)/2$. Notice that M is an invertible $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$-matrix because rows vectors of M are linearly independent. Since $MT = \frac{1}{2}(MS_1 + MS_2)$, $X_{(i_k,j_k)} \cdot T$ (which is the k-th component of MT) equals to the middle point of the k-th components of MS_1 and MS_2. Hence,

$$X_{(i_k,j_k)} \cdot T = \frac{1}{2}(X_{(i_k,j_k)} \cdot S_1 + X_{(i_k,j_k)} \cdot S_2) \text{ for all } k = 1, \ldots, n(n+1)/2.$$

Since

$$|X_{(i_k,j_k)} \cdot T| = 1 \text{ for all } k = 1, \ldots, n(n+1)/2,$$

we have

$$X_{(i_k,j_k)} \cdot T = X_{(i_k,j_k)} \cdot S_l \text{ for all } k = 1, \ldots, n(n+1)/2 \text{ and } l = 1, 2.$$

Hence, $MT = MS_l$ for $l = 1, 2$. Since M is invertible, $T = S_l$ for $l = 1, 2$. Therefore, T is extreme. □

Using Theorem 2.2, we completely describe $\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})}$.

Theorem 2.3. *Let $n \geq 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Then*

$$\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})} = \Big\{ M^{-1}(c_1, \ldots, c_{n(n+1)/2})^t \in S_{\mathcal{L}_s({}^2\mathbb{R}^n_{\|\cdot\|})} : c_k = \pm 1, \ M \text{ is the invertible } n(n+1)/2 \times n(n+1)/2\text{-matrix such that the } k\text{-the row of } M \text{ equals to } X_{(i_k,j_k)} \text{ for } (i_k, j_k) \in \Gamma \text{ and } k = 1, \ldots, n(n+1)/2 \Big\}.$$

Proof. ($\subseteq$) Let $T \in \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$. By Theorem 2.2, there are $\frac{n(n+1)}{2}$ linearly independent vectors

$$X_{(i_1,j_1)}, \ldots, X_{(i_{n(n+1)/2},\ j_{n(n+1)/2})} \in \mathbb{R}^{\frac{n(n+1)}{2}}$$

for some $(i_1,j_1), \ldots, (i_{n(n+1)/2},\ j_{n(n+1)/2}) \in \Gamma$. Let $c_k = X_{(i_k,j_k)} \cdot T$ for $1 \leq k \leq n(n+1)/2$. By Theorem 2.2, $|c_k| = 1$ for all $k = 1, \ldots, n(n+1)/2$. Notice that

$$T = M^{-1}(c_1, \ldots, c_{n(n+1)/2})^t.$$

($\supseteq$) Let $L := M^{-1}(c_1, \ldots, c_{n(n+1)/2})^t \in S_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$ such that $c_k = \pm 1$ and M is the invertible $n(n+1)/2 \times n(n+1)/2$-matrix such that the k-the row of M equals to $X_{(i_k,j_k)}$ for $(i_k,j_k) \in \Gamma$ and $k = 1, \ldots, n(n+1)/2$. It follows that

$$ML = M(M^{-1}(c_1, \ldots, c_{n(n+1)/2})^t) = (c_1, \ldots, c_{n(n+1)/2})^t,$$

which shows that

$$|X_{(i_k,j_k)} \cdot L| = |c_k| = 1 \text{ for all } k = 1, \ldots, n(n+1)/2.$$

By Theorem 2.2, $L \in \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$. □

Kim [23] showed the following theorem:

Theorem 2.4. *Let E be a real Banach space such that $\operatorname{ext} B_E$ is finite. Suppose that $x \in \operatorname{ext} B_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \operatorname{ext} B_E \backslash \{\pm x\}$. Then $x \in \exp B_E$.*

Using Theorem 2.4, we show that every extreme point of $B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$ is exposed.

Theorem 2.5. *Let $n \geq 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Then the equality $\exp B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$ holds.*

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$. By Theorem 2.2, there are

$$(i_1,j_1), \ldots, (i_{n(n+1)/2},\ j_{n(n+1)/2}) \in \Gamma$$

such that $X_{(i_1,j_1)}, \ldots, X_{(i_{n(n+1)/2},\ j_{n(n+1)/2})}$ are linearly independent in $\mathbb{R}^{\frac{n(n+1)}{2}}$ and $|X_{(i_k,j_k)} \cdot T| = |T(U_{i_k}, U_{j_k})| = 1$ for all $k = 1, \ldots, n(n+1)/2$. Let M be the invertible $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$-matrix such that the k-th row of M equals to $X_{(i_k,j_k)}$ for $k = 1, \ldots, n(n+1)/2$. Let $f \in \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})^*$ be such that

$$f = \frac{2}{n(n+1)} \sum_{k=1}^{\frac{n(n+1)}{2}} \operatorname{sign}(T(U_{i_k}, U_{j_k}))\delta_{(U_{i_k}, U_{j_k})},$$

where $\delta_{(U_i,U_j)}(S) := S(U_i,U_j)$ for $S \in \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$. Then $1 = \|f\| = f(T)$. Let $S \in \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})}$ be such that $|f(S)| = 1$. We will show that $S = T$ or

$S = -T$. It follows that

$$1 = |f(S)| = \Big| \frac{2}{n(n+1)} \sum_{k=1}^{\frac{n(n+1)}{2}} \mathrm{sign}(T(U_{i_k}, U_{j_k}))S(U_{i_k}, U_{j_k}) \Big|$$
$$\leq \frac{2}{n(n+1)} \sum_{k=1}^{\frac{n(n+1)}{2}} |S(U_{i_k}, U_{j_k})| \leq 1,$$

which shows that

$$S(U_{i_k}, U_{j_k}) = \mathrm{sign}(T(U_{i_k}, U_{j_k})) \text{ for } k = 1, \ldots, n(n+1)/2$$

or

$$S(U_{i_k}, U_{j_k}) = -\mathrm{sign}(T(U_{i_k}, U_{j_k})) \text{ for } k = 1, \ldots, n(n+1)/2.$$

Suppose that

$$S(U_{i_k}, U_{j_k}) = -\mathrm{sign}(T(U_{i_k}, U_{j_k})) \text{ for } k = 1, \ldots, n(n+1)/2.$$

Since $|S(U_{i_k}, U_{j_k})| = 1 = |T(U_{i_k}, U_{j_k})|$ for all $k = 1, \ldots, n(n+1)/2$,

$$S(U_{i_k}, U_{j_k}) = -T(U_{i_k}, U_{j_k}) \text{ for all } k = 1, \ldots, n(n+1)/2.$$

It follows that for all $k = 1, \ldots, n(n+1)/2$,

$$X_{(i_k, j_k)} \cdot S = S(U_{i_k}, U_{j_k}) = -T(U_{i_k}, U_{j_k}) = -X_{(i_k, j_k)} \cdot T,$$

which shows that $MS = -MT$. Since M is invertible, $S = -T$. Notice that if $S(U_{i_k}, U_{j_k}) = \mathrm{sign}(T(U_{i_k}, U_{j_k}))$ for $k = 1, \ldots, n(n+1)/2$, then $S = T$. By Theorem 2.4, T is exposed. □

Kim [18, 23, 28, 29, 35, 38, 40, 41] showed that if $n \geq 2$, $0 < w < 1$ and $X = l_\infty^n, \mathbb{R}^2_{*(w)}, \mathbb{R}^2_{h(w)}$ or $\mathcal{L}_s(^2 l_\infty^2)$, then $\exp B_{\mathcal{L}_s(^2 X)} = \mathrm{ext}\, B_{\mathcal{L}_s(^2 X)}$.

Using Theorem 2.5, we obtain the following:

Corollary 2.6. *Let $n \geq 2$, $0 < w < 1$ and $X = l_\infty^n, \mathbb{R}^2_{*(w)}, \mathbb{R}^2_{h(w)}$ or $\mathcal{L}_s(^2 l_\infty^2)$. Then the equality $\exp B_{\mathcal{L}_s(^2 X)} = \mathrm{ext}\, B_{\mathcal{L}_s(^2 X)}$ holds.*

3. Extreme and exposed points of $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$

Let $n \geq 2$ and $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm $\|\cdot\|$ such that $B_{\mathbb{R}^n_{\|\cdot\|}}$ has finitely many extreme points more than $2n$. Let $\mathrm{ext}\, B_{\mathbb{R}^n_{\|\cdot\|}} = \{\pm U_1, \ldots, \pm U_m\}$ for some $m \geq n$ and $U_i \neq U_j$ for $1 \leq i \neq j \leq m$. Notice that $\{x_l y_s : 1 \leq l, s \leq n\}$ is a basis for $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$. Hence, $\dim(\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})) = n^2$. By Mazur's theorem, $B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ is compact and convex. By the Krein-Milman theorem, $\mathrm{ext}\, B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ is nonempty.

Let $T \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$. Then

$$T\Big((x_1, \ldots, x_n),\ (y_1, \ldots, y_n)\Big) = \sum_{1 \leq l, s \leq n} a_{ls} x_l y_s$$

for some $a_{ls} \in \mathbb{R}$.

For simplicity, we denote

$$T = \Big(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}\Big)^t \in \mathbb{R}^{n^2}.$$

For $j = 1, \dots, m$, we let $U_j = \sum_{1\le k\le n} \lambda_k^{(j)} e_k$ for some $\lambda_k^{(j)} \in \mathbb{R}$.

It follows that for $1 \le i \le j \le m$,

$$\begin{aligned} T(U_i, U_j) &= T\Big(\sum_{1\le k\le n} \lambda_k^{(i)} e_k,\ \sum_{1\le k\le n} \lambda_k^{(j)} e_k\Big) = \sum_{1\le k_1,k_2\le n} \lambda_{k_1}^{(i)}\lambda_{k_2}^{(j)}\ T(e_{k_1}, e_{k_2}) \\ &= \sum_{1\le k_1,k_2\le n} \lambda_{k_1}^{(i)}\lambda_{k_2}^{(j)} a_{k_1k_2} = Y_{(i,j)} \cdot T, \end{aligned}$$

where $Y_{(i,j)} = \Big(\lambda_1^{(i)}\lambda_1^{(j)},\ \dots,\ \lambda_1^{(i)}\lambda_n^{(j)},\ \lambda_2^{(i)}\lambda_1^{(j)},\ \dots,\ \lambda_2^{(i)}\lambda_n^{(j)},\ \dots,\ \lambda_n^{(i)}\lambda_1^{(j)},\ \dots,\ \lambda_n^{(i)}\lambda_n^{(j)}\Big) \in \mathbb{R}^{n^2}$ and $Y_{(i,j)} \cdot T$ denotes the dot product of $Y_{(i,j)}$ and T on $\mathbb{R}^{n^2}$.

Let $\Lambda := \{(i,j) : 1 \le i,j \le m\}$. Then $|\Lambda| = m^2 \ge n^2$. Notice that there are at most n^2 linearly independent vectors in $\{Y_{(i,j)} : (i,j) \in \Lambda\}$ since $\{Y_{(i,j)} : (i,j) \in \Lambda\} \subseteq \mathbb{R}^{n^2}$.

In this section we characterize $\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ and $\exp B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$, which expand some results of [25, 35, 38, 39, 43]. First, we present an explicit formulae for the norm of $T \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$.

Theorem 3.1. *Let $n \ge 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2.*

(a) *If $T \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$, then*

$$\|T\| = \sup_{1\le i,j\le m} |T(U_i, U_j)| = \sup_{1\le k\le m^2} |Y_{(i_k,j_k)} \cdot T|.$$

(b) *If $c_{(i,j)} \in \mathbb{R}$ for $(i,j) \in \Lambda$, then there is a unique $S \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ such that $S(U_i, U_j) = c_{(i,j)}$ for all $(i,j) \in \Lambda$.*

Proof. It follows from the Krein-Milman theorem and bilinearity of T. □

We are in position to prove the main result in this section.

Theorem 3.2. *Let $n \ge 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Let $T \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ with $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ if and only if there are n^2 linearly independent vectors $Y_{(i_1,j_1)}, \dots, Y_{(i_{n^2},\ j_{n^2})}$ in $\mathbb{R}^{n^2}$ for some $(i_1, j_1), \dots, (i_{n^2},\ j_{n^2}) \in \Lambda$ such that $|Y_{(i_k,j_k)} \cdot T| = 1$ for all $k = 1, \dots, n^2$.*

Proof. ($\Rightarrow$) Suppose that T is extreme.

By similar arguments as in Theorems 2.2 and 3.1(b), there are n^2 linearly independent vectors $Y_{(i_1,j_1)}, \dots, Y_{(i_{n^2},\ j_{n^2})}$ in $\mathbb{R}^{n^2}$ for some $(i_1, j_1), \dots, (i_{n^2},\ j_{n^2}) \in \Lambda$ such that $|Y_{(i_k,j_k)} \cdot T| = 1$ for all $k = 1, \dots, n^2$.

($\Leftarrow$) Let $S_1, S_2 \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$ be such that $\|S_l\| = 1$ for $l = 1, 2$ and $T = \frac{1}{2}(S_1 + S_2)$.

Claim: $T = S_l$ for $l = 1, 2$.

Since $\|S_l\| = 1$ for $l = 1, 2$, by Theorem 3.1(a), $|Y_{(i_k,j_k)} \cdot S_l| \leq 1$ for all $k = 1, \dots, n^2$. Let M_1 be the $n^2 \times n^2$-matrix such that the k-th row of M_1 equals to $Y_{(i_k,j_k)}$ for $k = 1, \dots, n^2$. Notice that M_1 is an invertible $n^2 \times n^2$-matrix because rows vectors of M_1 are linearly independent. By similar arguments as in Theorem 2.2, $M_1 T = M_1 S_l$ for $l = 1, 2$. Since M_1 is invertible, $T = S_l$ for $l = 1, 2$. Therefore, T is extreme. $\square$

Using Theorem 3.2, we completely describe $\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$.

Theorem 3.3. *Let $n \geq 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Then*

$$\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} = \Big\{ M^{-1}(b_1, \dots, b_{n^2})^t \in S_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} : b_k = \pm 1,\ M \textit{ is the invertible } n^2 \times n^2\textit{-matrix such that the } k\textit{-the row of } M \textit{ equals to } Y_{(i_k,j_k)} \textit{ for } (i_k, j_k) \in \Lambda \textit{ and } k = 1, \dots, n^2 \Big\}.$$

Proof. By similar arguments as in Theorems 2.3 and 3.2, it follows. $\square$

Using Theorem 3.3, we show that every extreme point of $B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ is exposed.

Theorem 3.4. *Let $n \geq 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Then* $\exp B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$.

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$. By Theorem 3.2, there are

$$(i_1, j_1), \dots, (i_{n^2},\ j_{n^2}) \in \Lambda$$

such that $Y_{(i_1,j_1)}, \dots, Y_{(i_{n^2},\ j_{n^2})}$ are linearly independent in $\mathbb{R}^{n^2}$ and $|Y_{(i_k,j_k)} \cdot T| = |T(U_{i_k}, U_{j_k})| = 1$ for all $k = 1, \dots, n^2$. Let M_1 be the invertible $n^2 \times n^2$-matrix such that the k-th row of M_1 equals to $Y_{(i_k,j_k)}$ for $k = 1, \dots, n^2$. Let $f \in \mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})^*$ be such that

$$f = \frac{1}{n^2} \sum_{k=1}^{n^2} \operatorname{sign}(T(U_{i_k}, U_{j_k}))\delta_{(U_{i_k}, U_{j_k})}.$$

Then $1 = \|f\| = f(T)$. Let $S \in extB_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})}$ be such that $|f(S)| = 1$. By similar arguments as in Theorem 2.5, $S = T$ or $S = -T$. By Theorem 2.4, T is exposed. $\square$

Kim [25, 35, 38, 39] showed that if $n \geq 2$, $0 < w < 1$ and $X = l^n_\infty$ or $\mathbb{R}^2_{*(w)}$, then $\exp B_{\mathcal{L}(^2X)} = \operatorname{ext} B_{\mathcal{L}(^2X)}$.

Using Theorem 3.4, we obtain the following:

Corollary 3.5. *Let $n \geq 2$, $0 < w < 1$ and $X = l_\infty^n, \mathbb{R}^2_{*(w)}, \mathbb{R}^2_{h(w)}$ or $\mathcal{L}(^2 l_\infty^2)$. Then the equality $\exp B_{\mathcal{L}(^2X)} = \operatorname{ext} B_{\mathcal{L}(^2X)}$ holds.*

The following theorem shows a relation between the spaces $\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$ and $\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})$.

Theorem 3.6. *Let $n \geq 2$ and let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm $\|\cdot\|$ be the same as in Theorem 2.2. Then the following equalities hold*:

(a) $\operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$.

(b) $\exp B_{\mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})} = \exp B_{\mathcal{L}(^2\mathbb{R}^n_{\|\cdot\|})} \cap \mathcal{L}_s(^2\mathbb{R}^n_{\|\cdot\|})$.

Proof. (a) It follows from Theorems 2.2 and 3.2.

(b) It follows from Theorems 2.5, 3.4 and (a). □

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Sung Guen Kim
Department of Mathematics
Kyungpook National University
Daegu 41566, Korea
Email address: `sgk317@knu.ac.kr`