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TWO-WEIGHTED CONDITIONS AND CHARACTERIZATIONS FOR A CLASS OF MULTILINEAR FRACTIONAL NEW MAXIMAL OPERATORS

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Abstract. In this paper, two weight conditions are introduced and the multiple weighted strong and weak characterizations of the multilinear fractional new maximal operator $\mathcal{M}_{\varphi,\beta}$ are established. Meanwhile, we introduce the $S_{(\vec{p},q),\beta}(\varphi)$ and $B_{(\vec{p},q),\beta}(\varphi)$ conditions and obtain the characterization of two-weighted inequalities for $\mathcal{M}_{\varphi,\beta}$. Finally, the relationships of the conditions $S_{(\vec{p},q),\beta}(\varphi)$, $\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ and $B_{(\vec{p},q),\beta}(\varphi)$ and the characterization of the one-weight $A_{(\vec{p},q),\beta}(\varphi)$ are given.

1. Introduction and main results

In 1974, Muckenhoupt and Wheeden first introduced the $A_{(p,q)}(\mathbb{R}^n)$ weights and studied weighted estimates of fractional type operators in [12]. In 2010, a class of multiple fractional type weights $A_{(\vec{p},q)}(\mathbb{R}^n)$ was defined by Chen and Xue in [4] and the strong and weak type multiple weighted estimates of the multilinear fractional maximal operator $\mathcal{M}_\alpha$ were given, here $\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_m) \in A_{(\vec{p},q)}(\mathbb{R}^n)$ if and only if

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m \omega_i\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q \omega_i^{-p_i'}\mathrm{d}x\right)^{\frac{1}{p_i'}} < \infty,$$

where $\mathcal{Q}$ denotes the family of all cubes on $\mathbb{R}^n$ with sides parallel to the axes. Simultaneously, the multiple fractional type weights $A_{(\vec{p},q)}(\mathbb{R}^n)$ were also investigated independently by Moen in [10]. In order to study the two-weighted inequality for the multilinear fractional maximal operator $\mathcal{M}_\alpha$, the following

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$\mathcal{A}_{(\vec{p},q)}$ condition was also defined in [10]

$$[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q)}}=\sup_{Q\in\mathcal{Q}}|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q\nu\mathrm{d}x\right)^{\frac{1}{q}}\prod_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q\omega_i^{1-p_i'}\mathrm{d}x\right)^{\frac{1}{p_i'}}<\infty.$$

It is proved that the $\mathcal{A}_{(\vec{p},q)}$ condition characterizes the boundedness of $\mathcal{M}_\alpha$ from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ to $L^{q,\infty}(\nu)$. Since then, the two-weighted problem has been studied extensively, for example see [7, 14, 15]. In 2016, a multilinear analogue of Sawyer's two-weight test condition was given by Li and Sun to warrant the boundedness of $\mathcal{M}_\alpha$ as follows in [9]

$$\begin{aligned}[\vec{\omega},\nu]_{S_{(\vec{p},q)}}=&\sup_{Q\in\mathcal{Q}}\left(\int_Q\mathcal{M}_\alpha(\sigma_1\chi_Q,\sigma_2\chi_Q,\ldots,\sigma_m\chi_Q)^q\nu\mathrm{d}x\right)^{\frac{1}{q}}\\&\times\left(\prod_{i=1}^{m}\sigma_i(Q)^{\frac{1}{p_i}}\right)^{-1}<\infty,\end{aligned}$$

where $\sigma_i=\omega_i^{1-p_i'}$ $(i=1,2,\ldots,m)$. In the same year, in order to obtain the strong boundedness of $\mathcal{M}_\alpha$, Cao and Xue gave the following $B_{(\vec{p},q)}$ condition in [1]

$$\begin{aligned}[\vec{\omega},\nu]_{B_{(\vec{p},q)}}=&\sup_{Q\in\mathcal{Q}}|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q\nu\mathrm{d}x\right)^{\frac{1}{q}}\left(\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q\omega_i\mathrm{d}x\right)\\&\times\exp\left(\frac{1}{|Q|}\int_Q\log\prod_{i=1}^{m}\omega_i^{-\frac{1}{p_i}}\mathrm{d}x\right)<\infty.\end{aligned}$$

In 2012, the new maximal operator M_φ was introduced by Tang in [17] to investigate the weighted L^p inequalities for the pseudo-differential operators with smooth symbols and their commutators

$$M_\varphi f(x)=\sup_{x\in Q\in\mathcal{Q}}\frac{1}{\varphi(|Q|)|Q|}\int_Q|f(y)|\mathrm{d}y,$$

where $\varphi(t)=(1+t)^\gamma$ for $t\geq 0$ and $\gamma\geq 0$, and the supremum is taken over all the cubes Q. Now, we consider the following fractional new maximal operator in [6]

$$M_{\varphi,\beta}f(x)=\sup_{x\in Q\in\mathcal{Q}}\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}}\int_Q|f(y)|\mathrm{d}y,$$

where $0\leq\beta<1$ and the supremum is taken over all the cubes Q. As $\beta=0$, we denote $M_{\varphi,0}$ by M_φ for simplicity, which is just the new maximal operator. Obviously, as $\varphi(|Q|)=1$, $M_\varphi f(x)=:Mf(x)$ is called the Hardy-Littlewood maximal operator.

Recently, to study the weighted norm inequalities for $M_{\varphi,\beta}$, Hu and Cao introduced a class of new weight function $A_{p,\beta}(\varphi)$ in [6], which included the classical Muckenhoupt weight $A_p(\mathbb{R}^n)$ in [11]. Let ω be a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ at almost everywhere. We say that $\omega \in A_{p,\beta}(\varphi)$ with $1 < p < \infty$ and $0 \leq \beta < 1$ if there exists a constant $C > 0$ such that for every cube Q,

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \left(\int_Q \omega(x) \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_Q \omega(x)^{1-p'} \mathrm{d}x \right)^{\frac{1}{p'}} \leq C.$$

Also, ω belongs to $A_{1,\beta}(\varphi)$ $(0 \leq \beta < 1)$ means

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega(x) \mathrm{d}x \leq C\omega(x).$$

To study the two-weighted inequalities of $M_{\varphi,\beta}$ in the multilinear setting, we first give the following definition of the multilinear fractional new maximal operator.

Definition 1.1. Let $\vec{f} = (f_1, f_2, \ldots, f_m)$ be a collection of locally integrable functions and $0 \leq \beta < 1$. Then the multilinear fractional new maximal operator is defined by

$$\mathcal{M}_{\varphi,\beta}(\vec{f})(x) = \sup_{x \in Q \in \mathcal{Q}} \prod_{i=1}^{m} \frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q |f_i(y_i)| \mathrm{d}y_i,$$

where $Q = Q(x, r)$ is denoted as a cube with sides parallel to the axes, x and r denote its center and side length, and the supremum is taken over every cube Q.

Remark 1.2. As $\beta = 0$, $\mathcal{M}_{\varphi,0}$ is called the multilinear new maximal operator and will be denoted by $\mathcal{M}_\varphi$, which was introduced by Pan and Tang in [13]. Also, as $\varphi(|Q|) = 1$, $\mathcal{M}_\varphi(\vec{f})(x) =: \mathcal{M}(\vec{f})(x)$ is said the multilinear maximal operator (see [8]). Noting that if $\beta = \frac{\alpha}{mn}$ $(0 \leq \alpha < mn)$ and $\varphi(|Q|) = 1$, then $\mathcal{M}_{\varphi,\beta}(\vec{f})(x) =: \mathcal{M}_\alpha(\vec{f})(x)$ is just the multilinear fractional maximal operator in [4].

Next, we introduce a class of new multiple weight functions $A_{(\vec{p},q),\beta}(\varphi)$.

Definition 1.3. Let $0 \leq \beta < 1$, $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$ and $\vec{p} = (p_1, p_2, \ldots, p_m)$ such that $1 \leq p_1, p_2, \ldots, p_m < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$ and $q > 0$. Then $\vec{\omega}$ is said to satisfy the $A_{(\vec{p},q),\beta}(\varphi)$ condition if

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{i=1}^{m} \omega_i \right)^q \mathrm{d}x \right)^{\frac{1}{q}} \times \prod_{i=1}^{m} \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_i^{-p_i'} \mathrm{d}x \right)^{\frac{1}{p_i'}} < \infty,$$

where $\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_i^{-p_i'} \mathrm{d}x \right)^{\frac{1}{p_i'}}$ in the case $p_i = 1$ $(i = 1, 2, \ldots, m)$ is understood as $(\inf_Q \omega_i)^{-1}$.

Remark 1.4. In one-weight case, it's obvious that $A_{(\vec{p},q)}(\mathbb{R}^n) \subset A_{(\vec{p},q),\beta}(\varphi)$. Specially, if $m = 1$, $\beta = 0$, and $\varphi(|Q|) = 1$, then the $A_{(\vec{p},q),\beta}(\varphi)$ condition will be degenerated to the classical $A_{(p,q)}(\mathbb{R}^n)$ condition in [12]. Thus, the new multiple weights $A_{(\vec{p},q),\beta}(\varphi)$ are natural generalizations of the multiple weights $A_{(\vec{p},q)}(\mathbb{R}^n)$ (see [4]) and the classical $A_{(p,q)}(\mathbb{R}^n)$ weights (see [12]).

Now, we introduce the following two-weight conditions to study the characterization of two-weighted inequalities for $\mathcal{M}_{\varphi,\beta}$.

Definition 1.5. Let $0 \le \beta < 1$, $0 < q < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$ with $1 \le p_1, p_2, \ldots, p_m < \infty$. Suppose that $\sigma_i = \omega_i^{1-p_i'}$ $(i = 1, 2, \ldots, m)$, $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$ and each ω_i and ν are nonnegative locally integrable functions on $\mathbb{R}^n$. Then

(i) we say that $(\vec{\omega}, \nu) \in \mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ if

$$[\vec{\omega}, \nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)} = \sup_{Q\in\mathcal{Q}} (\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \nu \mathrm{d}x\right)^{\frac{1}{q}} \times \prod_{i=1}^m \left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega_i^{1-p_i'}\mathrm{d}x\right)^{\frac{1}{p_i'}} < \infty;$$

(ii) we say that $(\vec{\omega}, \nu)$ satisfies the $B_{(\vec{p},q),\beta}(\varphi)$ condition if

$$\begin{aligned}&[\vec{\omega}, \nu]_{B_{(\vec{p},q),\beta}(\varphi)} \\ &= \sup_{Q\in\mathcal{Q}} (\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \nu \mathrm{d}x\right)^{\frac{1}{q}} \\ &\times \left(\prod_{i=1}^m \frac{1}{\varphi(|Q|)|Q|}\int_Q \omega_i \mathrm{d}x\right) \exp\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \log\prod_{i=1}^m \omega_i^{-\frac{1}{p_i}}\mathrm{d}x\right) < \infty;\end{aligned}$$

(iii) we say that $(\vec{\omega}, \nu) \in S_{(\vec{p},q),\beta}(\varphi)$ if it satisfies

$$[\vec{\omega}, \nu]_{S_{(\vec{p},q),\beta}(\varphi)} = \sup_{Q\in\mathcal{Q}} \left(\int_Q \mathcal{M}_{\varphi,\beta}(\sigma_1\chi_Q, \sigma_2\chi_Q, \ldots, \sigma_m\chi_Q)^q \nu \mathrm{d}x\right)^{\frac{1}{q}} \times \left(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}}\right)^{-1} < \infty.$$

Remark 1.6. From the structure of $\mathcal{M}_\alpha$ and the definitions of the conditions $\mathcal{A}_{(\vec{p},q)}$, $S_{(\vec{p},q)}$ and $B_{(\vec{p},q)}$ in [10, 9, 1], it follows that the definitions of three multiple two-weight conditions for $\mathcal{M}_{\varphi,\beta}$ are natural. Taking $q = p$ and $\nu = \nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{\frac{p}{p_i}}$ in (i), we say that $\vec{\omega}$ satisfies the $A_{\vec{p},\beta}(\varphi)$ condition. As $\varphi(|Q|) = 1$ and $\beta = \frac{\alpha}{mn}$ $(0 \le \alpha < mn)$, then $(\vec{\omega}, \nu) \in \mathcal{A}_{(\vec{p},q)}$.

Definition 1.7 ([3])**.** Let ω_i $(i = 1, 2, \ldots, m)$ be nonnegative locally integrable function on $\mathbb{R}^n$, $1 < p_1, p_2, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$. We say that $\vec{\omega}$ belongs to $W_{\vec{p}}^{\infty}$ if

$$[\vec{\omega}]_{W_{\vec{p}}^{\infty}} = \sup_{Q\in\mathcal{Q}} \left(\int_Q \prod_{i=1}^m M(\omega_i\chi_Q)^{\frac{p}{p_i}}\mathrm{d}x\right)\left(\int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}}\mathrm{d}x\right)^{-1} < \infty.$$

The main aim of this paper is to consider the multiple weighted strong and weak type estimates for the multilinear fractional new maximal operator and study the characterization of two-weighted inequalities for $\mathcal{M}_{\varphi,\beta}$. What should be stressed is that the strong boundedness of $\mathcal{M}_{\alpha}$ cannot be obtained as $(\vec{\omega},\nu)\in\mathcal{A}_{(\vec{p},q)}$ (see [10]). Thus, $(\vec{\omega},\nu)$ has to satisfy the following certain power bump condition for some $h>1$,

$$\sup_{Q\in\mathcal{Q}} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q \nu^q\mathrm{d}x\right)^{\frac{1}{q}}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q \omega_i^{-hp_i'}\mathrm{d}x\right)^{\frac{1}{hp_i'}} < \infty.$$

To show the boundedness of $\mathcal{M}_{\alpha}$ from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ into $L^q(\nu)$ in the two-weight $\mathcal{A}_{(\vec{p},q)}$ case, Cao and Xue added the condition $\vec{\sigma}\in W_{\vec{p}}^{\infty}$ in [1]. However, we can prove that $\mathcal{M}_{\varphi,\beta}$ is bounded from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ to $L^q(\nu)$ if and only if $[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}<\infty$, which extends some results in [10] and [1]. Before to state our main results, we also need the following definition.

Definition 1.8 ([3])**.** Let ω_i $(i=1,2,\dots,m)$ be nonnegative locally integrable function on $\mathbb{R}^n$, $\sigma_i=\omega_i^{1-p_i'}$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$ with $1<p_1,p_2,\dots,p_m<\infty$. We say that $\vec{\omega}\in RH_{\vec{p}}$ if

$$[\vec{\omega}]_{RH_{\vec{p}}} = \sup_{Q\in\mathcal{Q}}\prod_{i=1}^m\left(\int_Q \sigma_i\mathrm{d}x\right)^{\frac{p}{p_i}}\left(\int_Q\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\mathrm{d}x\right)^{-1}<\infty.$$

In what follows, we always assume that C is a positive constant which is irrelevant to the main parameters, but it may take different values in different lines. Occasionally, we will use the notation $T: X\to Y$ to mean T is a bounded operator from X to Y.

Our main results are formulated as follows. We first give the multiple weighted strong and weak characterizations for $\mathcal{M}_{\varphi,\beta}$.

Theorem 1.9. *Let $0\le\beta<1$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$ with $1\le p_1,p_2,\dots,p_m<\infty$, and $0<p\le q<\infty$. Suppose that $\vec{\omega}=(\omega_1,\omega_2,\dots,\omega_m)$ and ν are nonnegative locally integrable functions on $\mathbb{R}^n$. Then $(\vec{\omega},\nu)\in\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ if and only if $\mathcal{M}_{\varphi,\beta}$ is bounded from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ into $L^{q,\infty}(\nu)$.*

Theorem 1.10. *Let $0\le\beta<1$, $0<p\le q<\infty$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$ with $1<p_1,p_2,\dots,p_m<\infty$. Suppose that ω_i $(i=1,2,\dots,m)$ and ν are weights. Then $\mathcal{M}_{\varphi,\beta}$ is bounded from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ into $L^q(\nu)$ if and only if $(\vec{\omega},\nu)\in\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$.*

In addition, the two-weighted inequalities of $\mathcal{M}_{\varphi,\beta}$ are obtained as follows.

Theorem 1.11. *Let ω_i $(i = 1,2,\dots,m)$ and ν be weights, $\sigma_i = \omega_i^{1-p_i'}$, and $\vec{\omega} \in RH_{\vec{p}}$. Suppose that $0 \le \beta < 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$ with $1 < p_1, p_2, \dots, p_m < \infty$, and $0 < p \le q < \infty$. Then the following statements are equivalent*:

(i) $(\vec{\omega}, \nu) \in S_{(\vec{p},q),\beta}(\varphi)$.

(ii) $\mathcal{M}_{\varphi,\beta} : L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \dots \times L^{p_m}(\omega_m) \to L^q(\nu)$.

Moreover, there exists a positive constant C such that

$$[\vec{\omega}, \nu]_{S_{(\vec{p},q),\beta}(\varphi)} \le \|\mathcal{M}_{\varphi,\beta}\|_{L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\dots\times L^{p_m}(\omega_m)\to L^q(\nu)} \le C[\vec{\omega}]_{RH_{\vec{p}}}^{\frac{1}{p}} [\vec{\omega}, \nu]_{S_{(\vec{p},q),\beta}(\varphi)}.$$

Theorem 1.12. *Let $0 \le \beta < 1$, $0 < p \le q < \infty$, $0 < r < 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$ with $1 < p_1, p_2, \dots, p_m < \infty$. Suppose that $\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_m)$ and ν are weights, $\sigma_i = \omega_i^{1-p_i'}$ $(i = 1,2,\dots,m)$, and $(\vec{\sigma}, \nu) \in B_{(\vec{p},q),\beta}(\varphi)$. Then there is a positive constant C such that*

$$\|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^q(\nu)} \le C[\vec{\sigma}, \nu]_{B_{(\vec{p},q),\beta}(\varphi)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}.$$

Now, we have the following relationships of the conditions $S_{(\vec{p},q),\beta}(\varphi)$, $\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ and $B_{(\vec{p},q),\beta}(\varphi)$.

Theorem 1.13. *Let $\omega_1, \omega_2, \dots, \omega_m$ and ν be weights, $\sigma_i = \omega_i^{1-p_i'}$ $(i = 1,2,\dots, m)$. Suppose that $0 \le \beta < 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$ with $1 < p_1, p_2, \dots, p_m < \infty$, and $0 < p \le q < \infty$. Then*

(i) *There exists a positive constant C such that for any $0 < r < 1$ there holds*

$$[\vec{\omega}, \nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)} \le [\vec{\omega}, \nu]_{S_{(\vec{p},q),\beta}(\varphi)} \le C[\vec{\sigma}, \nu]_{B_{(\vec{p},q),\beta}(\varphi)}.$$

(ii) *If $m = 1$, $\nu = \omega$, $\beta = 0$, and $q = p$, then*

$$[\vec{\sigma}, \nu]_{B_{(\vec{p},q),\beta}(\varphi)} \le [\omega]_{A_{p,\beta}(\varphi)}^{\frac{p}{p-1}}.$$

Finally, we obtain the following characterization of the one-weight $A_{(\vec{p},q),\beta}(\varphi)$.

Theorem 1.14. *Let $0 \le \beta \le \frac{\gamma}{1+\gamma}$, $0 < p \le q < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$ with $1 < p_1, p_2, \dots, p_m < \infty$. If $\vec{\omega} \in A_{(\vec{p},q),\beta}(\varphi)$, then for $i = 1,2,\dots,m$, $\omega_i^{-p_i'} \in A_{mp_i',\beta}(\varphi)$ and $(\prod_{i=1}^m \omega_i)^q \in A_{mq,\beta}(\varphi)$.*

The article is organized as follows: Section 2 is devoted to giving some major lemmas. The proofs of Theorems 1.9 and 1.10 will be given in Section 3. The proof of Theorem 1.11 is given in Section 4. In Section 5, we show the proof of Theorem 1.12 by using a key lemma. By combining Theorem 1.11 and Theorem 1.12, the proofs of Theorems 1.13 and 1.14 shall be given in Section 6.

2. Some preliminaries

In this section, we first recall the notions of tent spaces. Let X be the cone $[0,\infty)^n$ minus the set of dyadic points, that is

$$X = [0,\infty)^n \setminus \{(2^{-l}k_1, 2^{-l}k_2, \ldots, 2^{-l}k_n) : l \in \mathbb{Z}, k_i \in \mathbb{N}\}.$$

$\widetilde{X} = X \times \{2^{-l} : l \in \mathbb{Z}\}$ denotes the upper half-space. Then for every couple $(y,\rho) \in \widetilde{X}$, there exists only dyadic cube $Q = Q_{y\rho}$ which has the side length $\rho = 2^{-l}$ and contains y. Write $(y,\rho) \in \widetilde{\Gamma}(x)$ if and only if $x \in Q_{y\rho}$.

For any $E \subset [0,\infty)^n$, define

$$\widehat{E} = \left(\bigcup_{x \in E^c} \widetilde{\Gamma}(x) \right)^c.$$

Thus,

$$(y,\rho) \in \widehat{E} \ \text{ if and only if } \ Q_{y\rho} \subset E.$$

We need the following lemmas, which are vital to the proofs of our theorems.

Lemma 2.1 ([1])**.** *Let ν be a nonnegative locally integrable function on $\mathbb{R}^n$ and $0 < p < \infty$. Then for all functions $\widetilde{f}_i(y, \rho = 2^{-l})$ with a support contained on $\widehat{Q[0,\theta]}$, there exist nonnegative scalars $\{\lambda_j\}_{j=1}^{\infty}$, functions $\{\widetilde{a_j}(y,\rho)\}_{j=1}^{\infty}$ and dyadic cubes $\{Q_j\}_{j=1}^{\infty}$ such that*

$$\text{supp } \widetilde{a_j} \ \text{ are disjoint and } |\widetilde{a_j}(y,\rho)| \le \nu(Q_j)^{-\frac{1}{p}} \chi_{\widehat{Q_j}}(y,\rho); \tag{2.1}$$

$$\prod_{i=1}^{m} \widetilde{f}_i(y,\rho) = \sum_j \lambda_j \widetilde{a_j}(y,\rho) \ \text{ a.e..} \tag{2.2}$$

Lemma 2.2. *Let $0 \le \beta < 1$ and $x, t \in \mathbb{R}^n$. For any $k \ge 0$ and $\vec{f} = (f_1, f_2, \ldots, f_m) \ge 0$, we define the following truncated version maximal operator,*

$$\mathcal{M}_{\varphi,\beta}^{(k)}(\vec{f})(x) = \sup_{x \in Q \in \mathcal{Q}, |Q| \le 2^k} \prod_{i=1}^{m} \frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q |f_i(y_i)| \mathrm{d}y_i.$$

Then there is $C > 0$ such that

$$\mathcal{M}_{\varphi,\beta}^{(k)}(\vec{f})(x) \le \frac{C}{|B_k|} \int_{B_k} \tau_{-t} \circ \mathcal{M}_{\varphi,\beta}^{d} \circ \vec{\tau}_t(\vec{f})(x) \mathrm{d}t,$$

where $B_k = [-2^{k+2}, 2^{k+2}]^n$, $\tau_t f(x) = f(x-t)$, and $\vec{\tau}_t \vec{f} = (\tau_t f_1, \tau_t f_2, \ldots, \tau_t f_m)$.

Proof. Using the similar arguments as the proof of [2, Lemma 3.3], we can derive the conclusion of Lemma 2.2. Thus, the details are omitted here. □

3. Proofs of Theorems 1.9 and 1.10

Proof of Theorem 1.9. We only consider the case as $1<p_i<\infty$ $(i=1,2,\dots,m)$ because with the minor modifications for the case of some $p_i=1$ as in the linear situation. Let $(\vec{\omega},\nu)\in\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ and $\|f_i\|_{L^{p_i}(\omega_i)}=1$ $(i=1,2,\dots,m)$. Then using Hölder's inequality, we have

$$\begin{aligned}
&\left(\prod_{i=1}^m \frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}}\int_Q|f_i|\right)^q\nu(Q)\\
&\le(\varphi(|Q|)|Q|)^{q(m\beta+\frac{1}{q}-\frac{1}{p})}\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q\nu\right)\left[\prod_{i=1}^m\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q\omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}}\right.\\
&\qquad\left.\times\left(\int_Q|f_i|^{p_i}\omega_i\right)^{\frac{1}{p_i}}\right]^q\\
&\le C\prod_{i=1}^m\left(\int_Q|f_i|^{p_i}\omega_i\right)^{\frac{q}{p_i}}.
\end{aligned}$$

Since $\|f_i\|_{L^{p_i}(\omega_i)}=1$, then $\prod_{i=1}^m\left(\int_Q|f_i|^{p_i}\omega_i\right)^{\frac{1}{p_i}}\le 1$. Thus,

$$\prod_{i=1}^m\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}}\int_Q|f_i|\le\frac{C}{\nu(Q)^{\frac{1}{q}}}\left(\prod_{i=1}^m\left(\int_Q|f_i|^{p_i}\omega_i\right)^{\frac{1}{p_i}}\right)^{\frac{p}{q}}.$$

Therefore,

$$\mathcal{M}_{\varphi,\beta}(\vec{f})(x)\le C\left(\prod_{i=1}^m\left(M_\nu^c\left(\frac{|f_i|^{p_i}\omega_i}{\nu}\right)(x)\right)^{\frac{1}{p_i}}\right)^{\frac{p}{q}},$$

where M_ν^c is the weighted centered maximal function. Applying the fact that M_ν^c is of weak type (1,1) with respect to ν, and weak-type Hölder's inequality in [5], we have

$$\begin{aligned}
\|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{q,\infty}(\nu)}&\le C\left\|\prod_{i=1}^m\left(M_\nu^c\left(\frac{|f_i|^{p_i}\omega_i}{\nu}\right)\right)^{\frac{1}{p_i}}\right\|_{L^{p,\infty}(\nu)}^{\frac{p}{q}}\\
&\le C\left(\prod_{i=1}^m\left\|M_\nu^c\left(\frac{|f_i|^{p_i}\omega_i}{\nu}\right)\right\|_{L^{1,\infty}(\nu)}^{\frac{1}{p_i}}\right)^{\frac{p}{q}}\\
&\le C.
\end{aligned}$$

For general f_i, if we replace $f_i\to\frac{f_i}{\|f_i\|_{L^{p_i}(\omega_i)}}$ $(i=1,2,\dots,m)$, then

$$\|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{q,\infty}(\nu)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)}.$$

Conversely, suppose that $\mathcal{M}_{\varphi,\beta}$ is bounded from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \cdots \times L^{p_m}(\omega_m)$ into $L^{q,\infty}(\nu)$. Then for any $\lambda > 0$, there holds

$$\nu(\{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\beta}(\vec{f})(x) > \lambda\}) \leq \left(\frac{C}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}\right)^q.$$

Let $f_i \geq 0$. Fix a cube Q with $\prod_{i=1}^{m}(f_i)_{Q,\varphi,\beta} > 0$, where $(f_i)_{Q,\varphi,\beta} =: \frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q f_i \mathrm{d}x$. For $x \in Q$, from the definition of $\mathcal{M}_{\varphi,\beta}$, we have

$$\prod_{i=1}^{m}(f_i)_{Q,\varphi,\beta} \leq \mathcal{M}_{\varphi,\beta}(f_1\chi_Q, f_2\chi_Q, \ldots, f_m\chi_Q)(x).$$

If $\prod_{i=1}^{m}(f_i)_{Q,\varphi,\beta} > \lambda$, then using the above inequality, we have

$$Q \subset \{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\beta}(f_1\chi_Q, f_2\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}.$$

Hence,

$$\begin{aligned}\nu(Q) &\leq \nu(\{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\beta}(f_1\chi_Q, f_2\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}) \\ &\leq \left(\frac{C}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}\right)^q.\end{aligned}$$

Set $f_i = \omega_i^{1-p_i'}$. Then

$$(\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\varphi(|Q|)|Q|} \int_Q \nu\right)^{\frac{1}{q}} \prod_{i=1}^{m} \left(\frac{1}{\varphi(|Q|)|Q|} \int_Q \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}} \leq C.$$

Therefore, $(\vec{\omega}, \nu) \in \mathcal{A}_{(\vec{p},q),\beta}(\varphi)$. The proof is complete. □

Proof of Theorem 1.10. Let $\mathcal{M}_{\varphi,\beta}$ be bounded from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \times \cdots \times L^{p_m}(\omega_m)$ into $L^q(\nu)$. Then the following inequality holds

$$\left(\int_Q \left(\mathcal{M}_{\varphi,\beta}(\vec{f})\right)^q \nu\right)^{\frac{1}{q}} \leq C \prod_{i=1}^{m} \left(\int_Q |f_i|^{p_i} \omega_i\right)^{\frac{1}{p_i}}.$$

Assume that $f_i = \omega_i^{1-p_i'} \geq 0$. Then

$$(\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\varphi(|Q|)|Q|} \int_Q \nu\right)^{\frac{1}{q}} \prod_{i=1}^{m} \left(\frac{1}{\varphi(|Q|)|Q|} \int_Q \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}} \leq C.$$

Therefore, we can obtain that $(\vec{\omega}, \nu) \in \mathcal{A}_{(\vec{p},q),\beta}(\varphi)$.

Conversely, suppose that $(\vec{\omega}, \nu) \in \mathcal{A}_{(\vec{p},q),\beta}(\varphi)$. We first prove the boundedness for the dyadic version,

$$\mathcal{M}^d_{\varphi,\beta}(\vec{f})(x) = \sup_{x \in Q \in \mathcal{D}} \prod_{i=1}^{m} \frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q |f_i(y_i)| \mathrm{d}y_i,$$

where $\mathcal{D}$ denotes the standard dyadic grid on $\mathbb{R}^n$ consists of the cubes $2^{-k}([0,1)^n+j)$, $k\in\mathbb{Z}$, $j\in\mathbb{Z}^n$. Let a be a constant such that $a>2^{mn}$. We take the following set,

$$\Omega_k=\{x\in\mathbb{R}^n:\mathcal{M}^d_{\varphi,\beta}(\vec{f})(x)>a^k\}.$$

If Ω_k is non-empty, then we write $\Omega_k=\bigcup_j Q_{kj}$ and every Q_{kj} is a maximal dyadic cube satisfying

$$a^k<\prod_{i=1}^m\frac{1}{(\varphi(|Q_{kj}|)|Q_{kj}|)^{1-\beta}}\int_{Q_{kj}}|f_i(y_i)|\mathrm{d}y_i\le 2^{mn}a^k.$$

By the properties of dyadic cubes, we know that $\Omega_{k+1}\subseteq\Omega_k$ and $Q_{(k+1)j}\subset Q_{kj}$ for some j. Thus, it follows from Hölder's inequality and the $\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ condition that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left(\mathcal{M}^d_{\varphi,\beta}(\vec{f})\right)^q\nu\right)^{\frac{1}{q}} \qquad (3.1)\\
&=\left(\sum_k\int_{\Omega_k\setminus\Omega_{k+1}}\left(\mathcal{M}^d_{\varphi,\beta}(\vec{f})\right)^q\nu\right)^{\frac{1}{q}}\\
&\le a\left(\sum_{kj}\left(\prod_{i=1}^m\frac{1}{(\varphi(|Q_{kj}|)|Q_{kj}|)^{1-\beta}}\int_{Q_{kj}}|f_i|\omega_i^{\frac{p_i'-1}{p_i'}}\omega_i^{\frac{1-p_i'}{p_i'}}\right)^q\right.\\
&\qquad\left.\times\left(\int_{Q_{kj}}\nu\right)\right)^{\frac{1}{q}}\\
&\le C\sum_{kj}\left(\prod_{i=1}^m\left(\frac{1}{(\varphi(|Q_{kj}|)|Q_{kj}|)^{1-\beta}}\int_{Q_{kj}}|f_i|^{p_i}\omega_i\right)^{\frac{q}{p_i}}\right.\\
&\qquad\left.\times\left(\frac{1}{(\varphi(|Q_{kj}|)|Q_{kj}|)^{1-\beta}}\int_{Q_{kj}}\omega_i^{1-p_i'}\right)^{\frac{q}{p_i'}}\left(\int_{Q_{kj}}\nu\right)\right)^{\frac{1}{q}}\\
&\le C[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}\sum_{kj}\left(\prod_{i=1}^m\left(\int_{Q_{kj}}|f_i|^{p_i}\omega_i\right)^{\frac{q}{p_i}}\right)^{\frac{1}{q}}\\
&\le C[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}|f_i|^{p_i}\omega_i\right)^{\frac{1}{p_i}}.
\end{aligned}$$

For non-dyadic version, from Minkowski's inequality and Lemma 2.2, it follows that

$$\|\mathcal{M}^{(k)}_{\varphi,\beta}(\vec{f})\|_{L^q(\nu)}\le\frac{C}{|B_k|}\left\|\int_{B_k}\tau_{-t}\circ\mathcal{M}^d_{\varphi,\beta}\circ\vec{\tau}_t(\vec{f})\mathrm{d}t\right\|_{L^q(\nu)} \qquad (3.2)$$

$$\le \frac{C}{|B_k|}\int_{B_k}\|\tau_{-t}\circ\mathcal{M}^d_{\varphi,\beta}\circ\vec{\tau}_t(\vec{f})\|_{L^q(\nu)}\mathrm{d}t$$
$$\le \frac{C}{|B_k|}\int_{B_k}\|\mathcal{M}^d_{\varphi,\beta}\vec{\tau}_t(\vec{f})\|_{L^q(\tau_t\nu)}\mathrm{d}t.$$

Since $(\vec{\omega},\nu)\in\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$, then we can get that $(\vec{\tau}_t\vec{\omega},\tau_t\nu)\in\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ independently of t. Therefore, by the inequalities (3.1) and (3.2) we have

$$\|\mathcal{M}^{(k)}_{\varphi,\beta}(\vec{f})\|_{L^q(\nu)}\le C[\vec{\tau}_t\vec{\omega},\tau_t\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}\frac{1}{|B_k|}\int_{B_k}\prod_{i=1}^m\|\tau_t f_i\|_{L^{p_i}(\tau_t\omega_i)}\mathrm{d}t$$
$$\le C[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)}.$$

Finally, letting k tend to infinity, the proof of Theorem 1.10 is finished. □

4. Proof of Theorem 1.11

To obtain the conclusions of Theorem 1.11, we need the following lemma.

Lemma 4.1. *Let $0\le\beta<1$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$ with $1<p_1,p_2,\dots,p_m<\infty$. Suppose that ω_i $(i=1,2,\dots,m)$ is nonnegative locally integrable function on $\mathbb{R}^n$, $\sigma_i=\omega_i^{1-p_i'}$, and $\vec{\omega}\in RH_{\vec{p}}$. Then there exists a positive constant C such that for all $\vec{f}\in L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$, one can find dyadic cubes $\{Q_j\}_{j=1}^\infty$ and nonnegative scalars $\{\lambda_j\}_{j=1}^\infty$ satisfying*

$$\left(\sum_j\lambda_j^p\right)^{\frac{1}{p}}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)};\tag{4.1}$$

$$\mathcal{M}^{d,\theta}_{\varphi,\beta}(\vec{f})(x)\chi_{(0,\theta)^n}(x)\le[\vec{\omega}]^{\frac{1}{p}}_{RH_{\vec{p}}}\sum_j\lambda_j\prod_{i=1}^m\sigma_i(Q_j)^{-\frac{1}{p_i}}\times\mathcal{M}^d_{\varphi,\beta}(\sigma_1\chi_{Q_j},\dots,\sigma_m\chi_{Q_j})(x)\chi_{Q_j}(x)\tag{4.2}$$

for almost everywhere $x\in(0,\infty)^n$, where the truncated dyadic version maximal operator $\mathcal{M}^{d,\theta}_{\varphi,\beta}$ is defined by

$$\mathcal{M}^{d,\theta}_{\varphi,\beta}(\vec{f})(x)=\sup_{x\in Q\in\mathcal{D},Q\subset(0,\theta)^n}\prod_{i=1}^m\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}}\int_Q|f_i(y_i)|\mathrm{d}y_i.$$

Proof. Since the inequality (4.1) has already been proved in [1], so we only need to show the inequality (4.2). Write

$$\widetilde{f}_i(y,\rho)=\frac{1}{\sigma_i(Q_{y\rho})}\int_{Q_{y\rho}}|f_i/\sigma_i|\sigma_i\mathrm{d}x.$$

Then

$$\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})(x) = \sup_{x\in Q_{y\rho}\in\mathcal{D}, Q_{y\rho}\subset(0,\theta)^n} \prod_{i=1}^{m} \frac{\sigma_i(Q_{y\rho})}{(\varphi(|Q_{y\rho}|)|Q_{y\rho}|)^{1-\beta}} \widetilde{f}_i(y,\rho).$$

Set $\nu = \nu_{\vec{\sigma}} = \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}$ in Lemma 2.1. Then from the inequalities (2.1) and (2.2), we have

$$\begin{aligned}
&\prod_{i=1}^{m} \frac{\sigma_i(Q_{y\rho})}{(\varphi(|Q_{y\rho}|)|Q_{y\rho}|)^{1-\beta}} \widetilde{f}_i(y,\rho)\\
&= \sum_j \lambda_j \widetilde{a_j}(y,\rho) \prod_{i=1}^{m} \frac{\sigma_i(Q_{y\rho})}{(\varphi(|Q_{y\rho}|)|Q_{y\rho}|)^{1-\beta}}\\
&\le \sum_j \lambda_j \left(\int_{Q_j} \prod_{i=1}^{m} \sigma_i^{\frac{p}{p_i}}\right)^{-\frac{1}{p}} \prod_{i=1}^{m} \frac{\sigma_i(Q_{y\rho})}{(\varphi(|Q_{y\rho}|)|Q_{y\rho}|)^{1-\beta}} \widetilde{\chi}_{\widehat{Q_j}}(y,\rho)\\
&\le [\vec{\omega}]_{RH_{\vec{p}}}^{\frac{1}{p}} \sum_j \lambda_j \left(\prod_{i=1}^{m} \sigma_i(Q_j)^{-\frac{1}{p_i}}\right) \prod_{i=1}^{m} \frac{\sigma_i(Q_{y\rho})}{(\varphi(|Q_{y\rho}|)|Q_{y\rho}|)^{1-\beta}} \widetilde{\chi}_{\widehat{Q_j}}(y,\rho)\\
&\le [\vec{\omega}]_{RH_{\vec{p}}}^{\frac{1}{p}} \sum_j \lambda_j \prod_{i=1}^{m} \sigma_i(Q_j)^{-\frac{1}{p_i}} \mathcal{M}_{\varphi,\beta}^{d}(\sigma_1\chi_{Q_j},\dots,\sigma_m\chi_{Q_j})(x)\chi_{Q_j}(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})(x)\chi_{(0,\theta)^n}(x)\\
&\le [\vec{\omega}]_{RH_{\vec{p}}}^{\frac{1}{p}} \sum_j \lambda_j \prod_{i=1}^{m} \sigma_i(Q_j)^{-\frac{1}{p_i}} \mathcal{M}_{\varphi,\beta}^{d}(\sigma_1\chi_{Q_j},\dots,\sigma_m\chi_{Q_j})(x)\chi_{Q_j}(x).
\end{aligned}$$

The proof is finished. □

Proof of Theorem 1.11. (ii)⇒(i). By the boundedness of $\mathcal{M}_{\varphi,\beta}$ from $L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$ to $L^q(\nu)$ and $\sigma_i = \omega_i^{1-p_i'}$ $(i = 1,2,\dots,m)$, we have

$$\begin{aligned}
[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)} &\le \sup_{Q\in\mathcal{Q}} \|\mathcal{M}_{\varphi,\beta}\|_{L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)\to L^q(\nu)}\\
&\quad\times \prod_{i=1}^{m} \|\sigma_i\chi_Q\|_{L^{p_i}(\omega_i)} \left(\prod_{i=1}^{m} \sigma_i(Q)^{\frac{1}{p_i}}\right)^{-1}\\
&= \sup_{Q\in\mathcal{Q}} \|\mathcal{M}_{\varphi,\beta}\|_{L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)\to L^q(\nu)}.
\end{aligned}$$

Therefore, we can get that $(\vec{\omega},\nu) \in S_{(\vec{p},q),\beta}(\varphi)$.

(i)⇒(ii). We first prove the dyadic version, that is, we need to show

$$\|\mathcal{M}_{\varphi,\beta}^{d}(\vec{f})\|_{L^q(\nu)} \le C[\vec{\omega}]_{RH_{\vec{p}}}^{\frac{1}{p}} [\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}. \tag{4.3}$$

By using reflections and translations of the cone $[0,\infty)^n$ as done in [16], it is sufficient to show

$$\|\mathcal{M}^{d,\theta}_{\varphi,\beta}(\vec{f})\chi_{(0,\theta)^n}\|_{L^q(\nu)} \le C[\vec{\omega}]^{\frac{1}{p}}_{RH_{\vec{p}}}[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

From the $S_{(\vec{p},q),\beta}(\varphi)$ condition, the inequalities (4.1) and (4.2), and Minkowski's inequality, we have

$$\begin{aligned}
&\|\mathcal{M}^{d,\theta}_{\varphi,\beta}(\vec{f})\chi_{(0,\theta)^n}\|^p_{L^q(\nu)}\\
&= \|\mathcal{M}^{d,\theta}_{\varphi,\beta}(\vec{f})^p\chi_{(0,\theta)^n}\|_{L^{\frac{q}{p}}(\nu)}\\
&\le [\vec{\omega}]_{RH_{\vec{p}}}\sum_j \lambda_j^p\prod_{i=1}^m \sigma_i(Q_j)^{-\frac{p}{p_i}}\left(\int_{Q_j}\mathcal{M}^d_{\varphi,\beta}(\sigma_1\chi_{Q_j},\dots,\sigma_m\chi_{Q_j})^q\nu \mathrm{d}x\right)^{\frac{p}{q}}\\
&\le [\vec{\omega}]_{RH_{\vec{p}}}[\vec{\omega},\nu]^p_{S_{(\vec{p},q),\beta}(\varphi)}\sum_j\lambda_j^p\\
&\le C[\vec{\omega}]_{RH_{\vec{p}}}[\vec{\omega},\nu]^p_{S_{(\vec{p},q),\beta}(\varphi)}\prod_{i=1}^m\|f_i\|^p_{L^{p_i}(\omega_i)}.
\end{aligned}$$

For non-dyadic version, since $(\vec{\omega},\nu)\in S_{(\vec{p},q),\beta}(\varphi)$ and $\vec{\omega}\in RH_{\vec{p}}$, then we know that $(\vec{\tau}_t\vec{\omega},\tau_t\nu)\in S_{(\vec{p},q),\beta}(\varphi)$ and $\vec{\tau}_t\vec{\omega}\in RH_{\vec{p}}$ independently of t. Thus, by the inequalities (3.2) and (4.3), we have

$$\begin{aligned}
\|\mathcal{M}^{(k)}_{\varphi,\beta}(\vec{f})\|_{L^q(\nu)} &\le C[\vec{\tau}_t\vec{\omega}]^{\frac{1}{p}}_{RH_{\vec{p}}}[\vec{\tau}_t\vec{\omega},\tau_t\nu]_{S_{(\vec{p},q),\beta}(\varphi)}\frac{1}{|B_k|}\int_{B_k}\prod_{i=1}^m\|\tau_t f_i\|_{L^{p_i}(\tau_t\omega_i)}\mathrm{d}t\\
&= C[\vec{\omega}]^{\frac{1}{p}}_{RH_{\vec{p}}}[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)}\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)}.
\end{aligned}$$

Finally, letting k tend to infinity, this completes the proof of Theorem 1.11. □

5. Proof of Theorem 1.12

Before proving Theorem 1.12, we first give the following lemma, which can be obtained by applying the similar arguments as the proofs of Lemma 4.1 and [1, Lemma 5.1]. Thus, the details are omitted here.

Lemma 5.1. *Let $0\le\beta<1$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$ with $1<p_1,p_2,\dots,p_m<\infty$. Suppose that $\omega_1,\omega_2,\dots,\omega_m$ are weights and $0<r<1$. Then there is a positive constant C such that for almost everywhere $x\in(0,\infty)^n$ and $\vec{f}\in L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)$, one can find functions $\{\widetilde{a_j}(y,\rho)\}_{j=1}^\infty$, dyadic cubes $\{Q_j\}_{j=1}^\infty$, and nonnegative scalars $\{\lambda_j\}_{j=1}^\infty$ satisfying*

$$\begin{aligned}
&|\widetilde{a_j}(y,\rho)| \qquad (5.1)\\
&\le (\varphi(|Q_j|)|Q_j|)^{-\frac{1}{p}}\exp\left(\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j}\log\prod_{i=1}^m\sigma_i^{-\frac{1}{p_i}}\mathrm{d}x\right)\widetilde{\chi}_{\widehat{Q_j}}(y,\rho);
\end{aligned}$$

$$\prod_{i=1}^{m} \widetilde{f}_i(y,\rho) = \sum_j \lambda_j \widetilde{a_j}(y,\rho) \ \ a.e.; \tag{5.2}$$

$$\left(\sum_j \lambda_j^p\right)^{\frac{1}{p}} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}; \tag{5.3}$$

$$\begin{aligned} &\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})(x)\chi_{(0,\theta)^n}(x) \\ \le\ & \sum_j \lambda_j (\varphi(|Q_j|)|Q_j|)^{-\frac{1}{p}} \left(\prod_{i=1}^{m} \frac{\sigma_i(Q_j)}{(\varphi(|Q_j|)|Q_j|)^{1-\beta}}\right) \\ & \times \exp\left(\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j} \log \prod_{i=1}^{m} \sigma_i^{-\frac{1}{p_i}} \mathrm{d}x\right)\chi_{Q_j(x)}. \end{aligned} \tag{5.4}$$

Proof of Theorem 1.12. Proceeding as we did in the proof of Theorem 1.11, in order to prove Theorem 1.12, then it is enough to show the following inequality

$$\|\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})\chi_{(0,\theta)^n}\|_{L^q(\nu)} \le C[\vec{\sigma},\nu]_{B_{(\vec{p},q),\beta}(\varphi)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}.$$

From the $B_{(\vec{p},q),\beta}(\varphi)$ condition, Minkowski's inequality, and the inequalities (5.3) and (5.4), it follows that

$$\begin{aligned} &\|\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})\chi_{(0,\theta)^n}\|_{L^q(\nu)}^p \\ =\ & \|\mathcal{M}_{\varphi,\beta}^{d,\theta}(\vec{f})^p\chi_{(0,\theta)^n}\|_{L^{\frac{q}{p}}(\nu)} \\ \le\ & \Bigg\| \sum_j \lambda_j^p (\varphi(|Q_j|)|Q_j|)^{-1} \left(\left(\prod_{i=1}^{m} \frac{\sigma_i(Q_j)}{(\varphi(|Q_j|)|Q_j|)^{1-\beta}}\right)\right. \\ & \left.\times \exp\left(\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j} \log\prod_{i=1}^{m}\sigma_i^{-\frac{1}{p_i}}\mathrm{d}x\right)\right)^p \chi_{Q_j} \Bigg\|_{L^{\frac{q}{p}}(\nu)} \\ \le\ & \sum_j \lambda_j^p \left((\varphi(|Q_j|)|Q_j|)^{m\beta+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j}\nu\mathrm{d}x\right)^{\frac{1}{q}}\right. \\ & \left.\times \left(\prod_{i=1}^{m}\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j}\sigma_i\mathrm{d}x\right)\exp\left(\frac{1}{\varphi(|Q_j|)|Q_j|}\int_{Q_j}\log\prod_{i=1}^{m}\sigma_i^{-\frac{1}{p_i}}\mathrm{d}x\right)\right)^p \\ \le\ & C[\vec{\sigma},\nu]_{B_{(\vec{p},q),\beta}(\varphi)}^p \prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i)}^p. \end{aligned}$$

The proof is complete. ∎

6. Proofs of Theorems 1.13 and 1.14

Proof of Theorem 1.13. First, we prove (i) is true. By the $\mathcal{A}_{(\vec{p},q),\beta}(\varphi)$ condition and the $S_{(\vec{p},q),\beta}(\varphi)$ condition, we have

$$
\begin{aligned}
&(\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \nu \mathrm{d}x\right)^{\frac{1}{q}}\prod_{i=1}^{m}\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega_i^{1-p_i'}\mathrm{d}x\right)^{\frac{1}{p_i'}}\\
&=\left(\int_Q\left(\prod_{i=1}^{m}\frac{\sigma_i(Q)}{(\varphi(|Q|)|Q|)^{1-\beta}}\right)^q \nu \mathrm{d}x\right)^{\frac{1}{q}}\prod_{i=1}^{m}\left(\int_Q \sigma_i \mathrm{d}x\right)^{-\frac{1}{p_i}}\\
&\leq\left(\int_Q \mathcal{M}_{\varphi,\beta}(\sigma_1\chi_Q,\sigma_2\chi_Q,\ldots,\sigma_m\chi_Q)^q\nu \mathrm{d}x\right)^{\frac{1}{q}}\left(\prod_{i=1}^{m}\sigma_i(Q)^{\frac{1}{p_i}}\right)^{-1}\\
&\leq[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)}.
\end{aligned}
$$

Thus,

$$[\vec{\omega},\nu]_{\mathcal{A}_{(\vec{p},q),\beta}(\varphi)}\leq[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)}.$$

Combining Theorem 1.11 and Theorem 1.12, we have

$$[\vec{\omega},\nu]_{S_{(\vec{p},q),\beta}(\varphi)}\leq\|\mathcal{M}_{\varphi,\beta}\|_{L^{p_1}(\omega_1)\times L^{p_2}(\omega_2)\times\cdots\times L^{p_m}(\omega_m)\to L^q(\nu)}\leq C[\vec{\sigma},\nu]_{B_{(\vec{p},q),\beta}(\varphi)}.$$

Now, we prove (ii) holds. Let $\beta=0$, $q=p$, $m=1$, and $\nu=\omega$. Applying Jensen's inequality and $\sigma_i=\omega_i^{1-p_i'}$ $(i=1,2,\ldots,m)$, we have

$$
\begin{aligned}
&(\varphi(|Q|)|Q|)^{m\beta+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \nu \mathrm{d}x\right)^{\frac{1}{q}}\left(\prod_{i=1}^{m}\frac{1}{\varphi(|Q|)|Q|}\int_Q \sigma_i \mathrm{d}x\right)\\
&\times\exp\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \log\prod_{i=1}^{m}\sigma_i^{-\frac{1}{p_i}}\mathrm{d}x\right)\\
&=\left(\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega \mathrm{d}x\right)\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega^{1-p'}\mathrm{d}x\right)^{p-1}\right)^{\frac{1}{p-1}}\\
&\times\left(\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega \mathrm{d}x\right)^{-1}\exp\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \log\omega \mathrm{d}x\right)\right)^{\frac{1}{p(p-1)}}\\
&\leq[\omega]_{A_{p,\beta}(\varphi)}^{\frac{p}{p-1}}.
\end{aligned}
$$

Thus, $[\vec{\sigma},\nu]_{B_{(\vec{p},q),\beta}(\varphi)}\leq[\omega]_{A_{p,\beta}(\varphi)}^{\frac{p}{p-1}}$. The proof of Theorem 1.13 is complete. □

Proof of Theorem 1.14. We first consider $\omega_i^{-p_i'}\in A_{mp_i',\beta}(\varphi)$ $(i=1,2,\ldots,m)$. Let $h_i=p(m-\frac{1}{p_i'})$ and $h_j=\frac{h_i p_j'}{p}$ $(1\leq j\neq i\leq m)$. Then $1<h_j<\infty$ and

$\frac{h_i}{p} = m - \frac{1}{p_i'} = \frac{1}{p} + \sum_{1\le j\ne i\le m} \frac{1}{p_j'}$. Noting that $0 \le \beta \le \frac{\gamma}{1+\gamma}$, then

$$\frac{|Q|}{(\varphi(|Q|)|Q|)^{1-\beta}} = \frac{|Q|^{\beta}}{(1+|Q|)^{\gamma(1-\beta)}} \le \left(\frac{|Q|}{1+|Q|}\right)^{\beta} \le 1. \tag{6.1}$$

Hence,

$$\begin{aligned}
&\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \left(\int_Q \omega_i^{-p_i'}\right)^{\frac{1}{mp_i'}} \left(\int_Q \omega_i^{\frac{p_i'}{mp_i'-1}}\right)^{\frac{mp_i'-1}{mp_i'}} \\
&= \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_i^{-p_i'}\right)^{\frac{1}{mp_i'}} \\
&\quad \times \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{j=1}^m \omega_j\right)^{\frac{p}{h_i}} \left(\prod_{j\ne i} \omega_j\right)^{-\frac{p}{h_i}}\right)^{\frac{h_i}{mp}},
\end{aligned}$$

where,

$$\begin{aligned}
&\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{j=1}^m \omega_j\right)^{\frac{p}{h_i}} \left(\prod_{j\ne i} \omega_j\right)^{-\frac{p}{h_i}}\right)^{\frac{h_i}{mp}} \\
&\le \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \left(\int_Q \left(\prod_{j=1}^m \omega_j\right)^{p}\right)^{\frac{1}{h_i}} \prod_{j\ne i} \left(\int_Q \omega_j^{-p_j'}\right)^{\frac{1}{h_j}}\right)^{\frac{h_i}{mp}} \\
&= \left[\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{j=1}^m \omega_j\right)^{p}\right)^{\frac{1}{p}} \prod_{j\ne i} \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_j^{-p_j'}\right)^{\frac{1}{p_j'}}\right]^{\frac{1}{m}} \\
&\le \left[\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{j=1}^m \omega_j\right)^{q}\right)^{\frac{1}{q}} \prod_{j\ne i} \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_j^{-p_j'}\right)^{\frac{1}{p_j'}}\right]^{\frac{1}{m}},
\end{aligned}$$

here we have used that the inequality (6.1) and Hölder's inequality. Therefore,

$$\begin{aligned}
&\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \left(\int_Q \omega_i^{-p_i'}\right)^{\frac{1}{mp_i'}} \left(\int_Q \omega_i^{\frac{p_i'}{mp_i'-1}}\right)^{\frac{mp_i'-1}{mp_i'}} \\
&\le \left[\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{j=1}^m \omega_j\right)^{q}\right)^{\frac{1}{q}} \prod_{j=1}^m \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_j^{-p_j'}\right)^{\frac{1}{p_j'}}\right]^{\frac{1}{m}}
\end{aligned}$$

$\leq C.$

Now, we consider $(\prod_{i=1}^m \omega_i)^q \in A_{mq,\beta}(\varphi)$. Let $q_i = \frac{qp_i}{p}$ $(i = 1, 2, \ldots, m)$. Since $q \geq p$, then $q_i \geq p_i$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$, and $m - \frac{1}{q} = \frac{1}{q_1'} + \frac{1}{q_2'} + \cdots + \frac{1}{q_m'}$. Set $r_i = q_i'(m - \frac{1}{q}) > 1$. Thus, using the inequality (6.1) and Hölder's inequality, we obtain

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \left(\int_Q \left(\prod_{i=1}^m \omega_i \right)^q \right)^{\frac{1}{mq}} \left(\int_Q \left(\prod_{i=1}^m \omega_i \right)^{-\frac{q}{mq-1}} \right)^{\frac{mq-1}{mq}}$$
$$\leq \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{i=1}^m \omega_i \right)^q \right)^{\frac{1}{mq}} \left(\prod_{i=1}^m \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_i^{-q_i'} \right)^{\frac{1}{q_i'}} \right)^{\frac{1}{m}}$$
$$\leq \left(\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \left(\prod_{i=1}^m \omega_i \right)^q \right)^{\frac{1}{q}} \prod_{i=1}^m \left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\beta}} \int_Q \omega_i^{-p_i'} \right)^{\frac{1}{p_i'}} \right)^{\frac{1}{m}}$$
$$\leq C.$$

This completes the proof of Theorem 1.14. □

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