

LEFT INVARIANT LORENTZIAN METRICS AND CURVATURES ON NON-UNIMODULAR LIE GROUPS OF DIMENSION THREE

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ABSTRACT. For each connected and simply connected three-dimensional non-unimodular Lie group, we classify the left invariant Lorentzian metrics up to automorphism, and study the extent to which curvature can be altered by a change of metric. Thereby we obtain the Ricci operator, the scalar curvature, and the sectional curvatures as functions of left invariant Lorentzian metrics on each of these groups.

Our study is a continuation and extension of the previous studies done in [3] for Riemannian metrics and in [1] for Lorentzian metrics on unimodular Lie groups.

1. Introduction

Let G be a connected and simply connected, three-dimensional Lie group. The classification of all left invariant Riemannian metrics on G up to automorphism of G is completely carried out in [3]. For the Lorentzian case, a complete classification is done in [1] when G is unimodular.

We continue and extend the study in [1] to all non-unimodular Lie groups G . So, our basic references are [3] and [1]. We also refer to [8] for the classification of left invariant Riemannian metrics on 4-dimensional unimodular Lie groups. In this article, we will be concerned with two main problems:

- (1) to classify all the left invariant Lorentzian metrics on each connected, simply connected three-dimensional non-unimodular Lie group G up to automorphism, and
- (2) to study the extent to which curvature can be altered by a change of left-invariant Lorentzian metric.

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There are uncountably many nonisomorphic, connected and simply connected three-dimensional non-unimodular Lie groups. These are all solvable and of the form $\mathbb{R}^2 \rtimes_{\varphi} \mathbb{R}$ where \mathbb{R} acts on \mathbb{R}^2 via a linear map φ , see Section 2.2. Let G be such a Lie group. Let $\mathfrak{M}(G)$ be the space of left invariant Lorentzian metrics on G . Then there is a natural action of $\text{Aut}(G)$ on the space $\mathfrak{M}(G)$. Our first goal is determine the moduli space $\mathfrak{M}(G)/\text{Aut}(G)$.

Let $h \in \mathfrak{M}(G)$ and let \mathfrak{B} be an orthonormal basis for the Lie algebra \mathfrak{g} of G with respect to h . By [5], there is a unique linear transformation $L : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the formula $[u, v] = L(u \times v)$ for all $u, v \in \mathfrak{g}$, and G is unimodular if and only if such a linear transformation L is self adjoint, i.e., $h(L(u), v) = h(u, L(v))$ for all $u, v \in \mathfrak{g}$. In this case, the matrix $[L]_{\mathfrak{B}}$ is Lorentzian symmetric; that is, $[L]_{\mathfrak{B}}^t = J[L]_{\mathfrak{B}}J$, where $J = J_{2,1} = \text{diag}\{1, 1, -1\}$. The authors in [1] give a complete computation of the moduli space $\text{Aut}(G) \backslash \mathfrak{M}(G)$ using these Lorentzian symmetric matrices. However, this procedure is possible only when G is unimodular. In this article, for this reason, we shall give a complete computation of the moduli space $\mathfrak{M}(G)/\text{Aut}(G)$ in a direct way since G is non-unimodular.

In Section 2.1, we review some properties related to the left invariant Lorentzian metric on a connected Lie group. In Section 2.2, we recall the three-dimensional non-unimodular Lie algebras and their groups of automorphisms ([3]).

In Section 3, we classify all the left invariant Lorentzian metrics on the three-dimensional non-unimodular Lie group up to automorphism.

In Section 4, we review Ricci operator and curvatures, sectional curvatures, and scalar curvatures of the left invariant Lorentzian metric on a connected Lie group. We find out the relationship between the Ricci curvature and the Ricci operator. Furthermore, we find an identity which is the Lorentzian version of the formula given by ([5, p. 306]) about sectional curvature $\kappa(u, v)$ associated with u and v . In particular, we classify explicitly three-dimensional Lorentzian non-unimodular Lie groups whose metrics have constant sectional curvatures.

All the calculations were done using the program MATHEMATICA and hand-checked.

2. Preliminaries

2.1. Left invariant Lorentzian metrics on Lie groups

Let h be a Lorentzian metric on a connected Lie group G . That is, for each point $p \in G$, h_p is a nondegenerate symmetric bilinear form on the tangent space T_pG and one can find a basis e_1, \dots, e_n of T_pG such that

$$\left[h_p(e_i, e_j) \right] = \left[h(e_i, e_j)_p \right] = J.$$

A Lorentzian inner product on \mathbb{R}^3 is a bilinear product such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t J \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

From now on, G is a connected and simply connected *non-unimodular* Lie group of dimension three. A Lorentzian metric on \mathfrak{g} is a Lorentzian inner product on \mathfrak{g} . That is, there exists a linear basis $\mathfrak{B}_0 = \{e_1, e_2, e_3\}$ for \mathfrak{g} with respect to which $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t J \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{g}$. Once Lorentzian inner product is given on \mathfrak{g} , it can be extended to the whole group G by (the left translations):

$$\langle \mathbf{x}, \mathbf{y} \rangle_a = \langle (\ell_a)_*^{-1} \mathbf{x}, (\ell_a)_*^{-1} \mathbf{y} \rangle$$

for all $a \in G$ and $\mathbf{x}, \mathbf{y} \in T_a G$, where the right hand side is the Lorentzian inner product on \mathfrak{g} . This is a left invariant Lorentzian metric on G .

We now consider the totality of left invariant Lorentzian metrics on G . For any linear basis $\mathfrak{B} = \{x_1, x_2, x_3\}$ for \mathfrak{g} , one can declare it be an orthonormal basis. That is,

$$\left[\langle x_i, x_j \rangle' \right] = J.$$

The new inner product is denoted by \langle, \rangle' . Clearly, this gives rise to a new left invariant Lorentzian metric h on G . Let P be a transition matrix from the basis \mathfrak{B}_0 to the basis \mathfrak{B} . Then

$$h(e_i, e_j) = \langle e_i, e_j \rangle' = (P^t J P)_{(i,j)}, \quad (i, j)\text{-component of } P^t J P.$$

Hence $[h] = [h]_{\mathfrak{B}_0} = P^t J P$.

Therefore, with the fixed basis \mathfrak{B}_0 , and varying $P \in \text{GL}(3, \mathbb{R})$, we get the new metric $P^t J P$. Notice that some P 's are redundant. For example, if $P \in \text{O}(2, 1)$, then clearly $P^t J P = J$. Let

$$\mathfrak{M}(G) = \{P^t J P : P \in \text{GL}(3, \mathbb{R})\}.$$

Then the symmetric matrix $P^t J P = [h_{ij}]$ is a left-invariant Lorentzian metric h defined by:

$$h(\mathbf{x}, \mathbf{y}) = [\mathbf{x}]^t (P^t J P) [\mathbf{y}]$$

for all $\mathbf{x}, \mathbf{y} \in \mathfrak{g}$, where $[\mathbf{x}] = [\mathbf{x}]_{\mathfrak{B}_0}$. Note that $\mathfrak{B}' := P^{-1} \mathfrak{B}_0 = \{y_1, y_2, y_3\}$ is an h -orthonormal basis for \mathfrak{g} , because

$$\begin{aligned} [h]_{\mathfrak{B}'} &= \left[h(y_i, y_j) \right] = \left[[y_i]_{\mathfrak{B}_0}^t (P^t J P) [y_j]_{\mathfrak{B}_0} \right] \\ &= \left[(P^{-1} [e_i]_{\mathfrak{B}_0})^t (P^t J P) (P^{-1} [e_j]_{\mathfrak{B}_0}) \right] = J. \end{aligned}$$

This is equivalent to: \mathfrak{B}_0 yields global vector fields (by ℓ_a) and its dual forms ω_i . Then use the formula $h = \sum h_{ij} \omega_i \otimes \omega_j$ so that the 2-form h is globally defined. This way, for any $P \in \text{GL}(3, \mathbb{R})$, $P^t J P$ becomes a left-invariant Lorentzian metric on G .

From these observations, we have the following:

Theorem 2.1. *Let $[h_{ij}]$ be a symmetric, nondegenerate matrix, and let $h = \sum h_{ij} \omega_i \otimes \omega_j$. In order for h to be a left invariant Lorentzian metric, it is necessary and sufficient that there exists $P \in \text{GL}(3, \mathbb{R})$ for which $[h_{ij}] = P^t J P$.*

We remark that each left invariant Lorentzian metric on G is determined by an inner product at the tangent space $T_e G = \mathfrak{g}$ and the inner product is given by a symmetric, nondegenerate matrix of the form $P^t J P$, where $P \in \mathrm{GL}(3, \mathbb{R})$.

Notice that if $\theta : G \rightarrow G$ is a diffeomorphism on G , then θ induces a Lorentzian metric on G by the rule $h_\theta(x, y)_p = h(\theta_*^{-1}(x), \theta_*^{-1}(y))_{\theta^{-1}(p)}$, $p \in G$, $x, y \in T_p G$, where θ_* is the differential of θ . Even though h is left invariant, the induced metric h_θ is not necessarily left invariant.

Now consider two actions by $\mathrm{GL}(3, \mathbb{R})$ on $\mathfrak{M}(G)$. First, the group $\mathrm{GL}(3, \mathbb{R})$ acts on $\mathfrak{M}(G)$ (of course, from the left) by:

For $A \in \mathrm{GL}(3, \mathbb{R})$ and $P^t J P \in \mathfrak{M}(G)$,

$$\mu(A)(P^t J P) := (AP)^t J (AP) \quad (\text{left multiplication by } A \text{ on } P\text{-factor}).$$

Then clearly, $\mu(AB) = \mu(A) \circ \mu(B)$. Notice that, this action is transitive and the isotropy subgroup is

$$\begin{aligned} \mathrm{GL}(3, \mathbb{R})_{P^t J P} &= \{A \in \mathrm{GL}(3, \mathbb{R}) \mid \mu(A)P^t J P = P^t J P\} \\ &= \{A \in \mathrm{GL}(3, \mathbb{R}) \mid \mu(A)J = J\} = O(2, 1). \end{aligned}$$

On the other hand, $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{G})$ is also a subset of $\mathrm{GL}(3, \mathbb{R})$, and it acts on $\mathfrak{M}(G)$ as follows:

For $\varphi \in \mathrm{Aut}(G)$, and $P^t J P \in \mathfrak{M}(G)$,

$$\begin{aligned} \xi(\varphi)(P^t J P) &:= [\varphi^{-1}]^t (P^t J P) [\varphi^{-1}] \\ &= (P[\varphi^{-1}])^t J (P[\varphi^{-1}]) \\ &\quad (\text{right multiplication by } [\varphi^{-1}] \text{ on } P\text{-factor}), \end{aligned}$$

where $[\varphi^{-1}] = [\varphi^{-1}]_{\mathfrak{B}_0}$. Then clearly, $\xi(\psi \circ \varphi) = \xi(\psi) \circ \xi(\varphi)$ for $\varphi, \psi \in \mathrm{Aut}(G)$.

Furthermore, these two actions (being left/right multiplications) commute each other:

$$\begin{array}{ccc} \mathfrak{M}(G) & \xrightarrow{\mu(A)} & \mathfrak{M}(G) & \xrightarrow{P^t J P} & \mathfrak{M}(G) & \xrightarrow{(AP)^t J (AP)} & \mathfrak{M}(G) \\ \xi(\varphi) \downarrow & & \downarrow \xi(\varphi) & & \downarrow & & \downarrow \\ \mathfrak{M}(G) & \xrightarrow{\mu(A)} & \mathfrak{M}(G) & \xrightarrow{[\varphi^{-1}]^t (P^t J P) [\varphi^{-1}]} & \mathfrak{M}(G) & \xrightarrow{[\varphi^{-1}]^t ((AP)^t J (AP)) [\varphi^{-1}]} & \mathfrak{M}(G) \\ & & & & & = (AP[\varphi^{-1}])^t J (AP[\varphi^{-1}]) & \end{array}$$

If we write the metric $P^t J P$ simply by $P \in \mathrm{GL}(3, \mathbb{R})$, then the above diagram becomes:

$$\begin{array}{ccc} \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{\mu(A)} & \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{P} & \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{\ell_A} & \mathrm{GL}(3, \mathbb{R}) \\ \xi(\varphi^{-1}) \downarrow & & \downarrow \xi(\varphi^{-1}) & & \downarrow r_{[\varphi^{-1}]} & & \downarrow r_{[\varphi^{-1}]} \\ \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{\mu(A)} & \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{P[\varphi^{-1}]} & \mathrm{GL}(3, \mathbb{R}) & \xrightarrow{\ell_A} & \mathrm{GL}(3, \mathbb{R}) \\ & & & & & = (AP)[\varphi^{-1}] = A(P[\varphi^{-1}]) & \end{array}$$

Then we must calculate the quotient

$$O(2, 1) \backslash \mathrm{GL}(3, \mathbb{R}) / \mathrm{Aut}(G) = \mathfrak{M}(G) / \mathrm{Aut}(G).$$

Now we describe explicitly the action of $\text{Aut}(G) = \text{Aut}(\mathfrak{g})$ on $\mathfrak{M}(G)$. Let h be a left invariant Lorentzian metric on G and $\varphi \in \text{Aut}(G)$. Define a metric h_φ on G by

$$h_\varphi(x, y)_p = h(\varphi^{-1}(x), \varphi^{-1}(y))_{\varphi^{-1}(p)}, \quad p \in G, x, y \in T_p G.$$

Then for $x, y \in G_p$,

$$\begin{aligned} & h_\varphi((\ell_{\varphi(a)})_*^{-1}(x), (\ell_{\varphi(a)})_*^{-1}(y))_{\varphi(a^{-1})p} \\ &= h(\varphi^{-1}(\ell_{\varphi(a)})_*^{-1}(x), \varphi^{-1}(\ell_{\varphi(a)})_*^{-1}(y))_{\varphi^{-1}(\varphi(a^{-1})p)} \quad (\text{by definition of } h_\varphi) \\ &= h((\ell_a)_*^{-1}\varphi^{-1}(x), (\ell_a)_*^{-1}\varphi^{-1}(y))_{a^{-1}\varphi^{-1}(p)} \quad (\text{from } \varphi \circ \ell_a = \ell_{\varphi(a)} \circ \varphi) \\ &= h(\varphi^{-1}(x), \varphi^{-1}(y))_{\varphi^{-1}(p)} \quad (h \text{ is left-invariant}) \\ &= h_\varphi(x, y)_p \quad (\text{by definition of } h_\varphi \text{ again}). \end{aligned}$$

Thus, h_φ is also left-invariant.

We remark that if $[h] = P^t J P$ for some $P \in \text{GL}(3, \mathbb{R})$, then

$$[h_\varphi] = (P[\varphi^{-1}])^t J (P[\varphi^{-1}]) = [\varphi^{-1}]^t [h] [\varphi^{-1}].$$

A left invariant Lorentzian metric h' on G is *equivalent up to automorphism* to a left invariant Lorentzian metric h if there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $h' = h_{\varphi^{-1}}$, or equivalently, $[h'] = [\varphi]^t [h] [\varphi]$. In this case we write $h' \sim h$ or $[h'] \sim [h]$, and we say that $[h']$ is equivalent up to automorphism to $[h]$.

2.2. The three-dimensional non-unimodular Lie algebras

There are uncountably many nonisomorphic three-dimensional non-unimodular, solvable Lie algebras and a basis may be chosen so that

- (a) $[x, y] = 0$, $[z, x] = x$, $[z, y] = y$, or
- (b) $[x, y] = 0$, $[z, x] = y$, $[z, y] = -cx + 2y$,

where $c \in \mathbb{R}$. Note that $\text{ad}(z) = \begin{bmatrix} 0 & -c \\ 1 & -2 \end{bmatrix}$ has trace 2 and determinant c . A reference is [3]. Such a Lie algebra is isomorphic to either \mathfrak{g}_I or \mathfrak{g}_c for some $c \in \mathbb{R}$ where

$$\begin{aligned} \mathfrak{g}_I &\cong \mathbb{R}^2 \rtimes_{\sigma_I} \mathbb{R}, \text{ where } \sigma_I(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}; \\ \mathfrak{g}_c &\cong \mathbb{R}^2 \rtimes_{\sigma_c} \mathbb{R}, \text{ where } \sigma_c(t) = \begin{bmatrix} 0 & -ct \\ t & 2t \end{bmatrix} \end{aligned}$$

with a “natural” basis

$$x = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right), y = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right), z = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right).$$

Then they satisfy Lie bracket conditions (a) or (b), respectively.

For any non-unimodular solvable Lie algebra \mathfrak{g} , its group of automorphisms is given as follows:

(1) The Lie group $\text{Aut}(\mathfrak{g}_I)$ is isomorphic to

$$\left\{ \left[\begin{array}{cc|c} \text{GL}(2, \mathbb{R}) & * & \\ \hline 0 & & 1 \end{array} \right] \mid * \in \mathbb{R}^2 \right\}.$$

(2) For each $c \in \mathbb{R}$, the Lie group $\text{Aut}(\mathfrak{g}_c)$ is isomorphic to

$$\left\{ \left[\begin{array}{ccc|c} \beta - \alpha & -c\alpha & * & \\ \alpha & \beta + \alpha & * & \\ 0 & 0 & 1 & \end{array} \right] \mid \begin{array}{l} \alpha, \beta, * \in \mathbb{R}, \\ \beta^2 + (c-1)\alpha^2 \neq 0 \end{array} \right\}.$$

The three-dimensional non-unimodular Lie algebra \mathfrak{g}_I or \mathfrak{g}_c is the Lie algebra of the connected and simply connected three-dimensional Lie group

$$G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}, \text{ where } \varphi_I(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}, \text{ or}$$

$$G_c \cong \mathbb{R}^2 \rtimes_{\varphi_c} \mathbb{R}, \text{ where}$$

$$\varphi_c(t) = \begin{cases} e^t \frac{e^{wt} + e^{-wt}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \frac{e^{wt} - e^{-wt}}{2w} \begin{bmatrix} -1 & -c \\ 1 & 1 \end{bmatrix} & \text{if } w = \sqrt{1-c} \neq 0, \\ e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t t \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} & \text{if } c = 1. \end{cases}$$

Note that $\varphi_c(t)$ is real even when $c > 1$.

3. Left invariant Lorentzian metrics

In this section, we will compute the moduli space $\mathfrak{M}(G)/\text{Aut}(G)$ for the three-dimensional non-unimodular Lie groups G .

3.1. The case of G_I

Recalling from Section 2.2 that

$$\text{Aut}(\mathfrak{g}_I) = \left\{ \left[\begin{array}{cc|c} \text{GL}(2, \mathbb{R}) & * & \\ \hline 0 & & 1 \end{array} \right] \mid * \in \mathbb{R}^2 \right\},$$

we obtain the following result.

Theorem 3.1. *Any left invariant Lorentzian metric on G_I is equivalent up to automorphism to a metric whose associated matrix $(P^t J P = [h_{ij}])$ is one of the following forms:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where $\mu > 0$.

Proof. Let h be a left invariant Lorentzian metric on G_I . Then $[h] = [h_{ij}]$ is a symmetric and non-degenerate real 3×3 matrix. Since $[h]$ is symmetric and $\begin{bmatrix} \text{SO}(2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \subset \text{Aut}(\mathfrak{g})$, we may assume that $h_{12} = 0$. First assume $h_{11}h_{22} \neq$

0. Then $C = \begin{bmatrix} 1 & 0 & -\frac{h_{13}}{h_{11}} \\ 0 & 1 & -\frac{h_{23}}{h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_I)$ and $C^t B^t [h] B C = \text{diag}\{h_{11}, h_{22}, \mu\}$.

Since only one of h_{11}, h_{22} and μ is negative, either $h_{11}h_{22} < 0, \mu > 0$ or $h_{11} > 0, h_{22} > 0, \mu < 0$. If $h_{11}h_{22} < 0$ and $\mu > 0$, then we may assume that $h_{11} > 0$ and $h_{22} < 0$. Let $D = \text{diag}\{\frac{1}{\sqrt{h_{11}}}, \frac{1}{\sqrt{-h_{22}}}, 1\} \in \text{Aut}(\mathfrak{g}_I)$. Then $D^t C^t B^t [h] B C D = \text{diag}\{1, -1, \mu\}$. If $h_{11} > 0, h_{22} > 0$ and $\mu < 0$, then $D = \text{diag}\{\frac{1}{\sqrt{h_{11}}}, \frac{1}{\sqrt{h_{22}}}, 1\} \in \text{Aut}(\mathfrak{g}_I)$ and $D^t C^t B^t [h] B C D = \text{diag}\{1, 1, \mu\}$.

Now assume $h_{11}h_{22} = 0$. Since one of h_{11} and h_{22} is non-zero, we may assume $h_{11} \neq 0$ and $h_{22} = 0$. Since $\det[h] = -h_{11}h_{23}^2 < 0$, we have $h_{11} > 0$ and $h_{23} \neq 0$. Let $C = \begin{bmatrix} \frac{1}{\sqrt{h_{11}}} & 0 & -\frac{h_{13}}{h_{11}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_I)$. Then $C^t B^t [h] B C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & \nu \end{bmatrix}$.

Since $\det(C^t B^t [h] B C) = -\mu^2 < 0$, we have $\mu \neq 0$. Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\mu} & -\frac{\nu}{\mu} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_I)$. Then $D^t C^t B^t [h] B C D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Finally it is easy to see that any three such distinct matrices are not equivalent. \square

3.2. The case of $G_c, c > 1$

Recalling from Section 2.2 that

$$\text{Aut}(\mathfrak{g}_c) = \left\{ \left[\begin{array}{ccc} \beta - \alpha & -c\alpha & * \\ \alpha & \beta + \alpha & * \\ 0 & 0 & 1 \end{array} \right] \mid \alpha, \beta, * \in \mathbb{R}, \beta^2 + (c-1)\alpha^2 \neq 0 \right\},$$

we obtain the following result.

Theorem 3.2. *Any left invariant Lorentzian metric on G_c with $c > 1$ is equivalent up to automorphism to a metric whose associated matrix ($P^t J P = [h_{ij}]$) is one of the following forms:*

$$\begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & \nu & 0 \\ 0 & 0 & -\mu \end{bmatrix},$$

where $\mu > 0, \tau < 1$ and $1 < \nu \leq c$.

Proof. Let h be a left invariant Lorentzian metric on G_c with $c > 1$. Then $[h] = [h_{ij}]$ is a symmetric and non-degenerate real 3×3 matrix.

First suppose $h_{11}h_{22} - h_{12}^2 = 0$. If $h_{12} \neq 0$, then $h_{11} \neq 0, h_{22} \neq 0$. Since $(h_{11} - h_{12})^2 + (c-1)h_{11}^2 > 0$, $B = \begin{bmatrix} -h_{12} & -ch_{11} & 0 \\ h_{11} & 2h_{11} - h_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t [h] B = [h'_{ij}]$, where $h'_{11} = h'_{12} = 0, h'_{13} \neq 0$ and $h'_{22} \neq 0$. Thus we may assume that $h_{12} = 0$. Then either $h_{11} \neq 0, h_{22} = 0$ or $h_{11} = 0, h_{22} \neq 0$. Suppose

$h_{11} = 0$ and $h_{22} \neq 0$. Since $B = \begin{bmatrix} -2 & -c & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = [h'_{ij}]$, where $h'_{11} = h_{22} \neq 0$ and $h'_{12} = h'_{22} = 0$, we may assume that $h_{11} \neq 0$ and $h_{22} = 0$. Then we have $h_{23} \neq 0$, $C = \begin{bmatrix} \frac{1}{h_{23}} & 0 & -\frac{h_{13}}{h_{11}} \\ 0 & \frac{1}{h_{23}} & \frac{h_{13}^2 - h_{11}h_{33}}{2h_{11}h_{23}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $C^t[h]C = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, where $\mu > 0$.

Now suppose $h_{11}h_{22} - h_{12}^2 \neq 0$. First we may assume that $h_{13} = h_{23} = 0$ because $B = \begin{bmatrix} 1 & 0 & \frac{h_{13}h_{22} - h_{12}h_{23}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 1 & \frac{h_{11}h_{23} - h_{12}h_{13}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = [h']$, where $h'_{13} = h'_{23} = 0$.

If $h_{11} \neq h_{12}$, then the quadratic equation

$$(h_{11} - h_{12})z^2 + ((c-2)h_{11} + 2h_{12} - h_{22})z - (c-1)(h_{11} - h_{12}) = 0$$

has a real root, say β , because $-(c-1)(h_{11} - h_{12})^2 < 0$. Then $B = \begin{bmatrix} \beta-1 & -c & 0 \\ 1 & \beta+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = [h']$, where $h'_{11} = h'_{12}$ and $h'_{13} = h'_{23} = 0$. Thus we may also assume that $h_{11} = h_{12}$. Since $\det[h_{ij}] = -h_{11}h_{33}(h_{11} - h_{22}) < 0$, we have $h_{11} \neq 0$. Let $B = \text{diag}\{\frac{1}{\sqrt{|h_{11}|}}, \frac{1}{\sqrt{|h_{11}|}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B$ is of the form

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $\mu(\tau - 1) < 0$.

Now consider equivalences up to automorphism between the following three types of left invariant Lorentzian metrics:

$$\text{Type 1. } \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{Type 2. } \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \text{Type 3. } \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

(1) Two different matrices of type 1 are not equivalent. And every matrix of type 1 is not equivalent to that of type 2 or type 3.

(2) Suppose $[h] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $[h'] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau' & 0 \\ 0 & 0 & \mu' \end{bmatrix}$ and $[h] \sim [h']$. Then $B^t[h]B =$

$$[h'] \text{ for some } B = \begin{bmatrix} \beta-\alpha & -c\alpha & x \\ \alpha & \beta+\alpha & y \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c). \text{ Thus we have}$$

- (i) $\beta x + (\beta + \alpha(\tau - 1))y = 0,$
- (ii) $(\beta + \alpha - c\alpha)x + ((\beta + \alpha)\tau - c\alpha)y = 0,$
- (iii) $\beta^2 - \alpha^2 + \tau\alpha^2 = 1,$
- (iv) $\beta^2 - \alpha^2 + \tau\alpha^2 + \alpha\beta(\tau - c) = 1,$
 $\tau\beta^2 + 2(\tau - c)\alpha\beta + (c^2 - 2c + \tau)\alpha^2 = \tau',$

$$\mu + x^2 + 2xy + \tau y^2 = \mu'.$$

Considering the linear system of equations in the variables x, y given by (i), (ii), the determinant of the coefficient matrix is $(\beta^2 + (c-1)\alpha^2)(\tau - 1) \neq 0$. This implies $x = y = 0$ and $\mu' = \mu$. Using (iii) and (iv), we have $\alpha\beta(\tau - c) = 0$.

- If $\beta = 0$, then $\tau > 1$ and $\alpha = \pm \frac{1}{\sqrt{\tau-1}}$ and $\tau' = 1 + \frac{(c-1)^2}{\tau-1}$.
- If $\alpha = 0$, then $\beta = \pm 1$ and $\tau' = \tau$.
- If $\tau = c$, then $\beta^2 + (c-1)\alpha^2 = 1$ and $\tau' = \tau = c$.

Hence we have

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau' & 0 \\ 0 & 0 & \mu' \end{bmatrix} \\ \Leftrightarrow \mu' = \mu, \tau' &= \begin{cases} \tau, & \text{if } \tau < 1, \\ 1 + \frac{(c-1)^2}{\tau-1}, & \text{if } \tau > 1. \end{cases} \end{aligned}$$

(3) Similarly, we have

$$\begin{aligned} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & \mu \end{bmatrix} &\sim \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau' & 0 \\ 0 & 0 & \mu' \end{bmatrix} \\ \Leftrightarrow \mu' = \mu, \tau' &= \begin{cases} \tau, & \text{if } \tau < 1, \\ 1 + \frac{(c-1)^2}{\tau-1}, & \text{if } \tau > 1. \end{cases} \end{aligned}$$

(4) Suppose $[h] = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $[h'] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau' & 0 \\ 0 & 0 & \mu' \end{bmatrix}$ and $[h] \sim [h']$. Then there

is $B = \begin{bmatrix} \beta - \alpha & -c\alpha & x \\ \alpha & \beta + \alpha & y \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ such that $B^t[h]B = [h']$. That is,

$$\begin{aligned} \beta x + (\beta + \alpha(\tau - 1))y &= 0, \\ (\beta + \alpha - c\alpha)x + ((\beta + \alpha)\tau - c\alpha)y &= 0, \\ -\beta^2 + \alpha^2 - \tau\alpha^2 &= 1, \\ -\beta^2 + \alpha^2 - \tau\alpha^2 - \alpha\beta(\tau - c) &= 1, \\ -\tau\beta^2 - 2(\tau - c)\alpha\beta - (c^2 - 2c + \tau)\alpha^2 &= \tau', \\ \mu - x^2 - 2xy - \tau y^2 &= \mu'. \end{aligned}$$

This implies $x = y = 0$, $\mu' = \mu$ and $\alpha\beta(\tau - c) = 0$.

- If $\beta = 0$, then $\tau < 1$ and $\alpha = \pm \frac{1}{\sqrt{1-\tau}}$ and $\tau' = 1 - \frac{(c-1)^2}{1-\tau}$.
- If $\alpha = 0$, then $\beta^2 + 1 = 0$, which is a contradiction.
- If $\tau = c$, then $-\beta^2 - (c-1)\alpha^2 = 1$, which is a contradiction.

Hence we have

$$\begin{aligned} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & \mu \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau' & 0 \\ 0 & 0 & \mu' \end{bmatrix} \\ \iff \tau < 1, \mu' = \mu, \tau' = 1 - \frac{(c-1)^2}{1-\tau}. \end{aligned}$$

(5) Suppose $[h] = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\tau & 0 \\ 0 & 0 & -\mu \end{bmatrix}$ with $1 < \tau \leq c$ and $\mu > 0$. Let

$$y_1 = x, \quad y_2 = \frac{1}{\sqrt{\rho-1}}(x-y), \quad y_3 = \frac{1}{\sqrt{\mu}}z.$$

Then $\mathcal{B} = \{y_1, y_2, y_3\}$ is a basis of the Lie algebra of G_c such that $[h]_{\mathcal{B}} = \text{diag}\{-1, -1, -1\}$, which is a contradiction.

Hence the proof is completed. \square

3.3. The case of $G_c, c = 1$

Instead of the basis $\{x, y, z\}$ for \mathfrak{g}_1 given in Section 2.2, we choose a new basis $\{x_1, x_2, x_3\}$ for \mathfrak{g}_1 by putting $x_1 = -x + y$, $x_2 = -x + 2y$, $x_3 = z$. Then they satisfy

$$[x_1, x_2] = 0, \quad [x_3, x_1] = x_1, \quad [x_3, x_2] = x_1 + x_2.$$

With respect to this new basis, the Lie group $\text{Aut}(\mathfrak{g}_1)$ is isomorphic to

$$\left\{ \begin{bmatrix} \gamma & \delta & * \\ 0 & \gamma & * \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma, \delta, * \in \mathbb{R}, \gamma \neq 0 \right\}.$$

In fact, given γ, δ ,

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma & \delta & * \\ 0 & \gamma & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma - \delta & -\delta & * \\ \delta & \gamma + \delta & * \\ 0 & 0 & 1 \end{bmatrix},$$

where $[\text{id}]_{\{x,y,z\},\{x_1,x_2,x_3\}} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 3.3. *Any left invariant Lorentzian metric on G_1 is equivalent up to automorphism to a metric whose associated matrix ($P^t J P = [h_{ij}]$) is one of the following forms:*

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} M \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where M is of the form

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -\mu \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $\mu > 0$ and $\nu > 0$.

Proof. Let $\{x_1, x_2, x_3\}$ be the basis for \mathfrak{g}_c given by

$$x_1 = -x + y, \quad x_2 = -x + 2y, \quad x_3 = z.$$

Then

$$[x_1, x_2] = 0, \quad [x_3, x_1] = x_1, \quad [x_3, x_2] = x_1 + x_2$$

and the Lie group $\text{Aut}(\mathfrak{g}_1)$ is isomorphic to $\left\{ \begin{bmatrix} \gamma & \delta & * \\ 0 & \gamma & * \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma, \delta, * \in \mathbb{R}, \gamma \neq 0 \right\}$.

Let h be a left invariant Lorentzian metric on G_1 . Then with respect to the basis $\{x_1, x_2, x_3\}$, $[h] = [h_{ij}]$ is a symmetric and non-degenerate real 3×3 matrix.

First assume that $h_{11}h_{22} - h_{12}^2 = 0$. If $h_{12} \neq 0$, then $h_{11} \neq 0$, $h_{22} \neq 0$. Let $B = \begin{bmatrix} h_{11} & -h_{12} & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B = [h'_{ij}]$, where $h'_{12} = 0$. Thus we may assume that $h_{12} = 0$. Then either $h_{11} = 0$, $h_{22} \neq 0$ or $h_{11} \neq 0$, $h_{22} = 0$. When $h_{11} = 0$ and $h_{22} \neq 0$, $\det([h]) = -h_{22}h_{13}^2 < 0$ and so we have $h_{22} > 0$ and $h_{13} \neq 0$. Let $B = \begin{bmatrix} \frac{1}{h_{13}} & 0 & 0 \\ 0 & \frac{1}{h_{13}} & -\frac{h_{23}}{h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mu & 0 \\ 1 & 0 & \nu \end{bmatrix}$.

Let $C = \begin{bmatrix} 1 & 0 & -\frac{\nu}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^tB^t[h]BC = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix}$, where $\mu > 0$.

Similarly, when $h_{11} \neq 0$ and $h_{22} = 0$, then $h_{23} \neq 0$, $B = \begin{bmatrix} \frac{1}{h_{23}} & 0 & -\frac{h_{13}}{h_{11}} \\ 0 & \frac{1}{h_{23}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \nu \end{bmatrix}$. And let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\nu}{2} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^tB^t[h]BC = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, where $\mu > 0$.

Next suppose that $h_{11}h_{22} - h_{12}^2 \neq 0$. Then we may assume that $h_{13} = h_{23} = 0$ because $B = \begin{bmatrix} 1 & 0 & \frac{h_{13}h_{22} - h_{12}h_{23}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 1 & \frac{h_{11}h_{23} - h_{12}h_{13}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = [h']$, where $h'_{13} = h'_{23} = 0$.

If $h_{11} \neq 0$, then $B = \begin{bmatrix} h_{11} & -h_{12} & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = \text{diag}\{d_1, d_2, d_3\}$ where $d_1d_2d_3 < 0$. Since only one of d_1, d_2, d_3 is negative, either $d_1d_2 < 0$, $\lambda > 0$ or $d_1 > 0, d_2 > 0, \lambda < 0$. Let $C = \text{diag}\{\frac{1}{\sqrt{|d_1|}}, \frac{1}{\sqrt{|d_1|}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^tB^t[h]BC$ is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $\mu > 0$ and $\nu > 0$.

If $h_{11} = 0$, then $\det([h]) = -h_{33}(h_{12})^2$. Thus $h_{33} > 0$ and $h_{12} \neq 0$. Let $B = \text{diag}\{\frac{1}{\sqrt{|h_{12}|}}, \frac{1}{\sqrt{|h_{12}|}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B = \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm 1 & \nu & 0 \\ 0 & 0 & \mu \end{bmatrix}$. Let $C = \begin{bmatrix} 1 & \mp \frac{\nu}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^t B^t [h] B C$ is of the form

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $\mu > 0$. Finally it is easy to see that any four such distinct matrices are not equivalent. \square

3.4. The case of G_c , $c < 1$

Instead of the basis $\{x, y, z\}$ for \mathfrak{g}_c given in Section 2.2, we choose a new basis $\{x_1, x_2, x_3\}$ for \mathfrak{g}_c by putting $x_1 = -(1-w)x + y$, $x_2 = -(1+w)x + y$, $x_3 = z$ where $w = \sqrt{1-c}$. Then they satisfy

$$[x_1, x_2] = 0, \quad [x_3, x_1] = (1+w)x_1, \quad [x_3, x_2] = (1-w)x_2.$$

With respect to this new basis, the Lie group $\text{Aut}(\mathfrak{g}_c)$ is isomorphic to

$$\left\{ \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma, \delta, * \in \mathbb{R}, \gamma \delta \neq 0 \right\}.$$

In fact, given γ, δ ,

$$\begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \beta - \alpha & -c\alpha & * \\ \alpha & \beta + \alpha & * \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{where } \alpha = \frac{\delta - \gamma}{2z}, \beta = \frac{\delta + \gamma}{2} \text{ and } [\text{id}]_{\{x, y, z\}, \{x_1, x_2, x_3\}} = \begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 3.4. *Any left invariant Lorentzian metric on G_c with $c < 1$ is equivalent up to automorphism to a metric whose associated matrix ($P^t J P = [h_{ij}]$) is one of the following forms:*

$$\begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} M \begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where M is of the form

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \mu \\ 0 & \mu & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \nu \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -\eta & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $w = \sqrt{1-c}$, $\mu > 0$, $\nu(\tau-1) < 0$ and $\eta < 1$.

Proof. Let $\{x_1, x_2, x_3\}$ be the basis for \mathfrak{g}_c given by

$$x_1 = (-1+w)x + y, \quad x_2 = -(1+w)x + y, \quad x_3 = z,$$

where $w = \sqrt{1-c}$. Then

$$[x_1, x_2] = 0, \quad [x_3, x_1] = (1+w)x_1, \quad [x_3, x_2] = (1-w)x_2$$

and the Lie group $\text{Aut}(\mathfrak{g}_c)$ is isomorphic to $\left\{ \begin{bmatrix} \gamma & 0 & * \\ 0 & \delta & * \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma, \delta, * \in \mathbb{R}, \gamma\delta \neq 0 \right\}$.

Let h be a left invariant Lorentzian metric on G_c with $c < 1$. Then with respect to the basis $\{x_1, x_2, x_3\}$, $[h] = [h_{ij}]$ is a symmetric and non-degenerate real 3×3 matrix.

First assume that $h_{11}h_{22} - h_{12}^2 = 0$ and $h_{12} = 0$. Then either $h_{11} = 0$, $h_{22} \neq 0$ or $h_{11} \neq 0$, $h_{22} = 0$. Assume $h_{11} = 0$ and $h_{22} \neq 0$. Since $\det([h]) = -h_{22}h_{13}^2 < 0$, we have $h_{22} > 0$ and $h_{13} \neq 0$. Let $B = \begin{bmatrix} \frac{1}{h_{13}} & 0 & 0 \\ 0 & \frac{1}{h_{13}} & -\frac{h_{23}}{h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in$

$\text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mu & 0 \\ 1 & 0 & \nu \end{bmatrix}$. Since $\det(B^t[h]B) = -\mu < 0$, we have $\mu > 0$. Let $C = \begin{bmatrix} 1 & 0 & -\frac{\nu}{2} \\ 0 & \frac{1}{\sqrt{\mu}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^t B^t [h] B C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Similarly,

when $h_{11} \neq 0$ and $h_{22} = 0$, then $h_{23} \neq 0$, $B = \begin{bmatrix} \frac{1}{h_{23}} & 0 & -\frac{h_{13}}{h_{11}} \\ 0 & \frac{1}{h_{23}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$

and $B^t[h]B = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \nu \end{bmatrix}$. Since $\det(B^t[h]B) = -\mu < 0$, we have $\mu > 0$. Let $C = \begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & 1 & -\frac{\nu}{2} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^t B^t [h] B C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Now assume that $h_{11}h_{22} - h_{12}^2 = 0$ and $h_{12} \neq 0$. Then $h_{11} \neq 0$, $h_{22} \neq 0$. Let $B = \begin{bmatrix} \frac{h_{12}}{h_{11}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B = \begin{bmatrix} h_{22} & h_{22} & h'_{13} \\ h_{22} & h_{22} & h'_{23} \\ h'_{13} & h'_{23} & h'_{33} \end{bmatrix}$. Since $\det(B^t[h]B) = -h_{22}(h'_{13} - h'_{23})^2 < 0$, we have $h_{22} > 0$ and $h'_{13} \neq h'_{23}$. Let $C = \text{diag}\{\frac{1}{\sqrt{h_{22}}}, \frac{1}{\sqrt{h_{22}}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $C^t B^t [h] B C = \begin{bmatrix} 1 & 1 & \nu \\ 1 & 1 & \lambda \\ \nu & \lambda & \mu \end{bmatrix}$, where $-(\nu-\lambda)^2 <$

0 . Let $D = \begin{bmatrix} 1 & 0 & \frac{\nu^2 - 2\nu\lambda + \mu}{2(\lambda - \nu)} \\ 0 & 1 & \frac{\nu^2 - \mu}{2(\lambda - \nu)} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$. Then $D^t C^t B^t [h] B C D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \mu \\ 0 & \mu & 0 \end{bmatrix}$, where $\mu \neq 0$. Let $E = \text{diag}\{-1, -1, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $E^t D^t C^t B^t [h] B C D E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -\mu \\ 0 & -\mu & 0 \end{bmatrix}$.

Next suppose that $h_{11}h_{22} - h_{12}^2 \neq 0$. Then we may assume that $h_{13} = h_{23} = 0$ because $B = \begin{bmatrix} 1 & 0 & \frac{h_{13}h_{22} - h_{12}h_{23}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 1 & \frac{h_{11}h_{23} - h_{12}h_{13}}{h_{12}^2 - h_{11}h_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = [h']$, where $h'_{13} = h'_{23} = 0$.

If $h_{12} = 0$, then $h_{11} \neq 0$ and $h_{22} \neq 0$. Let $B = \text{diag}\{\frac{1}{\sqrt{|h_{11}|}}, \frac{1}{\sqrt{|h_{22}|}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$. Then $B^t[h]B$ is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where $\mu > 0$.

Now assume $h_{12} \neq 0$. When $h_{11} = h_{22} = 0$, $B = \text{diag}\{\frac{1}{h_{12}}, 1, 1\} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}$, where $\mu > 0$. When $h_{11} = 0$ and $h_{22} \neq 0$, $B = \text{diag}\{\frac{\sqrt{|h_{22}|}}{h_{12}}, \frac{1}{\sqrt{|h_{22}|}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \mu \\ 0 & 0 & \mu \end{bmatrix}$, where $\mu > 0$. When $h_{11} \neq 0$, $B = \text{diag}\{\frac{1}{\sqrt{|h_{11}|}}, \frac{\sqrt{|h_{11}|}}{h_{12}}, 1\} \in \text{Aut}(\mathfrak{g}_c)$ and $B^t[h]B$ is of the form

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \nu \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -\tau & 0 \\ 0 & 0 & \nu \end{bmatrix},$$

where $\nu(\tau - 1) < 0$.

Suppose $[h] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -\tau & 0 \\ 0 & 0 & \nu \end{bmatrix}$ with $1 < \tau$ and $\nu < 0$. Let

$$y_1 = x_1, \quad y_2 = \frac{1}{\sqrt{\tau - 1}}(x_1 + x_2), \quad y_3 = \frac{1}{\sqrt{-\nu}}x_3.$$

Then $\mathcal{B} = \{y_1, y_2, y_3\}$ is a basis of the Lie algebra of G_c such that $[h]_{\mathcal{B}} = \text{diag}\{-1, -1, -1\}$, which is a contradiction.

Finally it is easy to see that any eleven such distinct matrices are not equivalent. \square

4. Curvatures of left invariant Lorentzian metrics

In this section, we study the extent to which curvature can be altered by a change of left invariant metric. Given a left invariant Lorentzian metric on $G = G_I$ or G_c , we shall compute the following associated curvatures

- (1) Ricci operator and curvature,
- (2) sectional curvature,
- (3) scalar curvature.

Let $\varphi \in \text{Aut}(\mathfrak{g})$. Let $h' = h_\varphi$ and let ∇' be the Levi-Civita connection associated to (G, h') . Then for all $u, v, w \in \mathfrak{g}$, we have

$$2h'(\nabla'_{\varphi(u)}\varphi(v), \varphi(w)) = h'([\varphi(u), \varphi(v)], \varphi(w)) + h'([\varphi(w), \varphi(u)], \varphi(v))$$

$$\begin{aligned}
 & + h'([\varphi(w), \varphi(v)], \varphi(u)) \\
 = & h'(\varphi([u, v]), \varphi(w)) + h'(\varphi([w, u]), \varphi(v)) \\
 & + h'(\varphi([w, v]), \varphi(u)) \\
 = & h([u, v], w) + h([w, u], v) + h([w, v], u) \\
 = & 2h(\nabla_u v, w) \\
 = & 2h'(\varphi(\nabla_u v), \varphi(w)).
 \end{aligned}$$

This reduces to $\nabla'_{\varphi(u)}\varphi(v) = \varphi(\nabla_u v)$, or the following diagram is commutative:

$$\begin{array}{ccc}
 \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\nabla} & \mathfrak{g} \\
 \varphi \times \varphi \downarrow & & \downarrow \varphi \\
 \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\nabla'} & \mathfrak{g}
 \end{array}$$

Therefore, the classification of the left invariant Lorentzian metrics up to automorphism leads to the study of the left invariant Lorentzian metrics which leave all the curvature properties invariant.

We will begin with some necessary definitions. Let $\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ be the Levi-Civita connection associated to (G, h) . This is characterized by Koszul formula

$$2h(\nabla_u v, w) = h([u, v], w) + h([w, u], v) + h([w, v], u).$$

The *Riemann curvature tensor* of (G, h) is defined associated to each $u, v \in \mathfrak{g}$ to be the linear transformation

$$R_{uv} = \nabla_{[u, v]} - [\nabla_u, \nabla_v].$$

The *Ricci curvature* ric is the symmetric tensor defined, for any $u, v \in \mathfrak{g}$, as the trace of the linear transformation $w \mapsto R_{uv}v$. Notice that The Riemann curvature tensor is completely determined by the Ricci curvature ric . The *Ricci operator* $\text{Ric} : \mathfrak{g} \rightarrow \mathfrak{g}$ is given the relation $h(\text{Ric}(u), v) = \text{ric}(u, v)$. Because of the symmetries of the Ricci curvature, the Ricci operator Ric is h -self adjoint. The *scalar curvature* ρ is the trace of the Ricci operator. If $u, v \in \mathfrak{g}$ are linearly independent, the number

$$\kappa(u, v) = \frac{h(R_{uv}u, v)}{h(u, u)h(v, v) - h(u, v)^2}$$

is called the *sectional curvature* associated with u, v .

For an explicit computation, we fix an orthonormal basis $\mathfrak{B} = \{y_i\}$ for \mathfrak{g} with respect to h . So, $h(y_1, y_1) = h(y_2, y_2) = 1, h(y_3, y_3) = -1$ and $h(y_i, y_j) = 0$ for $i \neq j$. From definitions, we immediately obtain

$$\begin{aligned}
 \kappa(y_1, y_2) &= h(R_{y_1 y_2} y_1, y_2), \\
 \kappa(y_1, y_3) &= -h(R_{y_1 y_3} y_1, y_3), \\
 \kappa(y_2, y_3) &= -h(R_{y_2 y_3} y_2, y_3),
 \end{aligned}$$

and

$$\text{ric}(u, v) = h(R_{uy_1}v, y_1) + h(R_{uy_2}v, y_2) - h(R_{uy_3}v, y_3).$$

Remark 4.1. We recall that the Ricci curvature $r(u)$ and the Ricci transformation $\hat{r} : \mathfrak{g} \rightarrow \mathfrak{g}$ are given in the Riemannian case as follows

$$\begin{aligned} r(u) &= g(R_{uy_1}u, y_1) + g(R_{uy_2}u, y_2) + g(R_{uy_3}u, y_3), \\ &= g\left(\sum_i R_{y_i u} y_i, u\right), \\ \hat{r}(u) &= \sum_i R_{y_i u} y_i. \end{aligned}$$

From the definition $h(\text{Ric}(u), v) = \text{ric}(u, v)$, we remark that $\text{ric}(u, u)$ and Ric are the Lorentzian versions of $r(u)$ and \hat{r} , respectively.

Let

$$[\text{Ric}]_{\{y_i\}} = [R_{ij}].$$

Then

$$\begin{aligned} R_{ij} &= h(\text{Ric}(y_j), y_i) = \text{ric}(y_j, y_i) = \text{ric}(y_i, y_j) = R_{ji} \text{ for } i, j = 1, 2, \\ R_{3j} &= -h(\text{Ric}(y_j), y_3) = -\text{ric}(y_j, y_3) = -\text{ric}(y_3, y_j) = -R_{j3} \text{ for } j = 1, 2. \end{aligned}$$

This means that the matrix of the h -self adjoint operator Ric

$$[\text{Ric}] = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ -R_{13} & -R_{23} & R_{33} \end{bmatrix}$$

is Lorentzian symmetric.

Proposition 4.2 ([7, Ex. 19 (p. 261)]). *Let V be a real vector space with Lorentzian inner product h of signature $(2, 1)$. Every self adjoint operator T on V has a matrix of exactly one of the following four types:*

Relative to an h -orthonormal basis,

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix} \quad (\beta \neq 0);$$

Relative to a basis $\{e, u, v\}$ with non-trivial products $h(u, v) = 1 = h(e, e)$,

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & \epsilon \\ 0 & 0 & b \end{bmatrix} \quad (\epsilon = \pm 1), \quad \text{or} \quad \begin{bmatrix} a & 0 & 1 \\ 1 & a & 0 \\ 0 & 0 & a \end{bmatrix}.$$

The vectors

$$e_1 = e, \quad e_2 = \frac{1}{\sqrt{2}}(u + v), \quad e_3 = \frac{1}{\sqrt{2}}(u - v)$$

form an h -orthonormal basis with respect to which the last two matrices become respectively

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b + \frac{\epsilon}{2} & -\frac{\epsilon}{2} \\ 0 & \frac{\epsilon}{2} & b - \frac{\epsilon}{2} \end{bmatrix} (\epsilon = \pm 1), \quad \text{or} \quad \begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a & 0 \\ \frac{1}{\sqrt{2}} & 0 & a \end{bmatrix}.$$

These four types are called *O’Neill type* or *Segré type* $\{11, 1\}$, $\{1z\bar{z}\}$, $\{21\}$ and $\{3\}$, respectively. Note that the O’Neill type $\{21\}$ is of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b \pm 1 & \mp 1 \\ 0 & \pm 1 & b \mp 1 \end{bmatrix} \sim \begin{bmatrix} a & 0 & 0 \\ 0 & b \pm 1 & 1 \\ 0 & -1 & b \mp 1 \end{bmatrix},$$

because

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b \pm \frac{1}{2} & \mp \frac{1}{2} \\ 0 & \pm \frac{1}{2} & b \mp \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} a & 0 & 0 \\ 0 & b \pm 1 & \mp 1 \\ 0 & \pm 1 & b \mp 1 \end{bmatrix}$$

via $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(-\ln \sqrt{2}) & \sinh(-\ln \sqrt{2}) \\ 0 & \sinh(-\ln \sqrt{2}) & \cosh(-\ln \sqrt{2}) \end{bmatrix} \in O(2, 1)$. Notice further that $\begin{bmatrix} a & 0 & 0 \\ 0 & b+1 & 1 \\ 0 & -1 & b-1 \end{bmatrix} \sim \begin{bmatrix} a & 0 & 0 \\ 0 & b-1 & 1 \\ 0 & -1 & b+1 \end{bmatrix}$. The O’Neill type $\{3\}$ is of the form

$$\begin{bmatrix} a & 1 & -1 \\ 1 & a & 0 \\ 1 & 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & -\frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} a & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a & 0 \\ \frac{1}{\sqrt{2}} & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & -\frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \end{bmatrix}.$$

Thus we can always construct an orthonormal basis \mathfrak{B} for \mathfrak{g} such that $[\text{Ric}]_{\mathfrak{B}}$ takes exactly one of the four O’Neill types. The O’Neill type $\{11, 1\}$ (the comma is used to separate the spacelike and timelike eigenvectors) denotes a diagonalizable self adjoint operator. The O’Neill type $\{1z\bar{z}\}$ denotes a self adjoint operator with one real and two complex conjugate eigenvalues. A self adjoint operator of O’Neill type $\{21\}$ has two eigenvalues (one of which has multiplicity two), each associated to a one-dimensional eigenspace. A self adjoint operator of O’Neill type $\{3\}$ has three equal eigenvalues associated to a one-dimensional eigenspace. The eigenvalues of $[\text{Ric}]$ are called the *Ricci eigenvalues*. Hence, the O’Neill type of Ric is determined by its eigenvalues with associated eigenspaces.

Remark 4.3. In the Riemannian case, since $[\hat{r}]$ is symmetric, the eigenvalues of $[\hat{r}]$ are real, called the *principal Ricci curvatures*, and the signature of the symmetric matrix $[\hat{r}]$ is well-defined, called the *signature* of the Ricci curvature. However, in the Lorentzian case, the Lorentzian version $[\text{Ric}]$ of \hat{r} may have complex eigenvalues. In fact, complex eigenvalues occur only in the following cases:

- G_c with $c > 1$, $[h] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \tau & 0 \\ 0 & 0 & \mu \end{bmatrix}$ with $\mu > 0$, $\tau < 1$, $(c - \tau)^2 - 4c < 0$;

- G_1 , $[h] = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\mu > 0$, $\nu > \frac{1}{4}$;
- G_c with $c < 1$, $[h] = \begin{bmatrix} -\frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} -\frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\nu > 0$, $\tau < 1$, $\tau(\tau - c) < 0$ or $\begin{bmatrix} -\frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -\eta & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} -\frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\mu > 0$, $\eta < 1$, $\eta(\eta - c) < 0$, where $w = \sqrt{1 - c}$.

As a result, the principal Ricci curvatures are defined always and the signature of the Ricci curvature is defined whenever the eigenvalues are real numbers.

Note also that the matrix of ric with respect to the orthonormal basis $\{y_i\}$ is

$$(4.1) \quad [\text{ric}]_{\{y_i\}} = J_{2,1} [\text{Ric}]_{\{y_i\}} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & -R_{33} \end{bmatrix}$$

and the scalar curvature ρ is

$$(4.2) \quad \begin{aligned} \rho &= \text{trace}(\text{Ric}) = R_{11} + R_{22} + R_{33} \\ &= \text{ric}(y_1, y_1) + \text{ric}(y_2, y_2) - \text{ric}(y_3, y_3) \\ &= 2(\kappa(y_1, y_2) + \kappa(y_1, y_3) + \kappa(y_2, y_3)). \end{aligned}$$

Clearly, the Ricci eigenvalues of curvature operator Ric are independent of the choice of basis for \mathfrak{g} . Now show that the signature of the Ricci curvature tensor ric is independent of the choice of basis for \mathfrak{g} . Let $\{x_1, x_2, x_3\}$ be another, not necessarily orthonormal, basis for \mathfrak{g} . By writing $x_j = \sum_i a_{ij} y_i$ for some a_{ij} , we have

$$\begin{aligned} \text{ric}(x_i, x_j) &= \text{ric}\left(\sum_k a_{ki} y_k, \sum_\ell a_{\ell j} y_\ell\right) \\ &= \sum_{k, \ell} a_{ki} \cdot \text{ric}(y_k, y_\ell) \cdot a_{\ell j}. \end{aligned}$$

Hence

$$(4.3) \quad [\text{ric}]_{\{x_i\}} = A^t \cdot [\text{ric}]_{\{y_i\}} \cdot A,$$

where $A = (a_{ij})$ is the associated transition matrix $[\text{id}]_{\{x_i\}, \{y_i\}}$. This implies that $[\text{ric}]_{\{x_i\}}$ and $[\text{ric}]_{\{y_i\}}$ have the determinants of same sign, hence the products of their eigenvalues have the same sign. Moreover, we obtain

$$[\text{Ric}]_{\{x_i\}} = A^{-1} [\text{Ric}]_{\{y_i\}} A = A^{-1} J_{2,1} [\text{ric}]_{\{y_i\}} A = A^{-1} J_{2,1} (A^{-1})^t [\text{ric}]_{\{x_i\}}.$$

Lemma 4.4. $[\text{ric}]_{\{x_i\}}$ and $[\text{ric}]_{\{y_i\}}$ have the same signature. Consequently, the signature of $[\text{ric}]$ is defined.

Proof. Suppose $[\text{ric}]_{\{y_i\}}$ has the signature $(0, 0, 0)$. Since $[\text{ric}]_{\{y_i\}} = O$ is symmetric, $[\text{ric}]_{\{x_i\}} = A^t[\text{ric}]_{\{y_i\}}A = O$. Hence $[\text{ric}]_{\{x_i\}}$ and $[\text{ric}]_{\{y_i\}}$ have the same signature $(0, 0, 0)$. Now assume that the signature of $[\text{ric}]_{\{y_i\}}$ is not $(0, 0, 0)$. Since A is invertible and $[\text{ric}]_{\{x_i\}} = A^t[\text{ric}]_{\{y_i\}}A$, $[\text{ric}]_{\{x_i\}}$ is positive (negative) definite if and only if $[\text{ric}]_{\{y_i\}}$ is positive (negative) definite if and only if all the eigenvalues of $[\text{ric}]_{\{y_i\}}$ are positive (negative). It follows that the result holds when the signatures are $(+, +, +)$, $(-, -, -)$, $(+, +, -)$, $(+, 0, -)$ and $(+, -, -)$.

If $\lambda = 0$ is an eigenvalue of $[\text{ric}]_{\{y_i\}}$, then $[\text{ric}]_{\{y_i\}}\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. Thus $A^{-1}\mathbf{x} \neq \mathbf{0}$ and

$$[\text{ric}]_{\{x_i\}}(A^{-1}\mathbf{x}) = (A^t[\text{ric}]_{\{y_i\}}A)(A^{-1}\mathbf{x}) = A^t[\text{ric}]_{\{y_i\}}\mathbf{x} = \mathbf{0}.$$

It follows that 0 is an eigenvalue of $[\text{ric}]_{\{x_i\}}$. If \mathbf{x} and \mathbf{y} are linearly independent eigenvectors of $[\text{ric}]_{\{y_i\}}$ corresponding to the eigenvalue 0 , then $A^{-1}\mathbf{x}$ and $A^{-1}\mathbf{y}$ are linearly independent eigenvectors of $[\text{ric}]_{\{x_i\}}$ corresponding to the eigenvalue 0 . Based on this fact, the result holds when the signatures are $(+, +, 0)$, $(+, 0, 0)$, $(0, 0, 0)$, $(0, 0, -)$ and $(0, -, -)$. \square

Let g be a left invariant Riemannian metric on a connected Lie group G of dimension three. Let ρ denote the sectional curvature, and let r denote the Ricci curvature of (G, g) . For any orthogonal vectors u and v , the sectional curvature $\kappa(u, v)$ associated with u and v is given by the formula ([5, p. 306])

$$\kappa(u, v) = \|u \times v\|^2 \frac{\rho}{2} - r(u \times v).$$

Now, we give its Lorentzian version. Let h be a left-invariant Lorentzian inner product on a Lie algebra \mathfrak{g} of dimension three. We fix an orthonormal basis $\{y_1, y_2, y_3\}$ with respect to h . The Lorentzian cross product is defined as follows:

$$y_1 \times y_2 = -y_3, \quad y_2 \times y_3 = y_1, \quad y_3 \times y_1 = y_2.$$

Let u and v be orthogonal vectors, i.e., $h(u, v) = 0$. Write

$$u = \sum_i u_i y_i, \quad v = \sum_i v_i y_i.$$

Since the cross product is bilinear and skew-symmetric, we see that

$$u \times v = \sum_i w_i y_i, \\ \|u \times v\|^2 = w_1^2 + w_2^2 - w_3^2,$$

where $w_1 = u_2 v_3 - u_3 v_2$, $w_2 = u_3 v_1 - u_1 v_3$, $w_3 = -(u_1 v_2 - u_2 v_1)$.

Further, we have

$$\begin{aligned} \text{ric}(u \times v, u \times v) &= \text{ric} \left(\sum_i w_i y_i, \sum_i w_i y_i \right) \\ &= \sum_{i,j} w_i w_j (h(R_{y_i y_1} y_j, y_1) + h(R_{y_i y_2} y_j, y_2) - h(R_{y_i y_3} y_j, y_3)) \end{aligned}$$

$$\begin{aligned}
&= w_1^2 (h(R_{y_1 y_2} y_1, y_2) - h(R_{y_1 y_3} y_1, y_3)) \\
&\quad + w_2^2 (h(R_{y_1 y_2} y_1, y_2) - h(R_{y_2 y_3} y_2, y_3)) \\
&\quad + w_3^2 (h(R_{y_1 y_3} y_1, y_3) + h(R_{y_2 y_3} y_2, y_3)) \\
&\quad - 2w_1 w_2 h(R_{y_1 y_3} y_2, y_3) - 2w_1 w_3 h(R_{y_1 y_2} y_2, y_3) \\
&\quad + 2w_2 w_3 h(R_{y_1 y_2} y_1, y_3).
\end{aligned}$$

Hence, by substituting with (4.2), we get

$$\begin{aligned}
&\|u \times v\|^2 \frac{\rho}{2} - \text{ric}(u \times v, u \times v) \\
&= -w_3^2 h(R_{y_1 y_2} y_1, y_2) - w_2^2 h(R_{y_1 y_3} y_1, y_3) - w_1^2 h(R_{y_2 y_3} y_2, y_3) \\
&\quad + 2w_1 w_2 h(R_{y_1 y_3} y_2, y_3) + 2w_1 w_3 h(R_{y_1 y_2} y_2, y_3) - 2w_2 w_3 h(R_{y_1 y_2} y_1, y_3).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
h(R_{uv} u, v) &= \sum_{i,j,k,\ell} u_i v_j u_k v_\ell h(R_{y_i y_j} y_k, y_\ell) \\
&= \sum_{i < j, k < \ell} (u_i v_j - u_j v_i)(u_k v_\ell - u_\ell v_k) h(R_{y_i y_j} y_k, y_\ell) \\
&= -\left(\|u \times v\|^2 \frac{\rho}{2} - \text{ric}(u \times v, u \times v)\right).
\end{aligned}$$

In conclusion, we have the following result.

Lemma 4.5. *Let h be a left invariant Lorentzian metric on a connected Lie group G of dimension three. If u and v are orthogonal vectors, then*

$$(4.4) \quad h(u, u)h(v, v) \kappa(u, v) = -\left(\|u \times v\|^2 \frac{\rho}{2} - \text{ric}(u \times v, u \times v)\right).$$

In particular,

$$\begin{aligned}
\kappa(y_1, y_2) &= \frac{\rho}{2} + \text{ric}(y_3, y_3) = \frac{1}{2}(R_{11} + R_{22} - R_{33}), \\
\kappa(y_2, y_3) &= \frac{\rho}{2} - \text{ric}(y_1, y_1) = \frac{1}{2}(-R_{11} + R_{22} + R_{33}), \\
\kappa(y_3, y_1) &= \frac{\rho}{2} - \text{ric}(y_2, y_2) = \frac{1}{2}(R_{11} - R_{22} + R_{33}).
\end{aligned}$$

When the Ricci operator Ric is computed, (4.1) yields the Ricci curvature tensor ric, (4.2) yields the scalar curvature ρ , and Lemma 4.5 gives rise to the sectional curvatures $\kappa(y_i, y_j)$. For this reason, only the Ricci operator will be described in the following sections.

Procedure of calculating Ricci operators

The Ricci operator was calculated as follows: Let h be a left invariant Lorentzian metric on $G = G_I$ or $G = G_c$. Choose an orthonormal basis

$\mathfrak{B} = \{y_1, y_2, y_3\}$ for \mathfrak{g} such that $h(y_1, y_1) = h(y_2, y_2) = 1$ and $h(y_3, y_3) = -1$. Write

$$[y_i, y_j] = \sum_k \xi_{ijk} y_k, \quad \nabla_{y_i} y_j = \sum_k \Gamma_{ij}^k y_k.$$

Then the Christoffel symbols Γ_{ij}^k is obtained by the identity

$$\Gamma_{ij}^k = h(y_k, y_k) \cdot \frac{1}{2} \left(h([y_i, y_j], y_k) - h([y_j, y_k], y_i) + h([y_k, y_i], y_j) \right).$$

From the definition of the Riemann curvature tensor, we have

$$\begin{aligned} & h(R_{y_i y_k} y_j, y_k) \\ &= h \left(\nabla_{[y_i, y_k]} y_j - \nabla_{y_i} \nabla_{y_k} y_j + \nabla_{y_k} \nabla_{y_i} y_j, y_k \right) \\ &= h \left(\sum_{\ell} \xi_{ik\ell} \sum_n \Gamma_{\ell j}^n y_n - \sum_{\ell} \Gamma_{kj}^{\ell} \sum_n \Gamma_{i\ell}^n y_n + \sum_{\ell} \Gamma_{ij}^{\ell} \sum_n \Gamma_{k\ell}^n y_n, y_k \right) \\ &= h(y_k, y_k) \cdot \sum_{\ell} \left(\xi_{ik\ell} \Gamma_{\ell j}^k - \Gamma_{kj}^{\ell} \Gamma_{i\ell}^k + \Gamma_{ij}^{\ell} \Gamma_{k\ell}^k \right). \end{aligned}$$

The Ricci curvature $\text{ric}(y_i, y_j)$ and the Ricci operator Ric are given by

$$\begin{aligned} \text{ric}(y_i, y_j) &= \sum_k h(y_k, y_k) \cdot h(R_{y_i y_k} y_j, y_k), \\ [\text{Ric}]_{\{y_i\}} &= [R_{ij}], \quad \text{where } R_{ij} = h(y_j, y_j) \cdot \text{ric}(Y_i, Y_j). \end{aligned}$$

We state the curvature results in the **case** G_I . For each left invariant Lorentzian metric h on G_I whose associated matrix is given in Theorem 3.1, we obtain an orthonormal basis $\{y_i\}$ as follows:

- (1) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}$ with $\mu > 0$,

$$\begin{aligned} y_1 &= x, \quad y_2 = \frac{1}{\sqrt{\mu}} z, \quad y_3 = y; \\ [y_1, y_2] &= -\frac{1}{\sqrt{\mu}} y_1, \quad [y_2, y_3] = \frac{1}{\sqrt{\mu}} y_3, \quad [y_3, y_1] = 0. \end{aligned}$$

- (2) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix}$ with $\mu > 0$,

$$\begin{aligned} y_1 &= x, \quad y_2 = y, \quad y_3 = \frac{1}{\sqrt{\mu}} z; \\ [y_1, y_2] &= 0, \quad [y_2, y_3] = -\frac{1}{\sqrt{\mu}} y_2, \quad [y_3, y_1] = \frac{1}{\sqrt{\mu}} y_1. \end{aligned}$$

- (3) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$y_1 = x, \quad y_2 = \frac{1}{\sqrt{2}}(y + z), \quad y_3 = \frac{1}{\sqrt{2}}(y - z);$$

$$[y_1, y_2] = -\frac{1}{\sqrt{2}}y_1, [y_2, y_3] = \frac{1}{\sqrt{2}}(y_2 + y_3), [y_3, y_1] = -\frac{1}{\sqrt{2}}y_1.$$

With respect to the orthonormal basis $\{y_i\}$, we have the following:

Theorem 4.6. *The Ricci operator Ric of the metric h on G_I is expressed as follows:*

(1) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}$ with $\mu > 0$,

$$[\text{Ric}]_{\{y_i\}} = \text{diag}\left\{-\frac{2}{\mu}, -\frac{2}{\mu}, -\frac{2}{\mu}\right\} \quad (\text{O'Neill type } \{11, 1\}).$$

(2) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix}$ with $\mu > 0$,

$$[\text{Ric}]_{\{y_i\}} = \text{diag}\left\{\frac{2}{\mu}, \frac{2}{\mu}, \frac{2}{\mu}\right\} \quad (\text{O'Neill type } \{11, 1\}).$$

(3) If $[h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$[\text{Ric}]_{\{y_i\}} = \text{diag}\{0, 0, 0\} \quad (\text{flat}) \quad (\text{O'Neill type } \{11, 1\}).$$

All the other cases of G_c can be argued similarly. More details can be found in [4].

Remark 4.7. Three-dimensional Lie groups, having a flat Lorentzian metric, have been classified in [2] and [6]. We classify explicitly three-dimensional Lorentzian non-unimodular Lie groups whose metrics have constant sectional curvatures.

Let (G, h) be a connected, simply connected three-dimensional Lorentzian non-unimodular Lie group G . Then we have:

- (1) (G, h) is flat if and only if h is equivalent up to automorphism to a metric whose associated matrix is of the form

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ on G_I ;

(b) $\begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ with $\mu > 0$ on G_1 ;

(c) $\begin{bmatrix} \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on G_0 .

- (2) (G, h) has positive constant sectional curvature if and only if h is equivalent up to automorphism to a metric whose associated matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mu \end{bmatrix} \text{ with } \mu > 0 \text{ on } G_I.$$

- (3) (G, h) has negative constant sectional curvature if and only if h is equivalent up to automorphism to a metric whose associated matrix is of the form

$$(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix} \text{ with } \mu > 0 \text{ on } G_I;$$

$$(b) \begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \frac{1}{2w} & \frac{1+w}{2w} & 0 \\ -\frac{1}{2w} & \frac{w-1}{2w} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\mu > 0$ on G_c , $c < 1$, where $w = \sqrt{1-c}$.

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