

## BISECTORS IN THE HEISENBERG GROUP I

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ABSTRACT. We show that metric bisectors with respect to the Korányi metric in the Heisenberg group are spinal spheres and vice versa. We also calculate explicitly their horizontal mean curvature.

### 1. Introduction

A *metric bisector* in a metric space  $(X, d)$  is the subset  $B(x_1, x_2)$  of points  $x_1 \neq x_2$  of  $X$  that are equidistant from both  $x_1$  and  $x_2$ :

$$B(x_1, x_2) = \{x \in X \mid d(x_1, x) = d(x_2, x)\}.$$

Metric bisectors enjoy the following property: if  $f : X \rightarrow X$  is a similarity of  $X$ , that is, a mapping satisfying a relation of the form  $d(f(x), f(y)) = K_f d(x, y)$  for every  $x, y \in X$ , where  $K_f$  is a positive constant depending only on  $f$ , then the  $f$ -image of any bisector is again a bisector. In general, bisectors can be quite complicated objects in an arbitrary metric space  $X$ .

The same holds for Riemannian manifolds  $(M, g)$  with the metric  $d_g$  induced by the Riemannian tensor  $g$ . The most tractable example of a bisector in a Riemannian manifold is of course that of  $M = \mathbb{R}^n$ ,  $g = \sum_{i=1}^n dx_i^2$  and  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$  for each  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  in  $\mathbb{R}^n$ . Here,  $\|\cdot\|$  is the usual Euclidean norm. Then it follows at once that  $B(\mathbf{x}_1, \mathbf{x}_2)$  is the hyperplane comprising of  $\mathbf{x} \in \mathbb{R}^n$  such that  $2\mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2) = \|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2$ . We stress that due to invariance by similarities we would have chosen the points  $\mathbf{0} = (0, \dots, 0)$  and  $\mathbf{1} = (1, 0, \dots, 0)$ . The bisector of this points is the hyperplane  $x_1 = 1/2$  and then we would have concluded that all bisectors are hyperplanes since all images of  $x_1 = 1/2$  by Euclidean similarities are hyperplanes.

In this paper we study bisectors of the Heisenberg group  $\mathfrak{H}$  endowed with the Korányi metric  $d_K$ . The Heisenberg group  $\mathfrak{H}$  is the set  $\mathbb{C} \times \mathbb{R}$  with multiplication  $*$  given by

$$(z, t) * (w, s) = (z + w, t + s + 2\Im(z\bar{w}))$$

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for every  $(z, t)$  and  $(w, s)$  in  $\mathfrak{H}$ . The Korányi metric is then defined by  $d_K(p, q) = \|p * q^{-1}\|$  for each  $p, q \in \mathfrak{H}$ . Here  $\|p\|_K = (|z|^4 + t^2)^{1/4}$  for each  $p = (z, t) \in \mathfrak{H}$ , see Subsection 2.2 for details.

We note at this point that the Heisenberg group is a simple sub-Riemmanian manifold. Any such manifold is naturally equipped with the Carnot-Carathéodory metric  $d_{cc}$ , see Subsection 2.2 and the references within. The metric  $d_{cc}$  is related to the metric  $d_K$  in quite a few ways, for instance, among others we stress here that both metrics have the same isometry and similarity groups. However, bisectors with respect to  $d_{cc}$  are generally not the same objects as bisectors with respect to  $d_K$ ; details about the study of those objects will appear elsewhere.

In this paper, we are dealing with bisectors with respect to the Korányi metric. Let  $p_1, p_2 \in \mathfrak{H}$  be two distinct points and the Korányi bisector

$$\mathcal{B}(p_1, p_2) = \{p \in \mathfrak{H} \mid d_K(p_1, p) = d_K(p_2, p)\}.$$

We may normalize so that  $p_i, i = 1, 2$  lie either in the same finite or in the same infinite  $\mathbb{C}$ -circle, see Section 3 for details. Our main theorem is then the following.

**Theorem 1.1.** *A Korányi bisector is a spinal sphere. Conversely, every spinal sphere is a Korányi bisector.*

Spinal spheres are the boundaries of bisectors of complex hyperbolic plane with respect to the Bergman metric. Such bisectors have been used to construct Dirichlet fundamental domains or Ford fundamental domains of a discrete subgroup of  $\mathrm{PU}(n, 1)$ . In particular, Parker and Will used isometric spheres to construct Ford fundamental domains in [6].

Finally, as far as it concerns the horizontal geometry of Korányi bisectors/spinal spheres we prove that if  $p_1, p_2$  lie on an infinite  $\mathbb{C}$ -circle, then the Korányi bisector is a horizontal minimal surface (see Proposition 3.1), that is, its horizontal mean curvature vanishes everywhere. However, this is not the case if  $p_1, p_2$  lie on a finite  $\mathbb{C}$ -circle. The surface tends to be horizontally minimal away from its characteristic locus (see Proposition 3.2).

This paper is organized as follows. In Section 2, we shall review some preliminaries about the complex hyperbolic plane and the bisectors with respect to the Bergman metric, as well as some basic knowledge for the Heisenberg group and the Korányi metric. In Section 3, we prove Theorem 1.1 and in Subsection 3.2 we prove Propositions 3.1 and 3.2.

## 2. Preliminaries

### 2.1. Complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$ and its bisectors

Let  $\mathbb{C}^{2,1}$  be  $\mathbb{C}^3$  equipped with a non degenerate, Hermitian form  $\langle \cdot, \cdot \rangle$  of signature  $(2, 1)$ : If  $\mathbf{z} = (z_1, z_2, z_3)^T$  and  $\mathbf{w} = (w_1, w_2, w_3)^T$ , then

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1 = \mathbf{w}^* H \mathbf{z},$$

where

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We consider the subsets

$$\begin{aligned} V_- &= \{ \mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle < 0 \}, \\ V_0 &= \{ \mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle = 0 \}, \\ V_+ &= \{ \mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle > 0 \} \end{aligned}$$

and let also

$$\mathbb{P} : \mathbb{C}^{2,1} \ni \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_1/z_3 \\ z_2/z_3 \end{bmatrix} \in \mathbb{C}^2.$$

**Definition 2.1.** The complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$  is  $\mathbb{P}(V_-)$  and its boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$  is  $\mathbb{P}(V_0)$ .

The standard model for complex hyperbolic plane we use here is the *Siegel domain model*:

$$\mathbf{H}_{\mathbb{C}}^2 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid 2\Re(z_1) + |z_2|^2 < 0 \}.$$

Let  $(z_1, z_2) \in \mathbb{C}^2$ . The *standard lift* of  $z$  is  $\mathbf{z} = (z_1, z_2, 1)^T$ . In particular, the standard lift of  $\infty$  is  $(1, 0, 0)^T$ . The *Bergman metric* of  $\mathbf{H}_{\mathbb{C}}^2$  in terms of the distance function  $\rho(\cdot, \cdot)$  is given by

$$\cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle},$$

and this definition is independent of the choice of lifts. The followings hold:

- The holomorphic sectional curvature is  $-1$ .
- The sectional curvature is pinched between  $-1$  and  $-1/4$ .
- The group of holomorphic isometries is  $\text{PU}(2, 1)$ . The group  $\text{SU}(2, 1)$ , a three-fold covering of  $\text{PU}(2, 1)$ , is also used.

There are two types of geodesic submanifolds (of dimension 2): First, we have *complex lines* ( $\mathbb{C}$ -lines): Let  $z, w \in \overline{\mathbf{H}_{\mathbb{C}}^2}$  and let

$$\mathbf{C}(\mathbf{z}, \mathbf{w}) = \text{span}_{\mathbb{C}}(\mathbf{z}, \mathbf{w}),$$

with  $\mathbf{z}, \mathbf{w}$  being lifts of  $z, w$ , respectively. The  $\mathbb{C}$ -line  $\mathcal{C}(z, w)$  is the complex projection  $\mathbb{P}(\mathbf{C}(\mathbf{z}, \mathbf{w}))$  of  $\mathbf{C}(\mathbf{z}, \mathbf{w})$ .  $\mathbb{C}$ -lines are all isometric to

$$\mathbf{H}_{\mathbb{C}}^1 = \{ z \in \mathbb{C} \mid \Re(z) < 0 \}.$$

Second, we have *Lagrangian planes*  $\mathcal{R}$  (or  $\mathbb{R}$ -planes): Those are characterised by  $\langle \mathbf{z}, \mathbf{w} \rangle \in \mathbb{R}$  for all  $z, w \in \mathcal{R}$  and are all isometric to

$$\mathbf{H}_{\mathbb{R}}^2 = \{ (z_1, z_2) \in \mathbf{H}_{\mathbb{C}}^2 \mid \Im(z_1) = \Im(z_2) = 0 \}.$$

**2.1.1.  $\mathbf{H}_{\mathbb{C}}^2$ -bisectors.** In contrast to the real hyperbolic space case, there are no geodesic submanifolds of dimension 3. Bisectors are three dimensional submanifolds which are pretty close to be geodesic.

**Definition 2.2.** Let  $z, w \in \mathbf{H}_{\mathbb{C}}^2$  be two distinct points. The *bisector*  $\mathcal{B}_{\rho}(z, w)$  of  $z$  and  $w$  is

$$\mathcal{B}_{\rho}(z, w) = \{x \in \mathbf{H}_{\mathbb{C}}^2 \mid \rho(x, z) = \rho(x, w)\},$$

where  $\rho$  is the distance defined by the Bergman metric.

The followings are standard features of a bisector  $\mathcal{B}_{\rho}(z, w)$ :

- The *complex spine*  $\Sigma$  of  $\mathcal{B}_{\rho}(z, w)$  is the complex geodesic  $\mathcal{C}(z, w)$ .
- The *spine*  $\sigma$  of  $\mathcal{B}_{\rho}(z, w)$  is  $\mathcal{B}_{\rho}(z, w) \cap \Sigma$ , which is the geodesic corresponding to  $\Sigma$ .
- The endpoints of the spine  $\sigma$  are the *vertices* of the bisector and they determine it completely.

A bisector is foliated in two distinguished manners which are described in the following theorems, see [2].

**Theorem 2.3** (Slice decomposition). *Let  $\mathcal{B}$  be a bisector,  $\Sigma$  its complex spine and  $\sigma$  its spine. Then*

$$\mathcal{B} = \Pi_{\Sigma}^{-1}(\sigma) = \bigcup_{p \in \sigma} \Pi_{\Sigma}(p),$$

where  $\Pi_{\Sigma} : \mathbf{H}_{\mathbb{C}}^2 \rightarrow \Sigma$  is the orthogonal projection to  $\Sigma$ .

**Theorem 2.4** (Meridional decomposition). *Let  $\sigma$  be a geodesic in  $\mathbf{H}_{\mathbb{C}}^2$ . Then the bisector  $\mathcal{B}$ , which has  $\sigma$  as its spine, is the union of all Lagrangian planes containing  $\sigma$ .*

The group of holomorphic isometries  $\mathrm{PU}(2, 1)$  of  $\mathbf{H}_{\mathbb{C}}^2$  acts transitively on bisectors. Therefore we have:

**Corollary 2.5.** *All bisectors are isometric to the bisector  $\mathcal{B}_0$  whose spine is  $\sigma_0 = (0, \infty)$ .*

We will also consider for further use the bisector  $\mathcal{B}_1$  whose spine is  $\sigma_1 = ((-1, 0), (1, 0))$ .

### 2.1.2. Spinal spheres.

**Definition 2.6.** A spinal sphere  $\mathfrak{S}$  is the boundary of a bisector  $\mathcal{B}$  in  $\partial\mathbf{H}_{\mathbb{C}}^2$ .

The followings hold:

- A spinal sphere is fully determined by its vertices.
- $\mathrm{PU}(2, 1)$  acts transitively on spinal spheres.
- Each spinal sphere is the image of  $\mathfrak{S}_0 = \partial\mathcal{B}_0 = \mathbb{C}$  via an element of  $\mathrm{PU}(2, 1)$ .

We shall also denote by  $\mathfrak{S}_1$  the spinal sphere  $\partial\mathcal{B}_1$ . This is the hypersurface with equation

$$f(x, y, t) = x(x^2 + y^2 + 1) - yt = 0.$$

**2.2. Heisenberg group**

The boundary  $\partial\mathbf{H}_{\mathbb{C}}^2 \setminus \{\infty\}$  is in a bijection with the Heisenberg group  $\mathfrak{H}$ ; this is the set  $\mathbb{C} \times \mathbb{R}$  with multiplication  $*$  given by

$$(z, t) * (w, s) = (z + w, t + s + 2\Im(z\bar{w}))$$

for every  $(z, t)$  and  $(w, s)$  in  $\mathfrak{H}$ . There are two natural (left invariant) metrics defined in  $\mathfrak{H}$ . First, we have the *Korányi-Cygan metric* given by

$$d_K((z, t), (w, s)) = |(z, t)^{-1} * (w, s)|_K,$$

where  $|\cdot|_K$  is the Korányi gauge given by

$$|(z, t)|_K = ||z|^2 - it|^{1/2}$$

for each  $(z, t) \in \mathfrak{H}$ .

The similarity group  $\text{Sim}(\mathfrak{H})$  of  $\mathfrak{H}$  with respect to the Korányi-Cygan metric comprises the following transformations:

- (1) Left translations  $L_p, p \in \mathfrak{H}$ , defined by

$$L_p(q) = p * q$$

for each  $q \in \mathfrak{H}$ .

- (2) Rotations around the vertical axis  $R_\theta, \theta \in \mathbb{R}$ , defined by  $R_\theta(z, t) = (e^{i\theta}z, t)$ , for each  $(z, t) \in \mathfrak{H}$ .
- (3) Dilations  $D_\delta, \delta > 0$ , defined by  $D_\delta(z, t) = (\delta z, \delta^2 t)$ , for each  $q \in \mathfrak{H}$ .
- (4) Conjugation  $j$ , defined by  $j(z, t) = (\bar{z}, -t)$ , for each  $q \in \mathfrak{H}$ .

Left translations, rotations and conjugation are the isometry group of  $\mathfrak{H}$  for  $d_K$ . The similarity group  $\text{Sim}(\mathfrak{H})$  may be viewed as the isotropy subgroup of  $\infty$  in  $\text{SU}(2, 1)$ , see [4].

The following holds, see [6, Proposition 2.6] or [5, Proposition 3.1]:

**Proposition 2.7.** *The similarity group  $\text{Sim}(\mathfrak{H})$  acts doubly transitively on the Heisenberg group.*

For clarification purposes, we describe in brief the second metric although it is not of our interest in the present paper. The Heisenberg group  $\mathfrak{H}$  is a 2-step nilpotent Lie group; we consider the basis for the left invariant vector fields of  $\mathfrak{H}$  comprising

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

Denote by  $\mathfrak{h}$  the Lie algebra of  $\mathfrak{H}$ . There exists a decomposition:  $\mathfrak{h} = V^1 \oplus V^2$ , where

$$V^1 = \text{span}_{\mathbb{R}} \{X, Y\}, \quad V^2 = \text{span}_{\mathbb{R}} \{T\}.$$

The contact structure of  $\mathfrak{H}$  is induced by the 1-form

$$\omega = dt + 2(xdy - ydx) = dt + 2\Im(\bar{z}dz),$$

where  $z = x + iy$ . By the contact version of Darboux’s Theorem,  $\omega$  is the unique 1-form such that  $X, Y \in \ker \omega, \omega(T) = 1$ . For each point  $p \in \mathfrak{H}, V_p^1 = \mathcal{H}_p(\mathfrak{H})$

is the *horizontal tangent space of  $\mathfrak{H}$  at  $p$* . On the other hand, consider the relations

$$(X, X)_h = (Y, Y)_h = 1, (X, Y)_h = (Y, X)_h = 0.$$

From these relations we obtain the sub-Riemannian metric  $(\cdot, \cdot)_h$  in  $\mathfrak{H}$ ; its induced norm shall be denoted by  $|\cdot|_h$ . A smooth curve  $\gamma : [a, b] \rightarrow \mathfrak{H}$  with

$$\gamma(s) = (z(s), t(s)) \in \mathbb{C} \times \mathbb{R},$$

is called a *horizontal curve* if  $\dot{\gamma} \in \mathcal{H}_{\gamma(s)}(\mathfrak{H})$  for all  $s \in [a, b]$ . Equivalently,

$$\dot{t}(s) = -2\Im(\overline{z(s)}\dot{z}(s))$$

for  $s \in [a, b]$ . The *horizontal length* of a smooth rectifiable curve  $\gamma$  with respect to  $|\cdot|_h$  is given by

$$\begin{aligned} \ell_h(\gamma) &= \int_a^b |\dot{\gamma}(s)|_h ds = \int_a^b \left[ (\dot{\gamma}(s), X_{\gamma(s)})_h^2 + (\dot{\gamma}(s), Y_{\gamma(s)})_h^2 \right]^{1/2} ds \\ &= \int_a^b |\dot{z}(s)| ds. \end{aligned}$$

The *Carnot-Carathéodory distance*  $d_{cc}(p, q)$  between any two points  $p, q \in \mathfrak{H}$  is then the infimum of horizontal lengths of all horizontal curves joining  $p, q$ . We note the following:

- A neat way to write down explicitly the distance formula may be found in [1].
- There is a relation between the Korányi gauge and the Carnot-Carathéodory norm, see Proposition 2.1 in [3].
- The metrics  $d_K$  and  $d_{cc}$  are bi-Lipschitz equivalent; however, note that  $d_{cc}$  is a path metric whereas  $d_K$  is not.
- Both  $d_K$  and  $d_{cc}$  have the same isometry and similarity groups.

### 3. Bisectors in the Heisenberg group

Suppose that  $d_K$  is the Korányi-Cygan metric in the Heisenberg group  $\mathfrak{H}$ . Let  $p_1, p_2 \in \mathfrak{H}$  be two distinct points and the Korányi bisector be

$$\mathcal{B}(p_1, p_2) = \{p \in \mathfrak{H} \mid d_K(p_1, p) = d_K(p_2, p)\}.$$

From the properties of  $d_K$  it follows immediately that the image of a Korányi bisector under a Heisenberg similarity is a Korányi bisector. Now for any two points  $p_1$  and  $p_2$  in  $\mathfrak{H}$  there is always a unique  $\mathbb{C}$ -circle passing through  $p_1$  and  $p_2$  (see Theorem 4.3.5 in [2]). Heisenberg similarities map finite  $\mathbb{C}$ -circles to finite  $\mathbb{C}$ -circles and infinite  $\mathbb{C}$ -circles to infinite  $\mathbb{C}$ -circles. If  $\mathcal{B}(p_1, p_2)$  is a Korányi bisector, by Proposition 2.7, we may always normalize so that

- (1)  $p_1 = (0, -1)$  and  $p_2 = (0, 1)$  in the case where  $p_1, p_2$  lie in the same infinite  $\mathbb{C}$ -circle. We denote by  $\mathcal{B}^0$  the bisector  $\mathcal{B}((0, -1), (0, 1))$ .
- (2)  $p_1 = (-1, 0)$  and  $p_2 = (1, 0)$  in the case where  $p_1, p_2$  lie in the same finite  $\mathbb{C}$ -circle. We denote by  $\mathcal{B}^1$  the bisector  $\mathcal{B}((-1, 0), (1, 0))$ .

### 3.1. Korányi bisectors

We are now set to prove Theorem 1.1:

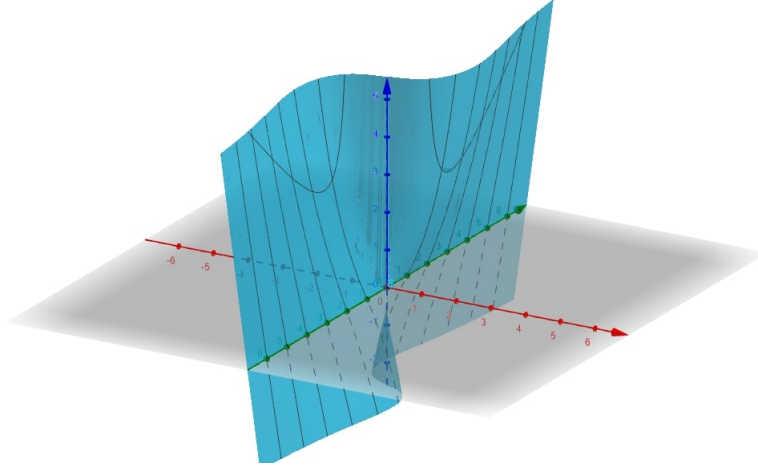


FIGURE 3.1. Korányi bisector, finite  $\mathbb{C}$ -circle case: The surface  $f(x, y, t) = x(x^2 + y^2 + 1) - yt = 0$ .

*Proof.* Let  $\mathcal{B}_K(p_1, p_2)$  be a Korányi bisector. It suffices to consider the cases where this is  $\mathcal{B}_K^0$  and  $\mathcal{B}_K^1$ , as above.

In the case of  $\mathcal{B}_K^0$ , the equation

$$d_K((0, -1), p) = d_K((0, 1), p)$$

is just

$$|z|^4 + (t + 1)^2 = |z|^4 + (t - 1)^2$$

from where it follows that

$$\mathcal{B}_K^0 = \{(z, t) \in \mathfrak{H} \mid t = 0\} = \mathbb{C}.$$

The complex plane  $\mathbb{C}$  is the spinal sphere  $\mathfrak{S}_0$  of the complex hyperbolic bisector given by  $\mathfrak{S}(z_2) = 0$ , see also Example 5.1.7 in [2].

In the case of  $\mathcal{B}_K^1$ , the equation

$$d_K((-1, 0), p) = d_K((1, 0), p)$$

is

$$|z + 1|^4 + (t - 2y)^2 = |z - 1|^4 + (t + 2y)^2.$$

After short calculations we obtain the hypersurface

$$(3.1) \quad f(x, y, t) = x(x^2 + y^2 + 1) - yt = 0,$$

which is also the spinal sphere,  $\mathfrak{S}_1$ .

Now conversely, any spinal sphere may be mapped to one of  $\mathfrak{S}_0$  or  $\mathfrak{S}_1$  under an element of  $\text{Sim}(\mathfrak{H})$ . The proof is thus complete.  $\square$

### 3.2. Horizontal geometry of Korányi bisectors

There is a major distinction in the horizontal geometry of Korányi bisectors. Before we state our result, we review some basic features of horizontal geometry of hypersurfaces in  $\mathfrak{H}$ . Let  $F : \mathfrak{H} \rightarrow \mathbb{R}$  be a  $C^2$  map and consider the hypersurface  $\mathcal{S}$  in  $\mathfrak{H}$  defined by the equation  $F(x, y, t) = 0$ . The *horizontal normal to  $\mathcal{S}$*  is the vector field

$$N_{\mathcal{S}}^h = XF \cdot X + YF \cdot Y.$$

The characteristic locus of  $\mathcal{S}$  is the set

$$C(\mathcal{S}) = \{p \in \mathcal{S} \mid X_p(F) = Y_p(F) = 0\}.$$

The *unit horizontal normal to  $\mathcal{S}$*  is

$$n_{\mathcal{S}}^h = \frac{N_{\mathcal{S}}^h}{|N_{\mathcal{S}}^h|_h}, \quad |N_{\mathcal{S}}^h|_h = [(XF)^2 + (YF)^2]^{1/2}.$$

Set  $n_{\mathcal{S}}^h = n_1 \cdot X + n_2 \cdot Y$ . The *horizontal mean curvature of  $\mathcal{S}$*  is then defined as

$$(3.2) \quad 2H^h = X(n_1) + Y(n_2).$$

Straightforward calculations deduce

$$\begin{aligned} X(n_1) &= \frac{XXF \cdot (YF)^2 - XF \cdot YF \cdot XYF}{[(XF)^2 + (YF)^2]^{3/2}}, \\ Y(n_1) &= \frac{YXF \cdot (YF)^2 - XF \cdot YF \cdot YYF}{[(XF)^2 + (YF)^2]^{3/2}}, \\ X(n_2) &= \frac{XYF \cdot (XF)^2 - XF \cdot YF \cdot XXF}{[(XF)^2 + (YF)^2]^{3/2}}, \\ Y(n_2) &= \frac{YYF \cdot (XF)^2 - XF \cdot YF \cdot YXF}{[(XF)^2 + (YF)^2]^{3/2}}. \end{aligned}$$

The above relations show that Eq. (3.2) also reads as

$$(3.3) \quad 2H^h = \frac{(YF)^2 \cdot XXF + (XF)^2 \cdot YYF - XF \cdot YF \cdot (XYF + YXF)}{[(XF)^2 + (YF)^2]^{3/2}}.$$

Horizontal mean curvature is invariant under Heisenberg similarities. The surface  $\mathcal{S}$  is horizontally minimal if  $H^h(\mathcal{S}) \equiv 0$ .

**Proposition 3.1.** *In the case where the points lie on an infinite  $\mathbb{C}$ -circle, a Korányi bisector is a horizontally minimal surface.*



*Proof.* The complex plane defined by  $F(x, y, t) = t = 0$  is well known to be horizontally minimal. For clarity, we carry out the details: We have

$$XF = 2y, \quad YF = -2x,$$

therefore the characteristic locus comprises the single point  $(0, 0, 0)$ . Away from this point,

$$n^h = \frac{y \cdot X - x \cdot Y}{\sqrt{x^2 + y^2}}.$$

Hence

$$\begin{aligned} 2H^h &= X(y/\sqrt{x^2 + y^2}) - Y(x/\sqrt{x^2 + y^2}) \\ &= \partial_x(y/\sqrt{x^2 + y^2}) - \partial_y(x/\sqrt{x^2 + y^2}) \\ &= 0. \end{aligned} \quad \square$$

This is not the case when the points defining the Korányi bisector lie in a finite  $\mathbb{C}$ -circle:

**Proposition 3.2.** *The horizontal mean curvature of the spinal sphere  $\mathfrak{S}_1$  diverge to infinity near the characteristic points  $(0, \pm 1, 0)$  and tends to 0 away from those points.*

*Proof.*  $\mathfrak{S}_1$  is the hypersurface given by

$$f(x, y, t) = x(x^2 + y^2 + 1) - yt = 0.$$

The partial derivatives of  $f$  are

$$\begin{aligned} f_x &= 3x^2 + y^2 + 1, \\ f_y &= 2xy - t, \\ f_t &= -y. \end{aligned}$$

There are no singular points here, thus  $\mathfrak{S}_1$  is a  $\mathcal{C}^2$  hypersurface. Now,

$$\begin{aligned} Xf &= 3x^2 - y^2 + 1, \\ Yf &= 4xy - t. \end{aligned}$$

Therefore the characteristic locus is the set of points belonging to both the curve defined by the equations

$$y^2 - 3x^2 = 1, \quad t = 4xy,$$

that is, the intersection of a hyperbolic cylinder and a saddle surface, and to the surface  $f(x, y, t) = 0$ . Plugging in the former two equations in the latter, we have

$$x(x^2 + 3x^2 + 2) - 4xy^2 = 0 \implies x = 0 \text{ or } 2x^2 - 2y^2 = 1.$$

If  $x = 0$ , then  $t = 0$  and  $y^2 = 1$  so we obtain the points  $(0, \pm 1, 0)$ . If  $2x^2 - 2y^2 = 1$ , then this together with  $y^2 - 3x^2 = 2$  gives  $-4x^2 = 3$  which is absurd. We conclude that the characteristic locus of the surface comprises the two points  $(0, \pm 1, 0)$ .

As now for the second derivatives, we have

$$\begin{aligned} XXf &= 6x, & YYf &= 6x, \\ XYf &= 2y, & YXf &= -2y. \end{aligned}$$

Therefore by formula (3.3) we immediately obtain

$$2H^h = \frac{XXf}{[(Xf)^2 + (Yf)^2]^{1/2}} = \frac{6x}{[(3x^2 - y^2 + 1)^2 + (4xy - t)^2]^{1/2}}.$$

The only points  $(x, y, t)$  on the surface with  $y = 0$  are points of the form  $(0, 0, t)$ . At those points  $H^h = 0$ . When  $y \neq 0$  we obtain from  $f(x, y, t) = 0$  that

$$t = \frac{x}{y}(x^2 + y^2 + 1).$$

In this manner  $H^h$  becomes a function  $H^h = H^h(x, y)$  with

$$H^h(x, y) = \frac{3xy}{[y^2(3x^2 - y^2 + 1)^2 + x^2(3y^2 - x^2 - 1)^2]^{1/2}}.$$

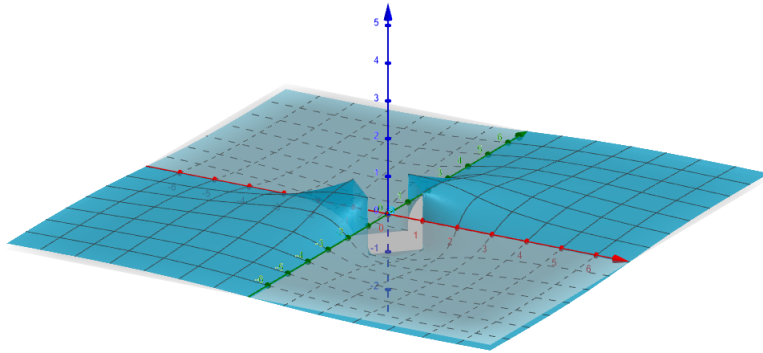


FIGURE 3.2. The horizontal mean curvature of the surface  $f = 0$ .

It is now straightforward to show (see also Figure 3.2) that the curvature tends to zero away from the critical points and it is bounded near them.  $\square$

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