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STABILITY AND TOPOLOGY OF TRANSLATING SOLITONS FOR THE MEAN CURVATURE FLOW WITH THE SMALL L^m NORM OF THE SECOND FUNDAMENTAL FORM

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Abstract. In this paper, we show that a complete translating soliton Σ^m in $\mathbb{R}^n$ for the mean curvature flow is stable with respect to weighted volume functional if Σ satisfies that the L^m norm of the second fundamental form is smaller than an explicit constant that depends only on the dimension of Σ and the Sobolev constant provided in Michael and Simon [12]. Under the same assumption, we also prove that under this upper bound, there is no non-trivial f-harmonic 1-form of L^2_f on Σ. With the additional assumption that Σ is contained in an upper half-space with respect to the translating direction then it has only one end.

1. Introduction

An orientable m-dimensional surface Σ^m in $\mathbb{R}^n$ is called a *translating soliton* (or *translator*) for the mean curvature flow (MCF) if it satisfies

$$H = V^{\perp}, \tag{1.1}$$

where H is the mean curvature vector of $\Sigma \subset \mathbb{R}^n$, V is a constant unit vector field in $\mathbb{R}^n$, and $(\cdot)^{\perp}$ denotes the projection onto the normal bundle of Σ. Translators arise as blow-up models at type II singularities of the MCF. A translator is a special solution of the MCF moving in the direction of V without deforming its shape under the flow. Moreover, it is a minimal submanifold in a conformally flat Riemannian manifold $(\mathbb{R}^n, e^{\frac{2}{m}\langle V,X\rangle}\langle,\rangle)$, where $\langle,\rangle$ is the standard Euclidean metric on $\mathbb{R}^n$ and X is the position vector. More precisely, a translator is a critical point of the following weighted volume functional:

$$\mathrm{Vol}_f(\Sigma) = \int_\Sigma e^{-f}\, d\mu, \tag{1.2}$$

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where $f = -\langle V, X\rangle$, and $d\mu$ is the induced volume form on $\Sigma \subset \mathbb{R}^n$. A translator is said to be *f-stable* if the second derivative of the weighted volume functional is always non-negative for any normal variation with compact support. Without weight, that is, when $f = 0$, a critical point of the usual volume functional is a minimal submanifold.

Let $\bar{\nabla}$ and ∇ be the standard connection on $\mathbb{R}^n$ and the induced Levi-Civita connection on Σ, respectively. The tangent and normal bundle of Σ are denoted by $T\Sigma$ and $N\Sigma$, respectively, and $(\cdot)^\top$ and $(\cdot)^\perp$ denote the projection of a vector field in $\mathbb{R}^n$ along the immersion onto $T\Sigma$ and $N\Sigma$, respectively. Then, the second fundamental form of an immersion $B : T\Sigma \times T\Sigma \to N\Sigma$ is defined by $B(Y, Z) = (\bar{\nabla}_Y Z)^\perp$, where Y and Z are tangent vector fields on Σ. Choose a local orthonormal frame field $\{e_i, e_\alpha\}$ of Σ, where $\{e_i : 1 \le i \le m\}$ is tangent to Σ and $\{e_\alpha : m + 1 \le \alpha \le n\}$ is normal to Σ. The mean curvature vector is given by the trace of the second fundamental form; $H = \mathrm{Trace}(B) = \sum_{i=1}^m B(e_i, e_i) \in \Gamma(N\Sigma)$. And the squared norm of the second fundamental form is defined by $|B|^2 = \sum_\alpha \sum_{i,j} \langle B(e_i, e_j), e_\alpha\rangle^2$.

For a submanifold $\Sigma^m \subset \mathbb{R}^n$, the *$L^m$ norm of the second fundamental form*, $\int_\Sigma |B|^m d\mu$ has been intensively studied. For a minimal submanifold Σ, it is equivalent to the total scalar curvature. Using an estimate on the L^m norm of the second fundamental form of Σ, it is possible to determine some properties of Σ, such as stability, topological properties, and shape. Among many significant results, Spruck [18] proved that $\Sigma^{m\ge3}$ is stable if the L^m norm of the second fundamental form of Σ is less than a constant that depends only on the dimension of Σ. Furthermore, Wang [19] proved that a stable minimal submanifold $\Sigma^{m\ge3}$ is an affine m-plane, if the second fundamental form satisfies $|B| \in L^m(\Sigma)$ (for the hypersurface case proved by Shen and Zhu [17]). With the similar assumption as [18], Ni [14] and Seo [16] deduced the topology of Σ (more precisely, the number of ends). In other directions, Palmer [15], Miyaoka [13] and Seo [16] studied the L^2 harmonic forms.

In this study, we further evaluate translators with the small L^m norm of the second fundamental form and determine three properties that hold even for translators of higher codimension. For the stability of translators, in Section 3, we first prove that:

Let $\Sigma^{m\ge3}$ be a complete translator immersed in $\mathbb{R}^n$. If Σ satisfies $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} \le C(m)$, then Σ is an f-stable translator. In fact, it is super f-stable.

In Section 4, based on the L^2 harmonic form theory developed by Palmer [15], Miyaoka [13] and Seo [16], we second prove that:

Let $\Sigma^{m\ge3}$ be a complete translator immersed in $\mathbb{R}^n$. If $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} < C(m)$, then Σ admits no non-trivial f-harmonic 1-form of L^2_f.

Since the height function in the given V direction has no local maximum, there is no compact translator. Thus, one significant topological property is the number of ends, i.e., the connected components outside of a compact geodesic ball, which is sufficiently large. For the topological ends of translators, in Section 5, we finally prove that:

Let $\Sigma^{m\geq 3}$ be a complete translator immersed in $\mathbb{R}^n$ with being contained in the half-space $\Pi_{V,a} = \{p \in \mathbb{R}^n : \langle p, V\rangle \geq a\}$ for some $a \in \mathbb{R}$. If $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} < C(m)$, then Σ has only one end.

There are many interesting results in translators analogous to minimal submanifolds. For the Bernstein-type theorem, Impera and Rimoldi [5] showed that if an f-stable translator Σ^m in $\mathbb{R}^{m+1}$ satisfies $|B| \in L^2_f(\Sigma)$, then Σ is a translator hyperplane parallel to the direction of translator, V. In previous works on higher codimensional translators, Xin [20] proved that an m-dimensional translator Σ^m in $\mathbb{R}^n$ satisfying both $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} \leq \tilde{C}(m)$ and $|B| \in L^m_f(\Sigma)$ is an affine m-plane parallel to V. Since the condition $|B| \in L^m_f(\Sigma)$ is too restrictive for the quantity $|B|$, the larger the height of Σ in the direction of V, it is important to note that, in the main theorems, we only assume the condition for the L^m norm of the second fundamental form of a given translator, which is smaller than an explicit constant. In other directions, Kunikawa [9, 10] showed rigidity results under natural geometric conditions, such as a flat normal bundle or parallel principal normal.

2. Preliminaries

From the first variation formula of the weighted volume functional (1.2), we obtain

$$\frac{d}{dt}\mathrm{Vol}_f(\Sigma)|_{t=0} = \int_\Sigma \langle V - H, E\rangle e^{-f} d\mu,$$

where $E = \varphi\nu$ is a normal variational vector field with compact support on Σ. More precisely, ν is a unit normal vector field of Σ in $\mathbb{R}^n$ and φ is any compactly supported smooth function on Σ.

From the second variation formula of the weighted volume functional, we obtain (see [20])

$$\frac{d^2}{dt^2}\mathrm{Vol}_f(\Sigma)|_{t=0} = \int_\Sigma \left(|\nabla^\perp E|^2 - \sum_{i,j} \langle B(e_i, e_j), E\rangle^2 \right) e^{-f} d\mu,$$

where $\nabla^\perp$ is the normal connection on Σ. If $\frac{d^2}{dt^2}\mathrm{Vol}_f(\Sigma)|_{t=0} \geq 0$ for any normal variation, then Σ is called *f-stable.*

A direct computation gives the following (see [18], [20] for more details)

$$\int_\Sigma \left(|\nabla^\perp E|^2 - \sum_{i,j} \langle B(e_i, e_j), E\rangle^2 \right) e^{-f} d\mu \geq \int_\Sigma \left(|\nabla\varphi|^2 - |B|^2\varphi^2 \right) e^{-f} d\mu.$$

Following Wang [19], we denote that if $\int_\Sigma \left(|\nabla\varphi|^2 - |B|^2\varphi^2 \right) e^{-f} d\mu \geq 0$, then Σ is called *super f-stable.* It is clear that if Σ is super f-stable, then it is

f-stable. The super f-stability coincides with the f-stability when Σ is a hypersurface.

Next, we recall the Sobolev inequality. In [12], Michael and Simon obtained the general Sobolev inequality for the C^2 submanifold Σ^m in $\mathbb{R}^n$:

$$\left(\int_\Sigma h^{\frac{m}{m-1}}d\mu\right)^{\frac{m-1}{m}} \leq S(m)\int_\Sigma (|\nabla h| + h|H|)d\mu,$$

where $0 \leq \forall h \in C_0^1(\Sigma)$, $S(m)$ is the Sobolev constant, and H is the mean curvature vector of Σ in $\mathbb{R}^n$. By substituting $h = u^{\frac{2(m-1)}{m-2}}$ and then using Hölder inequality and Young inequality, one can obtain the following L^2 Sobolev inequality (for example, see [20]):

$$S_0(m)\left(\int_\Sigma u^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} \leq \int_\Sigma |\nabla u|^2 d\mu + \frac{1}{2}\int_\Sigma |H|^2u^2d\mu, \tag{2.1}$$

where $0 \leq u \in C_0^1(\Sigma)$ and $S_0(m) = \frac{(m-2)^2}{(6m^2-14m+8)S(m)^2}$.

Given a complete translator $\Sigma \subset \mathbb{R}^n$, an end of Σ is a connected component of $\Sigma \backslash B_p(R)$, where $B_p(R) \subset \Sigma$ is the geodesic ball centered at $p \in \Sigma$ with a sufficiently large $R > 0$ as radius. Using the weighted L^1 Sobolev inequality on translators, we obtain:

Lemma 1. *Every end of a complete translator contained in the upper half-space* $\Pi_{V,a} = \{p \in \mathbb{R}^n : \langle p, V\rangle \geq a, a \in \mathbb{R}\}$ *is non-f-parabolic.*

Here, the condition of being in the upper half-space needs to apply the weighted L^1 Sobolev inequality on translators. See [5], [6] for more details.

The rotationally symmetric translators, *translating bowl*, and *winglike translators* [1,3,7], *grim-reaper cylinders*, and *Δ-wings* [4] are contained in the upper half-space $\Pi_{V,a}$. Kim and the second author [8] show that a half-space type theorem for translators.

Recall that the Bakry-Émery Ricci tensor of Σ is defined by

$$\mathrm{Ric}_f(Y,Y) = \mathrm{Ric}(Y,Y) + \mathrm{Hess}(f)(Y,Y),$$

where Y is a tangent vector field on Σ, Ric stands for the Ricci curvature of Σ and Hess(f) stands for the hessian of f on Σ. Using the Gauss equation, we obtain (see [5] for more details),

$$\mathrm{Ric}_f(Y,Y) \geq -|B|^2|Y|^2. \tag{2.2}$$

This gives a useful Bochner-type formula:

Lemma 2. *Let u be an f-harmonic function on Σ. Then*

$$\frac{1}{2}\Delta_f(|\nabla u|^2) \geq |\mathit{Hess}\ u|^2 - |B|^2|\nabla u|^2,$$

where $\Delta_f(\cdot) = \Delta(\cdot) - \langle\nabla f, \nabla(\cdot)\rangle$ is the weighted Laplacian on Σ.

This is derived from applying (2.2) to the weighted version of Bochner formula,

$$\frac{1}{2}\Delta_f(|\nabla u|^2) = |\text{Hess } u|^2 + \text{Ric}_f(\nabla u, \nabla u) + \langle \nabla \Delta_f u, \nabla u\rangle,$$

and using the fact that u is f-harmonic.

Finally, we study the f-harmonic 1-form of L^2_f. Let ω be a smooth 1-form on Σ. Recall that ω is called an f-harmonic 1-form of L^2_f on Σ if

$$\int_\Sigma |\xi|^2 e^{-f} d\mu < \infty \text{ and } \Delta_f \omega = 0,$$

where ξ is the dual vector field of ω on Σ, and $\Delta_f(\cdot)$ stands for the weighted Laplacian acting on the space of smooth 1-forms on Σ. In the particular case that Σ is a hypersurface contained in the upper half-space in the translating direction, if Σ has no non-trivial f-harmonic 1-form of L^2_f, then Σ admits no codimension one cycle which does not disconnect Σ. For more details about the f-harmonic form of L^2_f theory and codimension one cycle, see [11] and the references therein.

3. Stability of translators

Theorem 3. *Let $\Sigma^{m\geq 3}$ be a complete translator immersed in $\mathbb{R}^n$. If Σ satisfies $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} \leq C(m)$, then Σ is an f-stable translator. In fact, it is super f-stable. Here, $C(m) = \frac{\sqrt{2}(m-2)}{S(m)\sqrt{(6m^2-14m+8)(m+2)}}$.*

Proof. We prove by contradiction. If we suppose that Σ is not super f-stable, then for a suitable $\varphi \in C_0^\infty(\Sigma)$,

$$\int_\Sigma |\nabla\varphi|^2 e^{-f} d\mu < \int_\Sigma |B|^2\varphi^2 e^{-f} d\mu, \tag{3.1}$$

where $f = -\langle X, V\rangle$. By Hölder inequality, the RHS becomes

$$\int_\Sigma |B|^2\varphi^2 e^{-f} d\mu \leq \left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}} \left(\int_\Sigma (\varphi^2 e^{-f})^{\frac{m}{m-2}} d\mu\right)^{\frac{m-2}{m}}. \tag{3.2}$$

On the other hand, let $\psi = \varphi e^{-\frac{f}{2}}$, then

$$|\nabla\psi|^2 = |\nabla\varphi|^2 e^{-f} + \frac{1}{4}|\nabla f|^2\varphi^2 e^{-f} - \langle\nabla\varphi, \nabla f\rangle\varphi e^{-f}.$$

We claim that

$$\int_\Sigma \frac{1}{4}|\nabla f|^2\varphi^2 e^{-f} d\mu - \int_\Sigma \langle\nabla\varphi, \nabla f\rangle\varphi e^{-f} d\mu < 0, \tag{3.3}$$

that is,

$$\int_\Sigma |\nabla\psi|^2 d\mu < \int_\Sigma |\nabla\varphi|^2 e^{-f} d\mu. \tag{3.4}$$

Because φ is compactly supported in Σ, by applying the divergence theorem on $\int_\Sigma \operatorname{div}(\varphi^2 \nabla f e^{-f}) d\mu$, we obtain

$$\int_\Sigma (\langle \nabla(\varphi^2), \nabla f\rangle + \varphi^2 \Delta f - \varphi^2 |\nabla f|^2) e^{-f} d\mu = 0. \tag{3.5}$$

To analyze this equation, we consider the following identity from (1.1):

$$\Delta f = \operatorname{div}(-V^\top) = \operatorname{div}(V^\perp) = -\langle H, V^\perp\rangle = -|V^\perp|^2. \tag{3.6}$$

Applying (3.5) and (3.6) to the LHS of (3.3), we have

$$\begin{aligned} &\int_\Sigma \frac{1}{4}|\nabla f|^2 \varphi^2 e^{-f} d\mu - \int_\Sigma \langle \nabla\varphi, \nabla f\rangle \varphi e^{-f} d\mu \\ = & -\frac{1}{4}\int_\Sigma |\nabla f|^2 \varphi^2 e^{-f} d\mu - \frac{1}{2}\int_\Sigma \varphi^2 |V^\perp|^2 e^{-f} d\mu < 0. \end{aligned}$$

Thus, we obtain (3.4). Combining this result with (3.1) and (3.2), we obtain

$$\int_\Sigma |\nabla\psi|^2 d\mu < \left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}} \left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}.$$

Applying the previous Sobolev inequality (2.1) to ψ,

$$S_0(m)\left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \le \int_\Sigma |\nabla\psi|^2 d\mu + \frac{1}{2}\int_\Sigma |H|^2\psi^2 d\mu.$$

For the last term, by Hölder inequality,

$$\frac{1}{2}\int_\Sigma |H|^2\psi^2 d\mu \le \frac{1}{2}\left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}} \left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}.$$

Thus, we obtain

$$\begin{aligned} &S_0(m)\left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \\ < &\left(\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}} + \frac{1}{2}\left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}}\right) \left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}. \end{aligned}$$

Cauchy-Schwarz inequality gives

$$\left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}} \le m\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}.$$

Applying this to the preceding inequality and canceling $\left(\int_\Sigma \psi^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}$ on both sides, we obtain

$$S_0(m) < \left(1 + \frac{m}{2}\right)\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}.$$

Let $C(m) = \sqrt{\frac{2S_0(m)}{m+2}}$. Thus, $C(m) < \left(\int_\Sigma |B|^m d\mu\right)^{\frac{1}{m}}$. This contradicts to the prior assumption. Thus, the proof is complete. □

4. f-harmonic 1-forms of L_f^2 on translators

Theorem 4. *Let $\Sigma^{m\geq 3}$ be a complete translator immersed in $\mathbb{R}^n$. If*

$$\left(\int_\Sigma |B|^m d\mu\right)^{\frac{1}{m}} < C(m),$$

then Σ admits no non-trivial f-harmonic 1-form of L_f^2.

Proof. Let ω be an f-harmonic 1-form of L_f^2 on Σ, and ξ be the dual vector field of ω on Σ. From the weighted version of the Bochner formula and (2.2), we obtain

$$\frac{1}{2}\Delta_f(|\xi|^2) \geq |\nabla\xi|^2 - |B|^2|\xi|^2. \tag{4.1}$$

By a direct computation for the LHS,

$$\begin{aligned}\frac{1}{2}\Delta_f(|\xi|^2) &\geq \frac{1}{2}(\Delta(|\xi|^2) - \langle\nabla f, \nabla|\xi|^2\rangle)\\ &= |\nabla|\xi||^2 + |\xi|\Delta|\xi| - |\xi|\langle\nabla f, \nabla|\xi|\rangle.\end{aligned}$$

Based on (4.1),

$$|\xi|\Delta|\xi| + |B|^2|\xi|^2 = |\nabla\xi|^2 + |\xi|\langle\nabla f, \nabla|\xi|\rangle - |\nabla|\xi||^2 \geq |\xi|\langle\nabla f, \nabla|\xi|\rangle.$$

Here, we use the Kato inequality, that is,

$$|\nabla\xi|^2 - |\nabla|\xi||^2 \geq 0.$$

Let $\varphi = |\xi|$. Then, we can rewrite

$$\varphi\Delta\varphi + |B|^2\varphi^2 \geq \varphi\langle\nabla f, \nabla\varphi\rangle. \tag{4.2}$$

For a fixed point $p \in \Sigma$ and $R > 0$, we choose a suitable cut-off function η that satisfies

$$\eta = \begin{cases} 1 & \text{on } B_p(R) \\ 0 & \text{on } \Sigma\backslash B_p(2R) \end{cases} \quad \text{and} \quad |\nabla\eta| \leq \frac{1}{R} \text{ on } B_p(2R)\backslash B_p(R),$$

where $B_p(R) \subset \Sigma$ is the geodesic ball. Multiplying both sides by $\eta^2 e^{-f}$ on (4.2) and integrating over Σ,

$$\int_\Sigma \eta^2\varphi\Delta\varphi e^{-f}d\mu + \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq \int_\Sigma \eta^2\varphi\langle\nabla f, \nabla\varphi\rangle e^{-f}d\mu. \tag{4.3}$$

Because η is compactly supported on Σ, applying the divergence theorem on $\int_\Sigma \operatorname{div}(\eta^2\varphi\nabla\varphi e^{-f})d\mu$, we obtain

$$\begin{aligned}\int_\Sigma \eta^2\varphi\Delta\varphi e^{-f}d\mu = &\int_\Sigma \eta^2\varphi\langle\nabla\varphi, \nabla f\rangle e^{-f}d\mu - \int_\Sigma 2\eta\varphi\langle\nabla\eta, \nabla\varphi\rangle e^{-f}d\mu\\ &- \int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu.\end{aligned}$$

Using (4.3),

$$\int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq \int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu + \int_\Sigma 2\eta\varphi\langle\nabla\eta,\nabla\varphi\rangle e^{-f}d\mu.$$

By the Schwarz inequality, for any $a>0$, we obtain

$$\int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq (1-a)\int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu - \frac{1}{a}\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f}d\mu. \tag{4.4}$$

Because $\varphi\eta$ is compactly supported in Σ, we can apply (3.4) to $\varphi\eta$,

$$\int_\Sigma |\nabla(\varphi\eta)|^2 e^{-f}d\mu > \int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu. \tag{4.5}$$

Applying the previous Sobolev inequality (2.1) to $\varphi\eta e^{-\frac{f}{2}}$,

$$\begin{aligned}&\int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu\\ &\geq S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu.\end{aligned} \tag{4.6}$$

By a direct computation,

$$\int_\Sigma |\nabla(\varphi\eta)|^2 e^{-f}d\mu = \int_\Sigma (|\nabla\varphi|^2\eta^2 + 2\varphi\eta\langle\nabla\varphi,\nabla\eta\rangle + \varphi^2|\nabla\eta|^2)e^{-f}d\mu. \tag{4.7}$$

By the Schwarz inequality, for any $b>0$, we obtain

$$\begin{aligned}&\int_\Sigma (|\nabla\varphi|^2\eta^2 + 2\varphi\eta\langle\nabla\varphi,\nabla\eta\rangle + \varphi^2|\nabla\eta|^2)e^{-f}d\mu\\ &\leq (1+b)\int_\Sigma |\nabla\varphi|^2\eta^2 e^{-f}d\mu + (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu.\end{aligned} \tag{4.8}$$

Combining (4.5), (4.6), (4.7), and (4.8),

$$\begin{aligned}&(1+b)\int_\Sigma |\nabla\varphi|^2\eta^2 e^{-f}d\mu\\ &> \int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu - (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu\\ &\geq S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu\\ &\quad - (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu.\end{aligned} \tag{4.9}$$

For the LHS in (4.4), by Hölder inequality,

$$\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} \geq \int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu. \tag{4.10}$$

Combining (4.4) and (4.9), and (4.10), we obtain

$$\begin{aligned}
&\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}} \left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \\
> &-\frac{1}{a}\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
&+\frac{1-a}{1+b}\left(S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2\right. \\
&\qquad\left. -(1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}\right).
\end{aligned}$$

We can rewrite

$$\begin{aligned}
&\left(\frac{(1-a)(1+\frac{1}{b})}{1+b}+\frac{1}{a}\right)\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
> &\left(\frac{1-a}{1+b}S_0(m) - \left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \\
&-\frac{1-a}{2(1+b)}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu.
\end{aligned} \tag{4.11}$$

By Hölder inequality,

$$\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu \le \left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}}\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}. \tag{4.12}$$

Cauchy-Schwarz inequality gives

$$\left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}} \le m\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we obtain

$$\begin{aligned}
&\left(\frac{(1-a)(1+\frac{1}{b})}{1+b}+\frac{1}{a}\right)\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
> &\left(\frac{1-a}{1+b}S_0(m) - \left(1+\frac{m(1-a)}{2(1+b)}\right)\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}.
\end{aligned}$$

Next, a and b are chosen to be sufficiently small such that

$$\left(\frac{1-a}{1+b}S_0(m) - \left(1+\frac{m(1-a)}{2(1+b)}\right)\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right) \ge \epsilon > 0.$$

As $R \to \infty$, we obtain $\varphi \equiv 0$, that is, $\xi \equiv 0$. Since ξ is arbitrary, Σ has no non-trivial f-harmonic 1-form of L^2_f. □

5. Topology of translators

Theorem 5. *Let $\Sigma^{m\geq 3}$ be a complete translator immersed in $\mathbb{R}^n$ with being contained in the upper half-space $\Pi_{V,a} = \{p \in \mathbb{R}^n : \langle p, V\rangle \geq a\}$ for some a. If $(\int_\Sigma |B|^m d\mu)^{\frac{1}{m}} < C(m)$, then Σ has only one end.*

Proof. We reason by contradiction. Suppose that Σ has at least two ends. Because every end of Σ contained in $\Pi_{V,a}$ is non-f-parabolic, there exists a non-constant bounded f-harmonic function that has finite total weighted energy. See [2], [5] and [6] for details.

Let u be such an f-harmonic function. Then, we obtain

$$\frac{1}{2}\Delta_f(|\nabla u|^2) \geq |\text{Hess } u|^2 - |B|^2|\nabla u|^2. \tag{5.1}$$

By a direct computation for the LHS,

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla u|^2) &= \frac{1}{2}(\Delta(|\nabla u|^2) - \langle\nabla f, \nabla|\nabla u|^2\rangle)\\ &= |\nabla|\nabla u||^2 + |\nabla u|\Delta|\nabla u| - |\nabla u|\langle\nabla f, \nabla|\nabla u|\rangle.\end{aligned}$$

Based on (5.1),

$$\begin{aligned}|\nabla u|\Delta|\nabla u| + |B|^2|\nabla u|^2 &\geq |\text{Hess } u|^2 + |\nabla u|\langle\nabla f, \nabla|\nabla u|\rangle - |\nabla|\nabla u||^2\\ &\geq |\nabla u|\langle\nabla f, \nabla|\nabla u|\rangle.\end{aligned}$$

Here, we use the Kato inequality, that is,

$$|\text{Hess } u|^2 - |\nabla|\nabla u||^2 \geq 0.$$

Let $\varphi = |\nabla u|$. Then, we can rewrite

$$\varphi\Delta\varphi + |B|^2\varphi^2 \geq \varphi\langle\nabla f, \nabla\varphi\rangle. \tag{5.2}$$

For a fixed point $p \in \Sigma$ and $R > 0$, we choose a suitable cut-off function η that satisfies

$$\eta = \begin{cases} 1 & \text{on } B_p(R)\\ 0 & \text{on } \Sigma\backslash B_p(2R)\end{cases} \quad \text{and } |\nabla\eta| \leq \frac{1}{R} \text{ on } B_p(2R)\backslash B_p(R),$$

where $B_p(R) \subset \Sigma$ is the geodesic ball of centered at p with radius R. Multiplying both sides by $\eta^2 e^{-f}$ on (5.2) and integrating over Σ,

$$\int_\Sigma \eta^2\varphi\Delta\varphi e^{-f}d\mu + \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq \int_\Sigma \eta^2\varphi\langle\nabla f, \nabla\varphi\rangle e^{-f}d\mu. \tag{5.3}$$

Because η is compactly supported on Σ, applying the divergence theorem on $\int_\Sigma \text{div}(\eta^2\varphi\nabla\varphi e^{-f})d\mu$, we obtain

$$\begin{aligned}\int_\Sigma \eta^2\varphi\Delta\varphi e^{-f}d\mu = &\int_\Sigma \eta^2\varphi\langle\nabla\varphi, \nabla f\rangle e^{-f}d\mu - \int_\Sigma 2\eta\varphi\langle\nabla\eta, \nabla\varphi\rangle e^{-f}d\mu\\ &- \int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu.\end{aligned}$$

Using (5.3),

$$\int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq \int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu + \int_\Sigma 2\eta\varphi\langle\nabla\eta,\nabla\varphi\rangle e^{-f}d\mu.$$

By the Schwarz inequality, for any $a>0$, we obtain

$$\int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu \geq (1-a)\int_\Sigma \eta^2|\nabla\varphi|^2 e^{-f}d\mu - \frac{1}{a}\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f}d\mu. \tag{5.4}$$

Because $\varphi\eta$ is compactly supported in Σ, we can apply (3.4) to $\varphi\eta$,

$$\int_\Sigma |\nabla(\varphi\eta)|^2 e^{-f}d\mu > \int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu. \tag{5.5}$$

Applying the previous Sobolev inequality (2.1) to $\varphi\eta e^{-\frac{f}{2}}$,

$$\begin{aligned}&\int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu\\ &\geq S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu.\end{aligned} \tag{5.6}$$

By a direct computation,

$$\int_\Sigma |\nabla(\varphi\eta)|^2 e^{-f}d\mu = \int_\Sigma (|\nabla\varphi|^2\eta^2 + 2\varphi\eta\langle\nabla\varphi,\nabla\eta\rangle + \varphi^2|\nabla\eta|^2)e^{-f}d\mu. \tag{5.7}$$

By the Schwarz inequality, for any $b>0$, we obtain

$$\begin{aligned}&\int_\Sigma (|\nabla\varphi|^2\eta^2 + 2\varphi\eta\langle\nabla\varphi,\nabla\eta\rangle + \varphi^2|\nabla\eta|^2)e^{-f}d\mu\\ &\leq (1+b)\int_\Sigma |\nabla\varphi|^2\eta^2 e^{-f}d\mu + (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu.\end{aligned} \tag{5.8}$$

Combining (5.5), (5.6), (5.7), and (5.8),

$$\begin{aligned}&(1+b)\int_\Sigma |\nabla\varphi|^2\eta^2 e^{-f}d\mu\\ &> \int_\Sigma |\nabla(\varphi\eta e^{-\frac{f}{2}})|^2 d\mu - (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu\\ &\geq S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu\\ &\quad - (1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}d\mu.\end{aligned} \tag{5.9}$$

For the LHS in (5.4), by Hölder inequality,

$$\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}d\mu\right)^{\frac{m-2}{m}} \geq \int_\Sigma \eta^2|B|^2\varphi^2 e^{-f}d\mu. \tag{5.10}$$

Combining (5.4), (5.9), and (5.10), we obtain

$$\begin{aligned}
&\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}} \left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \\
> \; & -\frac{1}{a}\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
& + \frac{1-a}{1+b}\left(S_0(m)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} - \frac{1}{2}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 \right. \\
& \left. \qquad -(1+\frac{1}{b})\int_\Sigma |\nabla\eta|^2\varphi^2 e^{-f}\right).
\end{aligned}$$

We can rewrite

$$\begin{aligned}
&\left(\frac{(1-a)(1+\frac{1}{b})}{1+b} + \frac{1}{a}\right)\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
> & \left(\frac{1-a}{1+b}S_0(m) - \left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}} \\
& - \frac{1-a}{2(1+b)}\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu.
\end{aligned} \tag{5.11}$$

By Hölder inequality,

$$\int_\Sigma |H|^2(\varphi\eta e^{-\frac{f}{2}})^2 d\mu \le \left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}}\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}. \tag{5.12}$$

Cauchy-Schwarz inequality gives

$$\left(\int_\Sigma |H|^m d\mu\right)^{\frac{2}{m}} \le m\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}. \tag{5.13}$$

Combining (5.11), (5.12) and (5.13), we obtain

$$\begin{aligned}
&\left(\frac{(1-a)(1+\frac{1}{b})}{1+b} + \frac{1}{a}\right)\int_\Sigma \varphi^2|\nabla\eta|^2 e^{-f} d\mu \\
> & \left(\frac{1-a}{1+b}S_0(m) - \left(1+\frac{m(1-a)}{2(1+b)}\right)\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right)\left(\int_\Sigma (\varphi\eta e^{-\frac{f}{2}})^{\frac{2m}{m-2}} d\mu\right)^{\frac{m-2}{m}}.
\end{aligned}$$

Next, a and b are chosen to be sufficiently small such that

$$\left(\frac{1-a}{1+b}S_0(m) - \left(1+\frac{m(1-a)}{2(1+b)}\right)\left(\int_\Sigma |B|^m d\mu\right)^{\frac{2}{m}}\right) \ge \epsilon > 0.$$

As $R \to \infty$, we obtain $\varphi \equiv 0$, that is, $|\nabla u| \equiv 0$. This implies that u is constant, thereby contradicting the assumption of the existence of a non-trivial bounded f-harmonic function that has finite total weighted energy. Thus, Σ has only one end. □

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