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ON THE TOP LOCAL COHOMOLOGY AND FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a commutative Noetherian ring R and M a finitely generated R-module of finite dimension $d > 0$. In this paper, we obtain some results about the annihilators and attached primes of top local cohomology and top formal local cohomology modules. In particular, we determine $\operatorname{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, $\operatorname{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, $\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$ and $\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module of finite dimension $d > 0$. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$. For basic facts about local cohomology refer to [7]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$.

The basic properties of formal local cohomology modules are found in [1], [6], [10] and [14].

An important problem concerning local cohomology is determining the annihilators of the i-th local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$. This problem has been studied by several authors, see for example [2], [3], [4] and [5]. In [5], Bahmanpour et al. proved that if $(R, \mathfrak{m})$ is a complete local ring, then

$$\operatorname{Ann}(\mathrm{H}^{\dim(M)}_{\mathfrak{m}}(M)) = \operatorname{Ann}(M/T_R(M)),$$

where $T_R(M) = \cup\{N : N \leq M \text{ and} \dim N < \dim M\}$.

More recently, Atazadeh et al. in [2] generalized this main result by determining $\operatorname{Ann}(\mathrm{H}^{\dim(M)}_{\mathfrak{a}}(M))$ for an arbitrary Noetherian ring R. In [2, Theorem 2.3], by using the above main result, they proved the following main theorem.

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Theorem 1.1 ([2, Theorem 2.3])**.** *Let $\mathfrak{a}$ be an ideal of a Noetherianl ring R and M a non-zero finitely generated R-module such that $\mathrm{H}_{\mathfrak{a}}^{\dim(M)}(M) \neq 0$. Then*

$$\mathrm{Ann}(\mathrm{H}_{\mathfrak{a}}^{\dim(M)}(M)) = \mathrm{Ann}(M/T_R(\mathfrak{a}, M)),$$

where $T_R(\mathfrak{a}, M) = \cup\{N : N \leq M$ and $\mathrm{cd}(\mathfrak{a}, N) < \mathrm{cd}(\mathfrak{a}, M)\}$.

As a main result in the first section, we determine the annihilators and attached primes of top local cohomology module $\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{\dim(M)}(M)$ for two arbitrary ideals $\mathfrak{a}$ and $\mathfrak{b}$ of an arbitrary Noetherian ring R. In fact, we prove the following theorem, which is a generalization of [2, Theorem 2.3].

Theorem 1.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$ and $\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M) \neq 0$. Then*

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}(\mathfrak{b}M/(\mathfrak{b}M \cap T_R(\mathfrak{a}, M))),$$

where $T_R(\mathfrak{a}, M) = \cup\{N : N \leq M$ and $\mathrm{cd}(\mathfrak{a}, N) < \mathrm{cd}(\mathfrak{a}, M)\}$.

We obtain several corollaries of the above result. Among other things, in the following, we determine the set of attached primes of $\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{\dim M}(M)$.

Corollary 1.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$. If $\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M) \neq 0$, then*

$$\mathrm{Att}(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \{\mathfrak{p} \in \mathrm{Assh}(\mathfrak{b}M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

In Section 3, we obtain some results about the annihilators and attached primes of formal local cohomology modules. In the first main result, we will prove the following theorem which is an extension of [13, Theorem 1.2].

Theorem 1.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. of finite dimension d and $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) \neq 0$. Then*

$$\mathrm{Ann}(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}(\mathfrak{b}M/U_R(\mathfrak{a}, \mathfrak{b}M)),$$

where $U_R(\mathfrak{a}, \mathfrak{b}M)$ is the largest submodule of $\mathfrak{b}M$ such that

$$\dim(U_R(\mathfrak{a}, \mathfrak{b}M)/\mathfrak{a}U_R(\mathfrak{a}, \mathfrak{b}M)) < d.$$

In the another main result in Section 3, we determine the set of attached primes of $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{\dim M}(M)$. More precisely, we will prove the following result, which is an extension of [6, Theorem 3.1]:

Corollary 1.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. If $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) \neq 0$, then*

$$\mathrm{Att}(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \{\mathfrak{p} \in \mathrm{Ass}(\mathfrak{b}M) : \dim(R/\mathfrak{p}) = d, \mathfrak{p} \supseteq \mathfrak{a}\}.$$

2. Annihilators and attached primes of top local cohomology modules

A non-zero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary submodules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\operatorname{Att}(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \dots, n\}$ (see [12]). Yassemi [16] defined the cosupport of an R-module M, denoted by $\operatorname{Cosupp}(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}(L) \subseteq \mathfrak{p}$. It is well known that in case M is an Artinian R-module the equality $\operatorname{Cosupp}(M) = \operatorname{V}(\operatorname{Ann} M)$ is true.

A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann} T$ [16]. The set of coassociated primes of M is denoted by $\operatorname{Coass}(M)$. In [16] we can see that $\operatorname{Coass}(M) \subseteq \operatorname{Cosupp}(M)$ and every minimal element of the set $\operatorname{Cosupp}(M)$ belongs to $\operatorname{Coass}(M)$.

For the following proofs we need the following two next lemmas.

Lemma 2.1. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M and N be two finitely generated R-modules such that* $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. *Then* $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$.

Proof. See [9, Theorem 2.2]. □

Lemma 2.2. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M an R-module. Then* $\operatorname{Coass}(\mathfrak{a}M) \subseteq \operatorname{Coass} M$.

Proof. Since $\mathfrak{a}M$ is a homomorphic image of $\mathfrak{a} \otimes_R M$, we have $\operatorname{Coass}(\mathfrak{a}M) \subseteq \operatorname{Coass}(\mathfrak{a} \otimes_R M)$. On the other hand, by [16, Theorem 1.21] $\operatorname{Coass}(\mathfrak{a} \otimes_R M) \subseteq \operatorname{Coass} M$. Thus we conclude that $\operatorname{Coass}(\mathfrak{a}M) \subseteq \operatorname{Coass} M$. □

Theorem 2.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$. Then $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) = 0$ if and only if $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$.*

Proof. If $\operatorname{cd}(\mathfrak{a}, M) < d$, then $\operatorname{H}^d_{\mathfrak{a}}(M) = 0$ and so $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) = 0$. Also, by Lemma 2.1 $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) \leq \operatorname{cd}(\mathfrak{a}, M) < d$ and it follows that $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. Thus, we assume that $\operatorname{cd}(\mathfrak{a}, M) = d$. If $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) = 0$, then $\mathfrak{b} \subseteq \operatorname{Ann}(\operatorname{H}^d_{\mathfrak{a}}(M))$. By Theorem 1.1, $\operatorname{Ann}(\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(M/T_R(\mathfrak{a}, M))$ where $T_R(\mathfrak{a}, M) = \cup\{N : N \leq M$ and $\operatorname{cd}(\mathfrak{a}, N) < \operatorname{cd}(\mathfrak{a}, M)\}$. Thus $\mathfrak{b}M \subseteq T_R(\mathfrak{a}, M)$ and so $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) < \operatorname{cd}(\mathfrak{a}, M)$. Since $\operatorname{cd}(\mathfrak{a}, M) = d$ we conclude that $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) < d$ and so $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. Conversely, assume that $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. Thus $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) < d = \operatorname{cd}(\mathfrak{a}, M)$ and so

$\mathfrak{b}M \subseteq T_R(\mathfrak{a}, M)$. It follows that $\mathfrak{b} \subseteq \operatorname{Ann}(M/T_R(\mathfrak{a}, M)) = \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(M))$ and $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$. □

Corollary 2.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$. If $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$, then $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$ and $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = \dim(\mathfrak{b}M) = d$ and so $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is an Artinian R-module.*

Proof. Since $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$ we have $\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$ and so $\operatorname{cd}(\mathfrak{a}, M) = d$. Theorem 2.3 implies that $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$. Thus $d \leq \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M)$. But $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) \leq \dim(\mathfrak{b}M) \leq d$. Therefore

$$\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = \dim(\mathfrak{b}M) = d.$$

Since $\dim(\mathfrak{b}M) = d$ by using [7, 7.1.7] we conclude that $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is an Artinian R-module and the proof is complete. □

Theorem 2.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$. Then $\operatorname{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$.*

Proof. If $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$, then by Theorem 2.3, $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$ and the result follows in this case. Now, assume that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. By Corollary 2.4 we have

$$\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = \dim(\mathfrak{b}M) = d.$$

If $u \in \operatorname{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, then $u\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$ and so $u\mathfrak{b} \subseteq \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(M))$. But, by Theorem 1.1, $\operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann} M/T$ where $T = \cup\{N \leq M : \operatorname{cd}(\mathfrak{a}, N) < \operatorname{cd}(\mathfrak{a}, M) = d\}$. It follows that $u\mathfrak{b}M \subseteq T$ and so $\operatorname{cd}(\mathfrak{a}, u\mathfrak{b}M) < \operatorname{cd}(\mathfrak{a}, M)$. Hence $\operatorname{cd}(\mathfrak{a}, u\mathfrak{b}M) < d$. On the other hand, $\dim(\mathfrak{b}M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = d$ and so by Theorem 1.1 $\operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Ann}(\mathfrak{b}M/W)$ where

$$W = \cup\{N \leq \mathfrak{b}M : \operatorname{cd}(\mathfrak{a}, N) < \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = d\}.$$

Since $\operatorname{cd}(\mathfrak{a}, u\mathfrak{b}M) < d$ we get $u \in \operatorname{Ann}(\mathfrak{b}M/W) = \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Conversely, assume that $u \in \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Ann}(\mathfrak{b}M/W)$. Thus $u\mathfrak{b}M \leq W$ and so $\operatorname{cd}(\mathfrak{a}, u\mathfrak{b}M) < d$. We can see that $u\mathfrak{b}M \leq T$ and thus $u\mathfrak{b} \subseteq \operatorname{Ann}(M/T) = \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(M))$. Therefore $u\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$ and it follows that $u \in \operatorname{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, as required. □

Corollary 2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$ and $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. Then $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ is not finitely generated.*

Proof. Assume that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ is finitely generated. Since $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ is an Artinian R-module it follows that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ has finite length. Thus there exist $n \in \mathbb{N}$ and maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R such that $\mathfrak{m}_1 \cdots \mathfrak{m}_n \cdot \mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$. By Theorem 2.5 we conclude that $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \operatorname{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$. But by Corollary 2.4, $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is Artinian and so by [15, Theorem 7.30] $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is finitely generated which is a contradiction by [11, Remark 2.5]. □

Proposition 2.7. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$*. Then* $T_R(\mathfrak{a}, \mathfrak{b}M) = \mathfrak{b}M \cap T_R(\mathfrak{a}, M)$*.*

Proof. Since $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$ we have $\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$ and so $\mathrm{cd}(\mathfrak{a}, M) = d$. On the other hand, Theorem 2.3 implies that $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$ and thus $\mathrm{cd}(\mathfrak{a}, \mathfrak{b}M) = d$. By definition, $T_R(\mathfrak{a}, \mathfrak{b}M)$ is the largest submodule of $\mathfrak{b}M$ such that

$$\mathrm{cd}(\mathfrak{a}, T_R(\mathfrak{a}, \mathfrak{b}M)) < \mathrm{cd}(\mathfrak{a}, \mathfrak{b}M) = d.$$

But, $\mathfrak{b}M \cap T_R(\mathfrak{a}, M)$ is a submodule of $\mathfrak{b}M$ and by Lemma 2.1

$$\mathrm{cd}(\mathfrak{a}, \mathfrak{b}M \cap T_R(\mathfrak{a}, M)) \leq \mathrm{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < d.$$

It follows that $\mathfrak{b}M \cap T_R(\mathfrak{a}, M) \subseteq T_R(\mathfrak{a}, \mathfrak{b}M)$. Conversely, $T_R(\mathfrak{a}, M)$ is the largest submodule of M such that $\mathrm{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < \mathrm{cd}(\mathfrak{a}, M) = d$. Thus $T_R(\mathfrak{a}, \mathfrak{b}M) \subseteq T_R(\mathfrak{a}, M)$. But $T_R(\mathfrak{a}, \mathfrak{b}M) \subseteq \mathfrak{b}M$. Therefore $T_R(\mathfrak{a}, \mathfrak{b}M) \subseteq \mathfrak{b}M \cap T_R(\mathfrak{a}, M)$, as required. □

Theorem 2.8. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$*. Then*

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}(\mathfrak{b}M/(\mathfrak{b}M \cap T_R(\mathfrak{a}, M))).$$

Proof. By Theorem 2.5 and Theorem 1.1 we have

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \mathrm{Ann}(\mathfrak{b}M/T_R(\mathfrak{a}, \mathfrak{b}M)).$$

Now, the result follows by Proposition 2.7. □

Remark 2.9. Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R, and M a non-zero finitely generated R-module with finite cohomological dimension $c := \mathrm{cd}(\mathfrak{a}, M)$. By [3, Proposition 2.3] we have

$$T_R(\mathfrak{a}, M) = \cap_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j) = c} N_j,$$

where $0 = \cap_{j=1}^n N_j$ is a reduced primary decomposition of the zero submodule 0 in M and N_j is a $\mathfrak{p}_j$-primary submodule of M.

Corollary 2.10. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$*. Then*

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}(\mathfrak{b}M/(\mathfrak{b}M \cap (\cap_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j) = d} N_j))),$$

where $0 = \cap_{j=1}^n N_j$ *is a reduced primary decomposition of the zero submodule* 0 *in* M *and* N_j *is a* $\mathfrak{p}_j$*-primary submodule of* M*.*

Proof. The result follows by Theorem 2.8 and Remark 2.9. □

Corollary 2.11. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$*. Then* $\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}(\mathfrak{b}M)$ *whenever* $\mathrm{Ass}\, M \subseteq \{\mathfrak{p} \in \mathrm{Supp}\, M : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}$*.*

Proof. Assumption implies that $T_R(\mathfrak{a}, M) = 0$ and the result follows by Theorem 2.8. □

Corollary 2.12. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R*. Let* $d := \dim R$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(R) \neq 0$*. Then*

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(R)) = \mathrm{Ann}(\mathfrak{b}/(\mathfrak{b} \cap T_R(\mathfrak{a}, R))) = \mathrm{Ann}(\mathfrak{b}/(\mathfrak{b} \cap (\cap_{\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)=d}\mathfrak{q}_j))),$$

where $0 = \cap_{j=1}^{n}\mathfrak{q}_j$ *is a reduced primary decomposition of the zero ideal of* R*,* $\mathfrak{q}_j$ *is a* $\mathfrak{p}_j$*-primary ideal of* R *for all* $1 \leq j \leq n$*.*

Proof. The result follows by Theorem 2.8 and Corollary 2.10. □

Corollary 2.13. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian domain* R*. Let* $d := \dim R$ *and* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(R) \neq 0$*. Then* $\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(R)) = \mathrm{Ann}\,\mathfrak{b}$*.*

Proof. Since $\mathrm{Ass}\,R = 0$ we can see that $T_R(\mathfrak{a}, R) = 0$. Thus, by using Corollary 2.12 we have

$$\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(R)) = \mathrm{Ann}(\mathfrak{b}/(\mathfrak{b} \cap T_R(\mathfrak{a}, R))) = \mathrm{Ann}(\mathfrak{b}),$$

as required. □

Corollary 2.14. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$*. If* $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$*, then*

$$\mathrm{Cosupp}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Supp}(\mathfrak{b}M/(\mathfrak{b}M \cap T_R(\mathfrak{a}, M))).$$

Proof. By [7, Exercise 7.1.7] $\mathrm{H}^d_{\mathfrak{a}}(M)$ is Artinian. Thus $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ is Artinian and so by [16, Proposition 2.3] $\mathrm{Cosupp}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{V}(\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$ and by Theorem 2.8

$$\begin{aligned}\mathrm{V}(\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))) &= \mathrm{V}(\mathrm{Ann}(\mathfrak{b}M/(\mathfrak{b}M \cap T_R(\mathfrak{a}, M)))) \\ &= \mathrm{Supp}(\mathfrak{b}M/(\mathfrak{b}M \cap T_R(\mathfrak{a}, M)))\end{aligned}$$

and this completes the proof. □

Theorem 2.15. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian ring* R *and* M *a finitely generated* R*-module. Let* $d := \dim M$*. Then*

$$\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Att}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

Proof. If $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$, by Theorem 2.3 we conclude that $\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Att}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \phi$ and the result follows in this case. So, assume that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. By Lemma 2.2 and [8, Theorem 2.5]

$$\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) \subseteq \mathrm{Att}(\mathrm{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Assh}\,M : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

Thus, we conclude that

$$\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Min}\,\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)).$$

Since for an Artinian R-module A the set of all minimal prime ideals containing $\mathrm{Ann}\,A$ is exactly the set of all minimal elements of $\mathrm{Att}\,A$, by using Theorem 2.5 it follows that

$$\mathrm{Att}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M))).$$

But, by Corollary 2.4 $\dim(\mathfrak{b}M) = d$ and so by [8, Theorem 2.5] we have

$$\operatorname{Min} \operatorname{V}(\operatorname{Ann}(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M))) = \operatorname{Att}(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$$

and the proof is complete. □

Corollary 2.16. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring R and M a finitely generated R-module. Let $d := \dim M$. If $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) \neq 0$, then*

$$\operatorname{Att}(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Assh}(\mathfrak{b}M) : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

In particular, if $\mathfrak{b}\operatorname{H}^d_{\mathfrak{m}}(M) \neq 0$, then $\operatorname{Att}(\mathfrak{b}\operatorname{H}^d_{\mathfrak{m}}(M)) = \operatorname{Assh}(\mathfrak{b}M)$.

Proof. By Corollary 2.4 $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = \dim(\mathfrak{b}M) = d$ and so by [8, Theorem 2.5]

$$\operatorname{Att}(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \{\mathfrak{p} \in \operatorname{Assh}(\mathfrak{b}M) : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

Now the result follows by Theorem 2.15. □

3. Annihilators and attached primes of top formal local cohomology modules

In this section, we assume that $(R, \mathfrak{m})$ is a Noetherian local ring. The main results in this section are Theorems 3.7 and 3.13. We begin with:

Definition 3.1. Let $\mathfrak{a}$ be an ideal of R and M be a non-zero finitely generated R-module. We denote by $U_R(\mathfrak{a}, M)$ the largest submodule of M such that $\dim(U_R(\mathfrak{a}, M)/\mathfrak{a}U_R(\mathfrak{a}, M)) < \dim(M/\mathfrak{a}M)$. One can check that

$$U_R(\mathfrak{a}, M) = \cup\{N : N \leqslant M \text{ and } \dim(N/\mathfrak{a}N) < \dim(M/\mathfrak{a}M)\}.$$

The following theorem is a main result of [13] about the annihilators of the top formal local cohomology modules and plays a key role in the proof of main results.

Theorem 3.2 ([13, Theorem 1.2])**.** *Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(M/U_R(\mathfrak{a}, M)).$$

Here, by using the above result we calculate

$$\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^{\dim M}_{\mathfrak{a}}(M)) \text{ and } \operatorname{Att}(\mathfrak{b}\mathfrak{F}^{\dim M}_{\mathfrak{a}}(M)).$$

Theorem 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. Then $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$ if and only if $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$.*

Proof. If $\dim(M/\mathfrak{a}M) < d$, then $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) < d$ and so by [14, Theorem 4.5] $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = \mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. Thus we can assume that $\dim(M/\mathfrak{a}M) = d$.

Assume that $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. Since $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) \leq \dim(M/\mathfrak{a}M) = d$, by [14, Theorem 4.5] it follows that $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) < d$. Therefore

$$\mathfrak{b}M \subseteq U = \cup\{N \leq M : \dim N/\mathfrak{a}N < d\},$$

and so by Theorem 3.2 $\mathfrak{b} \subseteq \operatorname{Ann}(M/U) = \operatorname{Ann} \mathfrak{F}^d_{\mathfrak{a}}(M)$. Thus $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$. Conversely, if $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$, then $\mathfrak{b} \subseteq \operatorname{Ann} \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Ann}(M/U)$. It follows that $\mathfrak{b}M \subseteq U$ and so $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) < d$. Now [14, Theorem 4.5] implies that $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$. □

Corollary 3.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$, then $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$ and $\dim(M/\mathfrak{a}M) = \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = \dim(\mathfrak{b}M) = d$.*

Proof. $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ implies that $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ and so $\dim(M/\mathfrak{a}M) = d$ by [14, Theorem 4.5]. On the other hand, by Theorem 3.3 we have $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$. Thus by [14, Theorem 4.5] it follows that $d \leq \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M)$. But $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) \leq \dim(\mathfrak{b}M) \leq d$ by [14, Theorem 4.5] and so we conclude that $\dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = \dim(\mathfrak{b}M) = d$. □

Theorem 3.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. Then $\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$.*

Proof. If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$, then by Theorem 3.3 $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$ and the result follows in this case. Thus, we assume that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ and so by Corollary 3.4

$$\dim(M/\mathfrak{a}M) = \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = \dim \mathfrak{b}M = d.$$

Take $u \in \operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$. Thus $u\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$ and so $u\mathfrak{b} \subseteq \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(M))$. By Theorem 3.2 $\operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann} M/U$ where $U = \cup\{N \leq M : \dim N/\mathfrak{a}N < d\}$. Now, we have $u\mathfrak{b}M \subseteq U$ and so $\dim(u\mathfrak{b}M/u\mathfrak{a}\mathfrak{b}M) < d$. Set

$$W = \cup\{N \leq \mathfrak{b}M : \dim N/\mathfrak{a}N < d\}$$

and so $u \in \operatorname{Ann}(\mathfrak{b}M/W)$. Since by Theorem 3.2 $\operatorname{Ann}(\mathfrak{b}M/W) = \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$, it follows that $u \in \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Hence, $\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Conversely, assume that $u \in \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Ann}(\mathfrak{b}M/W)$. Since $W \subseteq U$ it follows that $u\mathfrak{b}M \subseteq U$ and so $u\mathfrak{b} \subseteq \operatorname{Ann}(M/U) = \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(M))$. Therefore $u \in \operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$ and the proof is complete. □

Corollary 3.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d and $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is not finitely generated.*

Proof. By [6, Lemma 2.2] $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and so $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian. Assume that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is finitely generated. Thus $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ has finite length and so there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$. By Theorem 3.5 we conclude that $\mathfrak{m}^n \subseteq \operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$. But by Corollary 3.4 $\dim(\mathfrak{b}M) = d$ and by [6, Lemma 2.2] $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is Artinian. Thus $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is finitely generated (see [15, Theorem 7.30]) which is a contradiction by [1, Theorem 2.6(ii)]. □

Theorem 3.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d and $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(\mathfrak{b}M/U_R(\mathfrak{a}, \mathfrak{b}M)),$$

where $U_R(\mathfrak{a}, \mathfrak{b}M)$ *is the largest submodule of* $\mathfrak{b}M$ *such that*

$$\dim(U_R(\mathfrak{a}, \mathfrak{b}M)/\mathfrak{a}U_R(\mathfrak{a}, \mathfrak{b}M)) < d.$$

Proof. The result follows by Theorems 3.5 and 3.2. □

Proposition 3.8. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module of finite dimension* d *such that* $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. *Then* $U_R(\mathfrak{a}, \mathfrak{b}M) = \mathfrak{b}M \cap U_R(\mathfrak{a}, M)$.

Proof. By Corollary 3.4

$$\dim(M/\mathfrak{a}M) = \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = d.$$

By definition, $U_R(\mathfrak{a}, \mathfrak{b}M)$ is the largest submodule of $\mathfrak{b}M$ such that

$$\dim(U_R(\mathfrak{a}, \mathfrak{b}M)/\mathfrak{a}U_R(\mathfrak{a}, \mathfrak{b}M)) < \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = d.$$

But, $\mathfrak{b}M \cap U_R(\mathfrak{a}, M)$ is a submodule of $\mathfrak{b}M$ and

$$\begin{aligned}\dim(\mathfrak{b}M \cap U_R(\mathfrak{a}, M)/\mathfrak{a}(\mathfrak{b}M \cap U_R(\mathfrak{a}, M))) &\leq \dim(U_R(\mathfrak{a}, M)/\mathfrak{a}U_R(\mathfrak{a}, M)) \\ &< \dim(M/\mathfrak{a}M) = d.\end{aligned}$$

It follows that $\mathfrak{b}M \cap U_R(\mathfrak{a}, M) \subseteq U_R(\mathfrak{a}, \mathfrak{b}M)$. Conversely, $U_R(\mathfrak{a}, M)$ is the largest submodule of M such that $\dim(U_R(\mathfrak{a}, M)/\mathfrak{a}U_R(\mathfrak{a}, M)) < \dim(M/\mathfrak{a}M) = d$. Thus $U_R(\mathfrak{a}, \mathfrak{b}M) \subseteq U_R(\mathfrak{a}, M)$. Clearly, $U_R(\mathfrak{a}, \mathfrak{b}M) \subseteq \mathfrak{b}M$. Therefore $U_R(\mathfrak{a}, \mathfrak{b}M) \subseteq \mathfrak{b}M \cap U_R(\mathfrak{a}, M)$, as required. □

Remark 3.9. Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d such that $\dim(M/\mathfrak{a}M) = d$. By [13, Theorem 2.6] we have

$$U_R(\mathfrak{a}, M) = \cap_{\mathfrak{p}_j \in \operatorname{Assh}_R M \cap \operatorname{V}(\mathfrak{a})} N_j,$$

where $0 = \cap_{j=1}^n N_j$ is a reduced primary decomposition of the zero submodule 0 in M and N_j is a $\mathfrak{p}_j$-primary submodule of M.

Corollary 3.10. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. *Then*

$$\begin{aligned}\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) &= \operatorname{Ann}(\mathfrak{b}M/\mathfrak{b}M \cap U_R(\mathfrak{a}, M)) \\ &= \operatorname{Ann}(\mathfrak{b}M/\mathfrak{b}M \cap (\cap_{\mathfrak{p}_j \in \operatorname{Assh} M \cap \operatorname{V}(\mathfrak{a})} N_j)),\end{aligned}$$

where $0 = \cap_{j=1}^n N_j$ *is a reduced primary decomposition of the zero submodule* 0 *in* M *and* N_j *is a* $\mathfrak{p}_j$*-primary submodule of* M.

Proof. By Theorem 3.7, Proposition 3.8 and Remark 3.9. □

Corollary 3.11. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a Noetherian local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module. Let* $d := \dim M$ *and* $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. *Then* $\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(\mathfrak{b}M)$ *whenever* $\operatorname{Ass} M \subseteq \operatorname{Assh} M \cap \operatorname{V}(\mathfrak{a})$.

Proof. It follows from Corollary 3.10. □

Corollary 3.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $d := \dim(M)$. If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$, then*

$$\operatorname{Cosupp}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}(\mathfrak{b}M/\mathfrak{b}M \cap U_R(\mathfrak{a}, M)).$$

Proof. By [6, Lemma 2.2] it follows that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian. Thus by [16, Proposition 2.3] and Corollary 3.10

$$\operatorname{Cosupp}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{V}(\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))) = \operatorname{V}(\operatorname{Ann}(\mathfrak{b}M/\mathfrak{b}M \cap U_R(\mathfrak{a}, M))).$$

But,

$$\operatorname{V}(\operatorname{Ann}(\mathfrak{b}M/\mathfrak{b}M \cap U_R(\mathfrak{a}, M))) = \operatorname{Supp}(\mathfrak{b}M/\mathfrak{b}M \cap U_R(\mathfrak{a}, M))$$

and the proof is complete. □

Theorem 3.13. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. Then*

$$\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

Proof. If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$, then $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$ by Theorem 3.3. Thus

$$\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \phi.$$

Thus we assume that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then in view of Corollary 3.4 we have

$$\dim M = \dim(M/\mathfrak{a}M) = \dim(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = \dim(\mathfrak{b}M) = d.$$

By [6, Lemma 2.2] $\mathfrak{F}^d_{\mathfrak{a}}(M)$ and $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$ are Artinian. Clearly, $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is a submodule of $\mathfrak{F}^d_{\mathfrak{a}}(M)$ and so is Artinian. On the other hand, by Lemma 2.2 $\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(M))$ and by [6, Theorem 3.1] $\operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Assh} M \cap \operatorname{V}(\mathfrak{a})$. Thus we conclude that $\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min}(\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)))$. But, $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$ is an Artinian R-module and so the set of all minimal elements of $\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$ is exactly the set of all minimal prime ideals containing $\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$. Thus $\operatorname{Min}(\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))) = \operatorname{Min}\operatorname{V}(\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)))$. Now by using Theorem 3.5 we have

$$\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min}\operatorname{V}(\operatorname{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))) = \operatorname{Min}\operatorname{V}(\operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))).$$

Since $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is Artinian we have

$$\operatorname{Min}\operatorname{V}(\operatorname{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))) = \operatorname{Min}\operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

On the other hand, $\dim(\mathfrak{b}M) = d$ and by [6, Theorem 3.1]

$$\operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Assh}(\mathfrak{b}M) \cap \operatorname{V}(\mathfrak{a}),$$

and so $\operatorname{Min}\operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Thus we conclude that $\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$, as required. □

Corollary 3.14. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of finite dimension d. If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$, then*

$$\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Assh}(\mathfrak{b}M) \cap \operatorname{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Ass}(\mathfrak{b}M) : \dim(R/\mathfrak{p}) = d, \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Proof. By Corollary 3.4 we have $\dim M = \dim(\mathfrak{b}M) = d$ and by Theorem 3.13

$$\operatorname{Att}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

But by [6, Theorem 3.1] $\operatorname{Att}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Assh}(\mathfrak{b}M) \cap \operatorname{V}(\mathfrak{a})$, as required. □

Theorem 3.15. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$ and k be an integer. Then*

$$\operatorname{Coass}(\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) \subseteq \operatorname{Coass}(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)).$$

In particular, if $\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \neq 0$, then $\operatorname{Assh}(\mathfrak{b}(M/\mathfrak{a}^k M)) \subseteq \operatorname{Coass}(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$.

Proof. The short exact sequence

$$0 \to \mathfrak{a}^k M \to M \to M/\mathfrak{a}^k M \to 0$$

induces the following exact sequence

$$\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}^k M) \to \mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}^k M).$$

Since $\dim(\mathfrak{a}^k M/\mathfrak{a}^{k+1} M) \leq \dim(M/\mathfrak{a}M) = l$ we have $\mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}^k M) = 0$. On the other hand, $M/\mathfrak{a}^k M$ is an $\mathfrak{a}$-torsion R-module and so by [6, Lemma 2.1] $\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}^k M) \simeq \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$. Thus from the above sequence we get the exact sequence $\mathfrak{F}^l_{\mathfrak{a}}(M) \to \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \to 0$. Thus $\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$ is a homomorphic image of $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ and so $\operatorname{Coass}(\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) \subseteq \operatorname{Coass}(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M))$. Now, assume that $\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \neq 0$. Since $\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$ is Artinian, by [16, Theorem 1.14] and Corollary 2.16 we have $\operatorname{Coass}(\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Att}(\mathfrak{b}\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Assh}(\mathfrak{b}(M/\mathfrak{a}^k M))$, as required. □

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