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INDUCTIVE LIMIT IN THE CATEGORY OF C^*-TERNARY RINGS

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Abstract. We show the existence of inductive limit in the category of C^*-ternary rings. It is proved that the inductive limit of C^*-ternary rings commutes with the functor $\mathcal{A}$ in the sense that if (M_n, ϕ_n) is an inductive system of C^*-ternary rings, then $\varinjlim \mathcal{A}(M_n) = \mathcal{A}(\varinjlim M_n)$. Some local properties (such as nuclearity, exactness and simplicity) of inductive limit of C^*-ternary rings have been investigated. Finally we obtain $\varinjlim M_n^{**} = (\varinjlim M_n)^{**}$.

1. Introduction

A C^*-ternary ring is a complex Banach space M, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of M^3 into M which is linear in the outer variables, conjugate linear in the middle variable, associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [u, z, y], v]$ satisfying $||[x, x, x]|| = ||x||^3$ and $||[x, y, z]|| \leq ||x||||y||||z||$. We refer to [13], [1], [9] and [11] for all necessary background related to C^*-ternary ring.

A closely related structure to C^*-ternary ring is the so-called ternary rings of operators (TROs) that is a norm closed subspace of $B(H, K)$, the set of all bounded operators from a Hilbert space H to a Hilbert space K which is closed under the ternary product $(x, y, z) \to xy^*z$. Clearly, the class of C^*-ternary rings includes TROs via the ternary product $[x, y, z] \to xy^*z$ and in particular C^*-algebras. In [5], Hamana showed that a TRO can be identified with the off diagonal corner of its linking C^*-algebra. Inductive limit in the category of TROs was studied in [7] and [3]. In [8], Kaur and Ruan studied TROs and their connections with their linking C^*-algebras. Using results obtained by Kaur and Ruan, authors in [7] showed that under certain restrictions inductive limit of TROs behaves well with some local properties such as simplicity, nuclearity and exactness. Pluta and Russo [11] extended the Hamana's notion of linking C^*-algebras to the category of C^*-ternary rings. Following construction is taken from [11].

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Given C^*-ternary ring M, let

$$E(M) = \mathrm{End}(M) \oplus \overline{[\mathrm{End}(M)]}^{\mathrm{op}},$$

where the notation $\overline{V}$ for a complex vector space means that the scalar multiplication in $\overline{V}$ is $(\lambda, v) \to \overline{\lambda} v$ and $\mathrm{End}(M)$ is a set of all endomorphisms on M equipped with the operator norm. For $\phi \oplus \psi \in E(M)$

$$\|\phi \oplus \psi\| = \max\{\|\phi\|, \|\psi\|\}.$$

For $g, h \in M$, define $L(g,h) = [g,h,\cdot], R(g,h) = [\cdot, h, g]$,

$$l(g,h) = (L(g,h), L(h,g)) \in E(M)$$

and

$$r(g,h) = (R(h,g), R(g,h)) \in E(M)^{\mathrm{op}}.$$

Next, let $L = L(M)$ and $R = R(M)$ denote the closures of $\mathrm{span}\{l(g,h) : g, h \in M\} \subset E(M)$ and $\mathrm{span}\{r(g,h) : g, h \in M\}$ in $E(M)^{\mathrm{op}}$, respectively. Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{\mathrm{op}}$, and $f \in M$. Then M is a left $E(M)$-module via

$$(A, f) \to A \cdot f = A_1 f$$

and a right $E(M)^{\mathrm{op}}$-module via

$$(f, B) \to f \cdot B = B_1 f.$$

Let $\overline{M}$ denote the vector space M with the element f denoted by $\overline{f}$ and with the scalar multiplication defined by $(\lambda, \overline{f}) \to \lambda \circ \overline{f} = \overline{\overline{\lambda} f}$. Then $\overline{M}$ is a left $E(M)^{\mathrm{op}}$-module via

$$(B, \overline{f}) \to B \cdot \overline{f} = \overline{B_2 f}$$

and a right $E(M)$-module via

$$(\overline{f}, A) \to \overline{f} \cdot A = \overline{A_2 f}.$$

Let

$$\mathcal{A} = \mathcal{A}(M) = L(M) \oplus M \oplus \overline{M} \oplus R(M)$$

and write the elements $a = (A, f, \overline{g}, B)$ of $\mathcal{A}$ as a matrix

$$a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}.$$

Define multiplication and involution in $\mathcal{A}$ by

$$aa' = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + l(f, g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B \cdot \overline{g'} & r(g, f') + B \circ B' \end{bmatrix}$$

and

$$a^{\#} = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}.$$

In [11, Proposition 2.7], it is shown that $\mathcal{A}(M)$ is a C^*-algebra and M is the off diagonal corner of C^*-algebra $\mathcal{A}(M)$. Moreover, if M is a TRO, then $\mathcal{A}(M)$ is *-isomorphic to the Hamana's linking C^*-algebra.

We show that $M \to \mathcal{A}(M)$, $(M \xrightarrow{\phi} N) \to (\mathcal{A}(M) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(N))$ where the map $\mathcal{A}(\phi)$ is defined in Proposition 2.4 is an exact functor from the category of C^*-ternary rings to the category of C^*-algebras. We then study the inductive limits in the category of C^*-ternary rings and prove its existence. The commutativity of the inductive limit with the functor $\mathcal{A}$ is proved. Using this commutativity property, it is shown that local properties such as nuclearity, exactness and simplicity behaves well with the inductive limit of C^*-ternary ring. In passing we obtain the ideal structure of inductive limits of C^*-ternary ring. Lastly, we show that inductive limit behaves well with biduals.

2. Inductive limits in the category of C^*-ternary rings

Definition 2.1. A linear mapping ϕ between C^*-ternary rings is called a (ternary) homomorphism if ϕ preserves the ternary structure, i.e.,

$$\phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)].$$

The following proposition is a restatement of ([1], Corollary 4.8).

Proposition 2.2. *Let M and N be two C^*-ternary rings and $\phi : M \to N$ a homomorphism. Then $\phi(M)$ is a norm-closed sub-C^*-ternary ring of N.*

If we are given two C^*-ternary rings M and N and a surjective homomorphism $\phi : M \to N$, then in ([11], Lemma 2.6), it was shown that we may define a *-homomorphism $L(\phi) : L(M) \to L(N)$ and $R(\phi) : R(M) \to R(N)$ by letting

$$L(\phi)\left(\sum_i ([g_i, h_i, \cdot][h_i, g_i, \cdot])\right) = \sum_i ([\phi(g_i), \phi(h_i), \cdot], [\phi(h_i), \phi(g_i), \cdot])$$

and

$$R(\phi)\left(\sum_i ([\cdot, g_i, h_i][\cdot, h_i, g_i])\right) = \sum_i ([\cdot, \phi(g_i), \phi(h_i)], [\cdot, \phi(h_i), \phi(g_i)]).$$

If in the above ϕ is not surjective, then we can replace N by $\phi(N)$, which is a norm-closed sub-C^*-ternary ring. Therefore we have:

Proposition 2.3. *Let M and N be two C^*-ternary rings and $\phi : M \to N$ be a homomorphism. Then there is a C^*-homomorphism*

$$\mathcal{A}(\phi)\left(\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right) = \begin{bmatrix} L(\phi)(A) & \phi(f) \\ \overline{\phi(g)} & R(\phi)(B) \end{bmatrix}$$

with $L(\phi) : L(M) \to L(N)$ and $R(\phi) : R(M) \to R(N)$ as defined above.

The following result which is an immediate consequence of the last proposition implies that $\mathcal{A}(M)$ is determined up to isomorphisms. We have a functor $M \to \mathcal{A}(M)$, $(M \xrightarrow{\phi} N) \to (\mathcal{A}(M) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(N))$ from the category of C^*-ternary rings to the category of C^*-algebras.

Proposition 2.4. *Let M and N be two C^*-ternary rings. Then if M is isomorphic to N as C^*-ternary rings, then $\mathcal{A}(M)$ is C^*-isomorphic to $\mathcal{A}(N)$.*

Proof. Let $\phi : M \to N$ be a ternary isomorphism. Then there is a unique C^*-homomorphism $\mathcal{A}(\phi)$ defined in the last proposition. Suppose $a \in \ker(\mathcal{A}(\phi))$ then $L(\phi)(A) = 0$, $\phi(f) = 0$, $\overline{\phi(g)} = 0$ and $R(\phi)(B) = 0$. Since ϕ is one to one, we have $f = 0$, $g = 0$ and $B \in \ker(R(\phi))$. Now we claim that $\phi(f' \cdot B) = 0$ for all $f' \in M$. Suppose first that $B = r(g, h)$. Then $\phi(f' \cdot B) = \phi(R(g, h)(f')) = 0$ since $R(\phi)(B) = 0$. By the same argument, if $B = \sum_i r(g_i, h_i)$, then $\phi(f' \cdot B) = 0$. Now suppose $B \in R(M)$, let $\epsilon > 0$ and choose $B' = \sum_i r(g_i, h_i)$ with $\|B - B'\| < \epsilon$. Then

$$\|\phi(f' \cdot B)\| = \|\phi(f' \cdot (B - B'))\| \leq \epsilon \|f'\|.$$

Thus $\phi(f' \cdot B) = 0$ for all $f' \in M$, and therefore using ([11], Proposition 2.3(iii)) we have $\mathrm{span}\langle M|M\rangle \circ B = 0$ where $\langle f|g\rangle = r(f, g)$. Since $\mathrm{span}\langle M|M\rangle$ is dense in $R(M)$, it follows that $R(M) \circ B = 0$, and hence $B = 0$. Similarly, $A = 0$. This shows that $\mathcal{A}(\phi)$ is one to one. It is easy to see that since ϕ is onto, so are $L(\phi)$ and $R(\phi)$, and hence $\mathcal{A}(\phi)$. □

Definition 2.5. A subspace I in a C^*-ternary ring M is called an ideal provided $[I, M, M] + [M, I, M] + [M, M, I] \subset I$. By an ideal, we shall always mean a closed ideal.

Let M be a C^*-ternary ring and I be an ideal of M. From ([6], Page 1135) it is known that every element of C^*-ternary ring I has a cube root. So using associativity and ([11], Lemma 1.1(iii)), the following is immediate.

Lemma 2.6. *For a C^*-ternary ring M and an ideal I of M, $\mathcal{A}(I)$ is an ideal of C^*-algebra $\mathcal{A}(M)$.*

Remark 2.7. Let I be a closed subspace of M satisfying $[M, M, I] + [I, M, M] \subset I$. Then as above using cube root in I, it can be shown that $\mathcal{A}(I)$ is an ideal in $\mathcal{A}(M)$. Therefore it will have an approximate unit. So we get $\{d_\lambda\}$ in $R(M)$ such that $xd_\lambda \to x$ for all $x \in I$. Thus we may approximate every $x \in I$ by sums of elements of the form $xr(g, h)$ with $g, h \in I$. Now using associativity, it follows that $[M, I, M] \subset I$.

For an ideal I of M, it follows from ([1], Proposition 4.5) that M/I is a C^*-ternary ring. This can also be concluded from the representation theorem of Zettl ([13], Theorem 3.1) and the fact that the quotient of a TRO is a TRO ([4], Proposition 2.2). Moreover, observe that $L(M/I) = L(M)/L(I)$ and $R(M/I) = R(M)/R(I)$ which gives the following.

Proposition 2.8. *Let I be an ideal of M. Then the quotient M/I is a C^*-ternary ring with $\mathcal{A}(M/I) = \mathcal{A}(M)/\mathcal{A}(I)$*

As a consequence of the above proposition, we have the following.

Proposition 2.9. *Let M be a C^*-ternary ring and I an ideal of M. The exact sequence*

$$0 \to I \xrightarrow{i} M \xrightarrow{\pi} M/I \to 0$$

induces an exact sequence of C^*-algebras

$$0 \to \mathcal{A}(I) \xrightarrow{\mathcal{A}(i)} \mathcal{A}(M) \xrightarrow{\mathcal{A}(\pi)} \mathcal{A}(M/I) \to 0.$$

Proof. Analogous to what we did in proof of Proposition 2.4, injectivity of i gives injectivitity of $\mathcal{A}(i)$ and surjectivity of π gives the surjectivity of $\mathcal{A}(\pi)$ so we only need to show exactness at $\mathcal{A}(M)$. Since

$$0 \to \mathcal{A}(I) \to \mathcal{A}(M) \xrightarrow{\overline{\pi}} \mathcal{A}(M)/\mathcal{A}(I) \to 0$$

is obviously an exact sequence of C^*-algebras where $\overline{\pi}$ is the natural quotient homomorphism, the exactness of our sequence follows from Proposition 2.8. $\square$

The following corollary is an immediate consequence of the last proposition.

Corollary 2.10. (1) *The functor $M \to \mathcal{A}(M)$, $(M \xrightarrow{\phi} N) \to (\mathcal{A}(M) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(N))$ is an exact functor from the category of C^*-ternary rings to the category of C^*-algebras.*

(2) *Every split exact sequence of C^*-ternary rings induces a split exact sequence of C^*-algebras.*

(3) *For all C^*-ternary rings M and N, $\mathcal{A}(M \oplus N) = \mathcal{A}(M) \oplus \mathcal{A}(N)$.*

We now proceed to show the existence of inductive limits in category of C^*-ternary rings. Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings. Since L, R and $\mathcal{A}$ are functors, $(L(M_n), L(\phi_n))$, $(R(M_n), R(\phi_n))$ and $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$ are inductive sequences of C^*-algebras. For convenience of the reader, we recall definition and universal property of inductive limits.

Inductive Limits. An inductive limit of an inductive sequence (X_n, ϕ_n) in a category C is a system (X, μ_n) where X_∞ is an object in C and $\mu_n : X_n \to X_\infty$ is a morphism in C for each $n \in \mathbb{N}$ satisfying the following properties:

- The following diagram commutes for all n

$$\begin{tikzcd} X_n \arrow[r, "\phi_n"] \arrow[dr, "\Phi^n"'] & X_{n+1} \arrow[d, "\Phi^{n+1}"] \\ & X_\infty \end{tikzcd}$$

- If $(Y, (\lambda_n))$ is an inductive system in C which is compatible with the system (X_n, ϕ_n) in the sense that $\lambda_n = \lambda_{n+1} \circ \phi_n$, then there exists a unique $\lambda : X \to Y$ such that $\lambda \circ \mu_n = \lambda_n$ for all n.

The existence of inductive limits in the category of C^*-algebras is well known (see e.g. ([12], Proposition 6.2.4)). Let $(\mathcal{A}_\infty, \mu_n)$ be the inductive limit of the inductive system $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$. Then $\mathcal{A}_\infty = \overline{\bigcup_n \mu_n(\mathcal{A}(M_n))}$. Let $i_n : M_n \to$

$\mathcal{A}(M_n)$ be the standard corner embedding of M_n. Let $M_\infty = \overline{\bigcup_n \lambda_n(M_n))} \subset \mathcal{A}_\infty$ where $\lambda_n = \mu_n \circ i_n : M_n \to M_\infty$ is a homomorphism.

Theorem 2.11. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings. Then $\varinjlim(M_n, \phi_n)$ exists.*

Proof. We identify M_n with its image in $\mathcal{A}(M_n)$. For every $x \in M_n$, we have

$$\begin{aligned}\lambda_{n+1} \circ \phi_n(x) &= \mu_{n+1} \circ i_{n+1} \circ \phi_n(x)\\ &= \mu_{n+1} \circ \mathcal{A}(\phi_n)(x)\\ &= \mu_n \circ i_n(x)\\ &= \lambda_n(x).\end{aligned}$$

Let (N, α_n) be another system satisfying $\alpha_{n+1} \circ \phi_n = \alpha_n$ where $\alpha_n : M_n \to N$ is a homomorphism for all n. Since $(\mathcal{A}_\infty, \mu_n)$ is the inductive limit of the inductive sequence $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$ and $\mathcal{A}(\alpha_{n+1}) \circ \mathcal{A}(\phi_n) = \mathcal{A}(\alpha_{n+1} \circ \phi_n) = \mathcal{A}(\alpha_n)$, there exists one and only one *-homomorphism $\mu : \mathcal{A}_\infty \to \mathcal{A}(N)$ satisfying $\mu \circ \mu_n = \mathcal{A}(\alpha_n)$, i.e., the following diagram

$$\begin{tikzcd}\mathcal{A}(M_n) \arrow[r, "\mu_n"] \arrow[dr, "\mathcal{A}(\alpha_n)"'] & \mathcal{A}_\infty \arrow[d, "\mu"] \\ & \mathcal{A}(N)\end{tikzcd}$$

is commutative for all n. Note that the restriction $\tilde{\mu}$ of μ to M_∞ is a homomorphism and satisfies $\tilde{\mu} \circ \lambda_n = \alpha_n$. Moreover, uniqueness of μ gives the uniqueness of $\tilde{\mu}$. Hence, (M_∞, λ_n) is the inductive limit of (M_n, ϕ_n). $\square$

If the connecting maps of an inductive system (M_n, ϕ_n) are injective, then we can assume that $M_n \subset M_{n+1}$ and that ϕ_n are inclusion maps. As an application of the last theorem, we have the following corollary.

Corollary 2.12. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings with injective connecting maps. Let $M_\infty = \overline{\bigcup_n M_n}$ and $i_n : M_n \to M_\infty$ be the inclusion map. Then (M_∞, i_n) is the inductive limit of the inductive system (M_n, ϕ_n).*

Given an inductive system (M_n, ϕ_n) of C^*-ternary rings, the ternary morphism $\phi_n : M_n \to M_{n+1}$ induces the C^*-morphism $L(\phi_n) : L(M_n) \to L(M_{n+1})$. Thus $(L(M_n), L(\phi_n))$ becomes an inductive system of C^*-algebras. Let $i_{L(M_n)} : L(M_n) \to \mathcal{A}(M_n)$ be the natural embedding. Let $\psi_n = \mu_n \circ i_{L(M_n)}$ and $L_\infty = \overline{\bigcup_n \psi_n(L(M_n))}$. The verification of the next proposition is straightforward.

Proposition 2.13. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings with inductive limit (M_∞, λ_n). Then (L_∞, ψ_n) is the inductive limit of the inductive system $(L(M_n), L(\phi_n))$.*

Next, we show that inductive limit of C^*-ternary rings behaves well with the functor L.

Theorem 2.14. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings. Then $\varinjlim L(M_n) = L(\varinjlim M_n)$.*

Proof. We have the following:

$$\begin{aligned}
L(M_\infty) &= \overline{\operatorname{span}\left\{ l\left(\overline{\bigcup_n \lambda_n(M_n)}, \overline{\bigcup_n \lambda_n(M_n)} \right) \right\}} \\
&= \overline{\operatorname{span}\left\{ \overline{l\left(\bigcup_n \lambda_n(M_n), \bigcup_n \lambda_n(M_n) \right)} \right\}} \\
&= \overline{\operatorname{span}\left\{ \overline{l\left(\bigcup_n \mu_n \circ i_n(M_n), \bigcup_n \mu_n \circ i_n(M_n) \right)} \right\}} \\
&= \overline{\bigcup_n \overline{\operatorname{span}} \left\{ l(\mu_n \circ i_n(M_n), \mu_n \circ i_n(M_n)) \right\}} \\
&= \overline{\bigcup_n \overline{\operatorname{span}} \left\{ \mu_n \circ i_n(l(M_n, M_n)) \right\}} \\
&= \overline{\bigcup_n \mu_n \circ i_{L(M_n)}(L(M_n))} \\
&= \overline{\bigcup_n \psi_n(L(M_n))} \\
&= L_\infty
\end{aligned}$$

which shows that $\varinjlim L(M_n) = L(\varinjlim M_n)$. Moreover, the homomorphism $\zeta_n : L(M_n) \to L(M_\infty)$ are given by $\zeta_n = L(\lambda_n) = L(\mu_n \circ i_{M_n}) = \mu_n \circ i_{L(M_n)}$ for all n. $\square$

Similarly we can obtain the following result by mimicking the proof of last theorem.

Theorem 2.15. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings. Then $\varinjlim R(M_n) = R(\varinjlim M_n)$.*

We refer the reader to [1] for a discussion on tensor product of C^*-ternary rings. We note that C^*-algebra M^r given in [1] which is defined as closed span of $\{[\cdot, g, h] : g, h \in M\}$ is *-isomorphic to $R(M)$. In [1, Section 5], it was shown that there exists a maximum C^*-norm $\|\cdot\|_{\max}$ on $M \otimes N$ and a minimum C^*-norm $\|\cdot\|_{\min}$ on $M \otimes N$ satisfying $(M \otimes_{\max} N)^r = M^r \otimes^{\max} N^r$ and $(M \otimes_{\min} N)^r = M^r \otimes^{\min} N^r$.

The following two definitions are from ([1], Definitions 5.7 and 5.15).

Definition 2.16. A C^*-ternary ring M will be called nuclear if for every C^*-ternary rings N, there is a unique C^*-norm on $M \otimes N$.

Definition 2.17. We say that a C^*-ternary ring M is exact if for every exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

of C^*-ternary rings

$$0 \to M \otimes_{\min} N_1 \to M \otimes_{\min} N_2 \to M \otimes_{\min} N_3 \to 0$$

is also exact.

As a consequence of Theorem 2.14, we have the following.

Corollary 2.18. (1) *Let (M_n, ϕ_n) be an inductive system of nuclear C^*-ternary rings. Then $M = \varinjlim(M_n, \phi_n)$ is also nuclear.*

(2) *Let (M_n, ϕ_n) be an inductive system of exact C^*-ternary rings. Then $M = \varinjlim(M_n, \phi_n)$ is also exact. Moreover, if the connecting maps are injective, then converse also holds.*

Proof. (1) In view of Theorem 2.11 and ([1], Corollary 5.14), we only need to show that M^r is nuclear which is clear by Theorem 2.15.

(2) It follows immediately from Theorem 2.15 and the fact that a C^*-ternary ring M is exact if and only if M^r is exact C^*-algebra. For converse, recall that every C^*-subalgebra of an exact C^*-algebra is exact. Now apply Corollary 2.12 to conclude the result. ☐

Our next aim is to see if the identity $\varinjlim \mathcal{A}(M_n) = \mathcal{A}(\varinjlim M_n)$ holds for an inductive sequence (M_n, ϕ_n) of C^*-ternary rings and the functor $\mathcal{A}$. Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings. If $(\mathcal{A}_\infty, \mu_n)$ is the inductive limit of the inductive system $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$, then from Theorem 2.11, it is known that (M_∞, λ_n) is the inductive limit of (M_n, ϕ_n) where $M_\infty = \overline{\bigcup_n \lambda_n(M_n))} \subset \mathcal{A}_\infty$ and $\lambda_n = \mu_n \circ i_n : M_n \to M_\infty$ is a homomorphism. From ([12], Proposition 6.2.4), $\|\mu_n(x)\| = \lim_{m\to\infty} \|\mathcal{A}(\phi_{m,n})(x)\|$ for all n. For $y \in M_n$, we have

$$\|\lambda_n(y)\| = \|\mu_n \circ i_n(y)\| = \lim_{m\to\infty} \|\mathcal{A}(\phi_{m,n})(i_n(y))\| = \lim_{m\to\infty} \|\phi_{m,n}(y)\|$$

which proves the following.

Lemma 2.19. *Let (M_n, ϕ_n) be an inductive system of C^*-ternary rings with inductive limit (M_∞, λ_n). Then*

$$\ker(\lambda_n) = \{x \in M_n : \lim_{m\to\infty} \|\phi_{m,n}(x)\| = 0\}.$$

Theorem 2.20. *If (M_n, ϕ_n) is an inductive system of C^*-ternary rings, then $\varinjlim \mathcal{A}(M_n) = \mathcal{A}(\varinjlim M_n)$ where the inductive limits are taken in the corresponding categories.*

Proof. First observe that for every $n \in \mathbb{N}$, $\mathcal{A}(\lambda_{n+1}) \circ \mathcal{A}(\phi_n) = \mathcal{A}(\lambda_n)$. Thus $(\mathcal{A}(M_\infty), \mathcal{A}(\lambda_n))$ is an inductive system which is compatible with the system $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$. Hence by universal property of inductive limit we get a unique *-homomorphism $\mu : \varinjlim \mathcal{A}(M_n) \to \mathcal{A}(\varinjlim M_n)$ and the following commutative diagram:

$$\begin{tikzcd} \mathcal{A}(M_n) \arrow[r, "\mu_n"] \arrow[dr, "\mathcal{A}(\lambda_n)"'] & \varinjlim \mathcal{A}(M_n) \arrow[d, "\mu"] \\ & \mathcal{A}(\varinjlim M_n) \end{tikzcd}$$

Since $\mathcal{A}(\varinjlim M_n) = \overline{\bigcup_n \mathcal{A}(\lambda_n)(\mathcal{A}(M_n))}$ therefore by ([12], Proposition 6.2.4(iv)), the map μ is surjective. Again by ([12], Proposition 6.2.4(iv)), to show, μ is injective we only need to show that $\ker(\mathcal{A}(\lambda_n)) \subset \ker(\mu_n)$. Let $x = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right]$ be an element of $\ker(\mathcal{A}(\lambda_n))$. Then by above lemma, $\lim_{m\to\infty} \|\phi_{m,n}(x_j)\| = 0$, $j = 2, 3$ and by ([12], Proposition 6.2.4(iii)),

$$\lim_{m\to\infty} \|L(\phi_{m,n})(x_1)\| = 0, \ \lim_{m\to\infty} \|R(\phi_{m,n})(x_4)\| = 0.$$

Thus we get,

$$\begin{aligned} \lim_{m\to\infty} \|\mathcal{A}(\phi_{m,n})(x)\| &= \lim_{m\to\infty} \left\| \begin{matrix} L(\phi_{m,n})(x_1) & \phi_{m,n}(x_2) \\ \overline{\phi_{m,n}(x_3)} & R(\phi_{m,n})(x_4) \end{matrix} \right\| \\ &\le \lim_{m\to\infty} \big(\|L(\phi_{m,n})(x_1)\| + \|\phi_{m,n}(x_2)\| \\ &\quad + \|\overline{\phi_{m,n}(x_3)}\| + \|R(\phi_{m,n})(x_4)\| \big) \\ &= 0 \end{aligned}$$

which implies $x \in \ker(\mu_n)$ and therefore μ is an isomorphism. $\square$

We shall next study the connection between ideals of C^*-ternary ring M and $\mathcal{A}(M)$. In [1, Proposition 4.2], it was shown that the map $I \to I^r$ is a one-to-one correspondence between closed ideals of M and M^r. It is not difficult to see that the map $I \to \mathcal{A}(I)$ is a one-to-one correspondence between closed ideals of M and $\mathcal{A}(M)$. Hence we have the following result.

Proposition 2.21. *Let M be a C^*-ternary ring. Then there are one-to-one correspondences between*

(1) *closed ideals in the C^*-ternary ring M.*
(2) *closed ideals in the C^*-algebra M^r.*
(3) *closed ideals in the C^*-algebra $\mathcal{A}(M)$.*

As a consequence of the above proposition and Theorem 2.15, we have the following.

Corollary 2.22. *Every closed ideal of inductive limit $M = \varinjlim(M_n, \phi_n)$ is an inductive limit of ideals of M_n.*

Proof. Let I be a closed ideal of M. Since every ideal of inductive limit of C^*-algebras is an inductive limit of ideals of C^*-algebras therefore $\mathcal{A}(I)$ being an ideal of the inductive limit of $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$ is an inductive limit of ideals of $\mathcal{A}(M_n)$. Now apply Theorem 2.20 to conclude the result. □

Recall that a C^*-ternary ring M is called simple if M has no non trivial closed ideal. Note that, M is simple if and only if $\mathcal{A}(M)$ is simple C^*-algebra.

Corollary 2.23. *Let (M_n, ϕ_n) be an inductive system of simple C^*-ternary rings. Then $M = \varinjlim(M_n, \phi_n)$ is also simple.*

Proof. In view of Theorem 2.15, it is enough to show that $\varinjlim \mathcal{A}(M_n)$ is simple C^*-algebra which follows from the fact that inductive limit of simple C^*-algebras is again simple. □

As an application of Proposition 2.21, we classify closed ideals of C^*-ternary ring of continuous functions vanishing at infinity and $M_n(M)$ space of $n \times n$ matrices with entries from M.

Example 2.24. Let X be a locally compact Hausdorff topological space and M be a C^*-ternary ring. Let $f : X \to M$ be a continuous function. Recall that f is said to vanish at infinity if for each $\epsilon > 0$, there exists a compact subset K of X such that $||f(x)|| < \epsilon$ whenever $x \notin K$. Denote, $C_0(X, M) := \{f : X \to M : f \text{ is continuous and vanishes at infinity}\}$. Note that $C_0(X, M)$ is a C^*-ternary ring with the ternary product defined as $[f_1, f_2, f_3](x) \to [f_1(x), f_2(x), f_3(x)]$. Note that by the map $\theta : (C_0(X, M))^r \to C_0(X, M^r)$ defined as $\theta([\cdot, f, g])(x) = [\cdot, f(x), g(x)]$, $(C_0(X, M))^r$ is isomorphic to $C_0(X, M^r)$ as C^*-algebras. For each $x \in X$, let I_x be an ideal of M. Then the set of $f \in C_0(X, M)$ satisfying $f(x) \in I_x$ is an ideal of $C_0(X, M)$. Conversely, let I be an ideal of $C_0(X, M)$. In view of ([10], V.26.2.1) and Proposition 2.21, it follows that that I^r is of the form $\{f \in C_0(X, M^r) : f(x) \in I_x^r, \forall x \in X\}$ where for every $x \in X$, I_x is a closed ideal of M. Since $\{f \in C_0(X, M^r) : f(x) \in I_x^r, \forall x \in X\} = \{f \in C_0(X, M) : f(x) \in I_x, \forall x \in X\}^r$ so every ideal of $C_0(X, M)$ is of the form $\{f \in C_0(X, M) : f(x) \in I_x, \forall x \in X\}$.

Example 2.25. For a C^*-ternary ring M, let $M_n(M)$ denote the space of $n \times n$ matrices with entries from M. Define,

$$[A, B, C]_{ij} = \sum_{l,k=1}^{n} [A_{il}, B_{lk}, C_{kj}].$$

$M_n(M)$ with this ternary operation is a C^*-ternary ring. Moreover by Corollary 2.10, $\mathcal{A}(M_n(M)) = M_n(\mathcal{A}(M))$. From Proposition 2.21, it follows that closed ideals of $M_n(M)$ are of the form $M_n(I)$ where I is a closed ideal of M. In particular, if M is simple C^*-ternary ring, then $M_n(M)$ is also simple.

Now, we consider biduals of C^*-ternary rings and study the commutativity of biduals with inductive limits. Let M be a C^*-ternary ring. In [9], it was proved that second dual of a C^*-ternary ring is again a C^*-ternary ring. Our aim in this section is to see if the identity $\varinjlim M_n^{**} = (\varinjlim M_n)^{**}$ holds for an inductive system (M_n, ϕ_n) of C^*-ternary rings. Keeping in mind that every injective homomorphism of C^*-ternary rings is an isometry, the idea of the proof of next proposition is similar to the proof of ([2], Lemma 2.1), we shall sketch an outline of a proof.

Proposition 2.26. *If (M_n, ϕ_n) is an inductive system of C^*-ternary rings with injective connecting maps, then $\varinjlim M_n^{**} = (\varinjlim M_n)^{**}$.*

Proof. Observe that for every $n \in \mathbb{N}$, the injective map $\phi_n : M_n \to M_{n+1}$ induces the canonical injective map $\phi_n^{**} : M_n^{**} \to M_{n+1}^{**}$. Thus (M_n^{**}, ϕ_n^{**}) becomes an inductive system of C^*-ternary rings with injective connecting maps. Let $((M^{**})_\infty, \mu_n)$ be the inductive limit of this inductive system and (M_∞, λ_n) be the inductive limit of (M_n, ϕ_n). Since the connecting maps ϕ'_ns are injective therefore λ_n and μ_n are also injective. Note that $\lambda_{n+1}^{**} \circ \phi_n^{**} = {\lambda_n}^{**}$. Thus $(M_\infty^{**}, \lambda_n^{**})$ is an inductive system which is compatible with the system (M_n^{**}, ϕ_n^{**}). Hence by universal property of inductive limit we get a unique homomorphism $\mu : \varinjlim M_n^{**} \to (\varinjlim M_n)^{**}$ and the following commutative diagram:

$$\begin{tikzcd} M_n^{**} \arrow[r, "\mu_n"] \arrow[dr, "\lambda_n^{**}"'] & \varinjlim M_n^{**} \arrow[d, "\mu"] \\ & (\varinjlim M_n)^{**} \end{tikzcd}$$

Finally it is not difficult to check that μ is an isomorphism and therefore $\varinjlim M_n^{**} = (\varinjlim M_n)^{**}$. □

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