

**INDUCTIVE LIMIT IN THE CATEGORY OF C*-TERNARY RINGS**

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**Abstract.** We show the existence of inductive limit in the category of C*-ternary rings. It is proved that the inductive limit of C*-ternary rings commutes with the functor $A$ in the sense that if $(M_n, \phi_n)$ is an inductive system of C*-ternary rings, then $\lim_{\to} A(M_n) = A(\lim_{\to} M_n)$. Some local properties (such as nuclearity, exactness and simplicity) of inductive limit of C*-ternary rings have been investigated. Finally we obtain $\lim_{\to} M_n^{**} = (\lim_{\to} M_n)^{**}$.

**1. Introduction**

A C*-ternary ring is a complex Banach space $M$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $M^3$ into $M$ which is linear in the outer variables, conjugate linear in the middle variable, associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [u, z, y], v]$ satisfying $||[x, x, x]|| = ||x||^3$ and $||[x, y, z]|| \leq ||x|| ||y|| ||z||$. We refer to [13], [1], [9] and [11] for all necessary background related to C*-ternary ring.

A closely related structure to C*-ternary rings is the so-called ternary rings of operators (TROs) that is a norm closed subspace of $B(H, K)$, the set of all bounded operators from a Hilbert space $H$ to a Hilbert space $K$ which is closed under the ternary product $(x, y, z) \mapsto xy^*z$. Clearly, the class of C*-ternary rings includes TROs via the ternary product $[x, y, z] \mapsto xy^*z$ and in particular C*-algebras. In [5], Hamana showed that a TRO can be identified with the off diagonal corner of its linking C*-algebra. Inductive limit in the category of TROs was studied in [7] and [3]. In [8], Kaur and Ruan studied TROs and their connections with their linking C*-algebras. Using results obtained by Kaur and Ruan, authors in [7] showed that under certain restrictions inductive limit of TROs behaves well with some local properties such as simplicity, nuclearity and exactness. Pluta and Russo [11] extended the Hamana’s notion of linking C*-algebras to the category of C*-ternary rings. Following construction is taken from [11].

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Given $C^*$-ternary ring $M$, let
\[ E(M) = \text{End}(M) \oplus \overline{\text{End}(M)}^\text{op}, \]
where the notation $\overline{\mathbf{V}}$ for a complex vector space means that the scalar multiplication in $\mathbf{V}$ is $(\lambda, v) \mapsto \overline{\lambda}v$ and $\text{End}(M)$ is a set of all endomorphisms on $M$ equipped with the operator norm. For $\phi \oplus \psi \in E(M)$
\[ \|\phi \oplus \psi\| = \max\{|\|\phi\|, \|\psi\|\}. \]

For $g, h \in M$, define $L(g, h) = [g, h, \cdot], R(g, h) = [\cdot, h, g], l(g, h) = (L(g, h), L(h, g)) \in E(M)$ and $r(g, h) = (R(h, g), R(g, h)) \in E(M)^\text{op}$. Next, let $L = L(M)$ and $R = R(M)$ denote the closures of span$\{l(g, h) : g, h \in M\} \subset E(M)$ and span$\{r(g, h) : g, h \in M\}$ in $E(M)^\text{op}$, respectively. Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^\text{op}$, and $f \in M$. Then $M$ is a left $E(M)$-module via
\[ (A, f) \rightarrow A \cdot f = A_1 f \]
and a right $E(M)^\text{op}$-module via
\[ (f, B) \rightarrow f \cdot B = B_1 f. \]

Let $\overline{M}$ denote the vector space $M$ with the element $f$ denoted by $\overline{f}$ and with the scalar multiplication defined by $(\lambda, \overline{f}) \rightarrow \lambda \circ \overline{f} = \overline{\lambda f}$. Then $\overline{M}$ is a left $E(M)^\text{op}$-module via
\[ (B, \overline{f}) \rightarrow B \cdot \overline{f} = \overline{B_2 f} \]
and a right $E(M)$-module via
\[ (\overline{f}, A) \rightarrow \overline{f} \cdot A = \overline{A_2 f}. \]

Let
\[ A = A(M) = L(M) \oplus M \oplus \overline{M} \oplus R(M) \]
and write the elements $a = (A, f, \overline{g}, B)$ of $A$ as a matrix
\[ a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}. \]

Define multiplication and involution in $A$ by
\[ aa' = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + l(f, g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B \cdot \overline{g'} & r(g, f') + B \circ B' \end{bmatrix} \]
and
\[ a^\# = \begin{bmatrix} A & g \\ \overline{f} & B \end{bmatrix}. \]

In [11, Proposition 2.7], it is shown that $A(M)$ is a $C^*$-algebra and $M$ is the off diagonal corner of $C^*$-algebra $A(M)$. Moreover, if $M$ is a TRO, then $A(M)$ is $^*$-isomorphic to the Hamana’s linking $C^*$-algebra.
We show that \( M \to A(M), (M \xrightarrow{\phi} N) \to (A(M) \xrightarrow{A(\phi)} A(N)) \) where the map \( A(\phi) \) is defined in Proposition 2.4 is an exact functor from the category of \( C^* \)-ternary rings to the category of \( C^* \)-algebras. We then study the inductive limits in the category of \( C^* \)-ternary rings and prove its existence. The commutativity of the inductive limit with the functor \( A \) is proved. Using this commutativity property, it is shown that local properties such as nuclearity, exactness and simplicity behaves well with the inductive limit of \( C^* \)-ternary ring. In passing we obtain the ideal structure of inductive limits of \( C^* \)-ternary ring. Lastly, we show that inductive limit behaves well with biduals.

2. Inductive limits in the category of \( C^* \)-ternary rings

**Definition 2.1.** A linear mapping \( \phi \) between \( C^* \)-ternary rings is called a \((\text{ternary}) \) homomorphism if \( \phi \) preserves the ternary structure, i.e.,

\[
\phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)].
\]

The following proposition is a restatement of ([1], Corollary 4.8).

**Proposition 2.2.** Let \( M \) and \( N \) be two \( C^* \)-ternary rings and \( \phi : M \to N \) a homomorphism. Then \( \phi(M) \) is a norm-closed sub-\( C^* \)-ternary ring of \( N \).

If we are given two \( C^* \)-ternary rings \( M \) and \( N \) and a surjective homomorphism \( \phi : M \to N \), then in ([11], Lemma 2.6), it was shown that we may define a \(*\)-homomorphism \( L(\phi) : L(M) \to L(N) \) and \( R(\phi) : R(M) \to R(N) \) by letting

\[
L(\phi) \left( \sum_i ([g_i, h_i, [h_i, g_i, :]]) \right) = \sum_i ([\phi(g_i), \phi(h_i), :], [\phi(h_i), \phi(g_i), :])
\]

and

\[
R(\phi) \left( \sum_i ([, g_i, h_i[, [h_i, g_i]]) \right) = \sum_i ([, \phi(g_i), \phi(h_i)], [, \phi(h_i), \phi(g_i)]).
\]

If in the above \( \phi \) is not surjective, then we can replace \( N \) by \( \phi(N) \), which is a norm-closed sub-\( C^* \)-ternary ring. Therefore we have:

**Proposition 2.3.** Let \( M \) and \( N \) be two \( C^* \)-ternary rings and \( \phi : M \to N \) be a homomorphism. Then there is a \( C^* \)-homomorphism

\[
A(\phi) \left( \begin{array}{c}
A \\
B
\end{array} \right) = \left[ \begin{array}{c}
L(\phi)(A) \\
\phi(f)
\end{array} \right] = \left[ \begin{array}{c}
\phi(g) \\
R(\phi)(B)
\end{array} \right]
\]

with \( L(\phi) : L(M) \to L(N) \) and \( R(\phi) : R(M) \to R(N) \) as defined above.

The following result which is an immediate consequence of the last proposition implies that \( A(M) \) is determined up to isomorphisms. We have a functor \( M \to A(M), (M \xrightarrow{\phi} N) \to (A(M) \xrightarrow{A(\phi)} A(N)) \) from the category of \( C^* \)-ternary rings to the category of \( C^* \)-algebras.
Proposition 2.4. Let $M$ and $N$ be two $C^*$-ternary rings. Then if $M$ is isomorphic to $N$ as $C^*$-ternary rings, then $\mathcal{A}(M)$ is $C^*$-isomorphic to $\mathcal{A}(N)$.

Proof. Let $\phi : M \to N$ be a ternary isomorphism. Then there is a unique $C^*$-homomorphism $\mathcal{A}(\phi)$ defined in the last proposition. Suppose $a \in \ker(\mathcal{A}(\phi))$ then $L(\phi)(A) = 0$, $\phi(f) = 0$, $\phi(g) = 0$ and $R(\phi)(B) = 0$. Since $\phi$ is one to one, we have $f = 0$, $g = 0$ and $B \in \ker(R(\phi))$. Now we claim that $\phi(f' \cdot B) = 0$ for all $f' \in M$. Suppose first that $B = r(g, h)$. Then $\phi(f' \cdot B) = \phi(R(g, h)(f')) = 0$ since $R(\phi)(B) = 0$. By the same argument, if $B = \sum_i r(g_i, h_i)$, then $\phi(f' \cdot B) = 0$. Now suppose $B \in R(M)$, let $\epsilon > 0$ and choose $B' = \sum_i r(g_i, h_i)$ with $\|B - B'\| < \epsilon$. Then

$$\|\phi(f' \cdot B)\| = \|\phi(f' \cdot (B - B'))\| \leq \epsilon\|f'\|.$$ 

Thus $\phi(f' \cdot B) = 0$ for all $f' \in M$, and therefore using ([11], Proposition 2.3(iii)) we have span$(M|M) \odot B = 0$ where $(f|g) = r(f, g)$. Since span$(M|M)$ is dense in $R(M)$, it follows that $R(M) \odot B = 0$, and hence $B = 0$. Similarly, $A = 0$. This shows that $\mathcal{A}(\phi)$ is one to one. It is easy to see that since $\phi$ is onto, so are $L(\phi)$ and $R(\phi)$, and hence $\mathcal{A}(\phi)$. ☐

Definition 2.5. A subspace $I$ in a $C^*$-ternary ring $M$ is called an ideal provided $[I, M, M] + [M, I, M] + [M, M, I] \subset I$. By an ideal, we shall always mean a closed ideal.

Let $M$ be a $C^*$-ternary ring and $I$ be an ideal of $M$. From ([6], Page 1135) it is known that every element of $C^*$-ternary ring $I$ has a cube root. So using associativity and ([11], Lemma 1.1(iii)), the following is immediate.

Lemma 2.6. For a $C^*$-ternary ring $M$ and an ideal $I$ of $M$, $\mathcal{A}(I)$ is an ideal of $C^*$-algebra $\mathcal{A}(M)$.

Remark 2.7. Let $I$ be a closed subspace of $M$ satisfying $[M, M, I] + [I, M, M] \subset I$. Then as above using cube root in $I$, it can be shown that $\mathcal{A}(I)$ is an ideal in $\mathcal{A}(M)$. Therefore it will have an approximate unit. So we get $\{d_\lambda\}$ in $R(M)$ such that $xd_\lambda \to x$ for all $x \in I$. Thus we may approximate every $x \in I$ by sums of elements of the form $xr(g, h)$ with $g, h \in I$. Now using associativity, it follows that $[M, I, M] \subset I$. 

For an ideal $I$ of $M$, it follows from ([1], Proposition 4.5) that $M/I$ is a $C^*$-ternary ring. This can also be concluded from the representation theorem of Zettl ([13], Theorem 3.1) and the fact that the quotient of a TRO is a TRO ([4], Proposition 2.2). Moreover, observe that $L(M/I) = L(M)/L(I)$ and $R(M/I) = R(M)/R(I)$ which gives the following.

Proposition 2.8. Let $I$ be an ideal of $M$. Then the quotient $M/I$ is a $C^*$-ternary ring with $\mathcal{A}(M/I) = \mathcal{A}(M)/\mathcal{A}(I)$

As a consequence of the above proposition, we have the following.
Proposition 2.9. Let $M$ be a $C^*$-ternary ring and $I$ an ideal of $M$. The exact sequence

$$0 \rightarrow I \xrightarrow{i} M \xrightarrow{\pi} M/I \rightarrow 0$$

induces an exact sequence of $C^*$-algebras

$$0 \rightarrow A(I) \xrightarrow{A(i)} A(M) \xrightarrow{A(\pi)} A(M/I) \rightarrow 0.$$

Proof. Analogous to what we did in proof of Proposition 2.4, injectivity of $i$ gives injectivity of $A(i)$ and surjectivity of $\pi$ gives the surjectivity of $A(\pi)$ so we only need to show exactness at $A(M)$. Since

$$0 \rightarrow A(I) \rightarrow A(M) \xrightarrow{\pi} A(M)/A(I) \rightarrow 0$$

is obviously an exact sequence of $C^*$-algebras where $\pi$ is the natural quotient homomorphism, the exactness of our sequence follows from Proposition 2.8. □

The following corollary is an immediate consequence of the last proposition.

Corollary 2.10. (1) The functor $M \rightarrow A(M), (M \xrightarrow{\phi} N) \rightarrow (A(M) \xrightarrow{A(\phi)} A(N))$ is an exact functor from the category of $C^*$-ternary rings to the category of $C^*$-algebras.

(2) Every split exact sequence of $C^*$-ternary rings induces a split exact sequence of $C^*$-algebras.

(3) For all $C^*$-ternary rings $M$ and $N$, $A(M \oplus N) = A(M) \oplus A(N)$.

We now proceed to show the existence of inductive limits in category of $C^*$-ternary rings. Let $(M_n, \phi_n)$ be an inductive system of $C^*$-ternary rings. Since $L, R$ and $A$ are functors, $(L(M_n), L(\phi_n)), (R(M_n), R(\phi_n))$ and $(A(M_n), A(\phi_n))$ are inductive sequences of $C^*$-algebras. For convenience of the reader, we recall definition and universal property of inductive limits.

Inductive Limits. An inductive limit of an inductive sequence $(X_n, \phi_n)$ in a category $C$ is a system $(X, \mu_n)$ where $X_\infty$ is an object in $C$ and $\mu_n : X_n \rightarrow X_\infty$ is a morphism in $C$ for each $n \in \mathbb{N}$ satisfying the following properties:

- The following diagram commutes for all $n$

$$\xymatrix{ X_n \ar[r]^-{\phi_n} \ar[d]_-{\phi_n^{n+1}} & X_{n+1} \ar[d]^-{\phi_{n+1}^{n+1}} \\ X_\infty & }$$

- If $(Y, (\lambda_n))$ is an inductive system in $C$ which is compatible with the system $(X_n, \phi_n)$ in the sense that $\lambda_n = \lambda_{n+1} \circ \phi_n$, then there exists a unique $\lambda : X \rightarrow Y$ such that $\lambda \circ \mu_n = \lambda_n$ for all $n$.

The existence of inductive limits in the category of $C^*$-algebras is well known (see e.g. ([12], Proposition 6.2.4)). Let $(A_\infty, \mu_n)$ be the inductive limit of the inductive system $(A(M_n), A(\phi_n))$. Then $A_\infty = \bigcup_n \mu_n(A(M_n))$. Let $i_n : M_n \rightarrow$
\( \mathcal{A}(M_n) \) be the standard corner embedding of \( M_n \). Let \( M_\infty = \bigcup_n \lambda_n(M_n) \subset \mathcal{A}_\infty \) where \( \lambda_n = \mu_n \circ i_n : M_n \to M_\infty \) is a homomorphism.

**Theorem 2.11.** Let \( (M_n, \phi_n) \) be an inductive system of \( C^* \)-ternary rings. Then \( \lim_{\to} (M_n, \phi_n) \) exists.

**Proof.** We identify \( M_n \) with its image in \( \mathcal{A}(M_n) \). For every \( x \in M_n \), we have

\[
\lambda_{n+1} \circ \phi_n(x) = \mu_{n+1} \circ i_{n+1} \circ \phi_n(x) = \mu_{n+1} \circ \mathcal{A}(\phi_n)(x) = \mu_n \circ i_n(x) = \lambda_n(x).
\]

Let \( (N, \alpha_n) \) be another system satisfying \( \alpha_{n+1} \circ \phi_n = \alpha_n \) where \( \alpha_n : M_n \to N \) is a homomorphism for all \( n \). Since \( (\mathcal{A}_\infty, \mu_n) \) is the inductive limit of the inductive sequence \( (\mathcal{A}(M_n), \mathcal{A}(\phi_n)) \) and \( \mathcal{A}(\alpha_{n+1}) \circ \mathcal{A}(\phi_n) = \mathcal{A}(\alpha_{n+1} \circ \phi_n) = \mathcal{A}(\alpha_n) \), there exists one and only one \( * \)-homomorphism \( \mu : \mathcal{A}_\infty \to \mathcal{A}(N) \) satisfying \( \mu \circ \mu_n = \mathcal{A}(\alpha_n) \), i.e., the following diagram

\[
\begin{array}{ccc}
\mathcal{A}(M_n) & \xrightarrow{\mu_n} & \mathcal{A}_\infty \\
\downarrow{\mathcal{A}(\alpha_n)} & & \downarrow{\mu} \\
\mathcal{A}(N) & & 
\end{array}
\]

is commutative for all \( n \). Note that the restriction \( \tilde{\mu} \) of \( \mu \) to \( M_\infty \) is a homomorphism and satisfies \( \tilde{\mu} \circ \lambda_n = \alpha_n \). Moreover, uniqueness of \( \mu \) gives the uniqueness of \( \tilde{\mu} \). Hence, \( (M_\infty, \lambda_n) \) is the inductive limit of \( (M_n, \phi_n) \). \( \square \)

If the connecting maps of an inductive system \( (M_n, \phi_n) \) are injective, then we can assume that \( M_n \subset M_{n+1} \) and that \( \phi_n \) are inclusion maps. As an application of the last theorem, we have the following corollary.

**Corollary 2.12.** Let \( (M_n, \phi_n) \) be an inductive system of \( C^* \)-ternary rings with injective connecting maps. Let \( M_\infty = \bigcup_n M_n \) and \( i_n : M_n \to M_\infty \) be the inclusion map. Then \( (M_\infty, i_n) \) is the inductive limit of the inductive system \( (M_n, \phi_n) \).

Given an inductive system \( (M_n, \phi_n) \) of \( C^* \)-ternary rings, the ternary morphism \( \phi_n : M_n \to M_{n+1} \) induces the \( C^* \)-morphism \( L(\phi_n) : L(M_n) \to L(M_{n+1}) \). Thus \( (L(M_n), L(\phi_n)) \) becomes an inductive system of \( C^* \)-algebras. Let \( i_{L(M_n)} : L(M_n) \to \mathcal{A}(M_n) \) be the natural embedding. Let \( \psi_n = \mu_n \circ i_{L(M_n)} \) and \( L_\infty = \bigcup_n \psi_n(L(M_n)) \). The verification of the next proposition is straightforward.

**Proposition 2.13.** Let \( (M_n, \phi_n) \) be an inductive system of \( C^* \)-ternary rings with inductive limit \( (M_\infty, \lambda_n) \). Then \( (L_\infty, \psi_n) \) is the inductive limit of the inductive system \( (L(M_n), L(\phi_n)) \).
Next, we show that inductive limit of $C^*$-ternary rings behaves well with the functor $L$.

**Theorem 2.14.** Let $(M_n, \phi_n)$ be an inductive system of $C^*$-ternary rings. Then $\varinjlim L(M_n) = L(\varinjlim M_n)$.

**Proof.** We have the following:

\[
L(M_\infty) = \text{span}\left\{ l \left( \bigcup_n \lambda_n(M_n), \bigcup_n \lambda_n(M_n) \right) \right\}
\]

\[
= \text{span}\left\{ l \left( \bigcup_n \lambda_n(M_n), \bigcup_n \lambda_n(M_n) \right) \right\}
\]

\[
= \text{span}\left\{ l \left( \bigcup_n \mu_n \circ i_n(M_n), \bigcup_n \mu_n \circ i_n(M_n) \right) \right\}
\]

\[
= \bigcup_n \text{span}\left\{ l(\mu_n \circ i_n(M_n), \mu_n \circ i_n(M_n)) \right\}
\]

\[
= \bigcup_n \text{span}\left\{ \mu_n \circ i_n(l(M_n, M_n)) \right\}
\]

\[
= \bigcup_n \mu_n \circ i_{L(M_n)}(L(M_n))
\]

\[
= \bigcup_n \psi_n(L(M_n))
\]

\[
= L_\infty
\]

which shows that $\varinjlim L(M_n) = L(\varinjlim M_n)$. Moreover, the homomorphism $\zeta_n : L(M_n) \to L(M_\infty)$ are given by $\zeta_n = \bar{L}(\lambda_n) = L(\mu_n \circ i_{M_n}) = \mu_n \circ i_{L(M_n)}$ for all $n$. \qed

Similarly we can obtain the following result by mimicking the proof of last theorem.

**Theorem 2.15.** Let $(M_n, \phi_n)$ be an inductive system of $C^*$-ternary rings. Then $\varinjlim R(M_n) = R(\varinjlim M_n)$.

We refer the reader to [1] for a discussion on tensor product of $C^*$-ternary rings. We note that $C^*$-algebra $M^r$ given in [1] which is defined as closed span of $\{[\cdot, g, h] : g, h \in M\}$ is $^*$-isomorphic to $R(M)$. In [1, Section 5], it was shown that there exists a maximum $C^*$-norm $\| \cdot \|_{\max}$ on $M \otimes N$ and a minimum $C^*$-norm $\| \cdot \|_{\min}$ on $M \otimes N$ satisfying $(M \otimes_{\max} N)^r = M^r \otimes_{\max} N^r$ and $(M \otimes_{\min} N)^r = M^r \otimes_{\min} N^r$.

The following two definitions are from ([1], Definitions 5.7 and 5.15).
Definition 2.16. A \( C^* \)-ternary ring \( M \) will be called nuclear if for every \( C^* \)-ternary rings \( N \), there is a unique \( C^* \)-norm on \( M \otimes N \).

Definition 2.17. We say that a \( C^* \)-ternary ring \( M \) is exact if for every exact sequence
\[
0 \to N_1 \to N_2 \to N_3 \to 0
\]
of \( C^* \)-ternary rings
\[
0 \to M \otimes_{\min} N_1 \to M \otimes_{\min} N_2 \to M \otimes_{\min} N_3 \to 0
\]
is also exact.

As a consequence of Theorem 2.14, we have the following.

Corollary 2.18. (1) Let \( (M_n, \phi_n) \) be an inductive system of nuclear \( C^* \)-ternary rings. Then
\[
M = \varinjlim (M_n, \phi_n)
\]
is also nuclear.

(2) Let \( (M_n, \phi_n) \) be an inductive system of exact \( C^* \)-ternary rings. Then
\[
M = \varinjlim (M_n, \phi_n)
\]
is also exact. Moreover, if the connecting maps are injective, then converse also holds.

Proof. (1) In view of Theorem 2.11 and ([1], Corollary 5.14), we only need to show that \( M^r \) is nuclear which is clear by Theorem 2.15.

(2) It follows immediately from Theorem 2.15 and the fact that a \( C^* \)-ternary ring \( M \) is exact if and only if \( M^r \) is exact \( C^* \)-algebra. For converse, recall that every \( C^* \)-subalgebra of an exact \( C^* \)-algebra is exact. Now apply Corollary 2.12 to conclude the result. \( \square \)

Our next aim is to see if the identity \( \varinjlim A(M_n) = A(\varinjlim M_n) \) holds for an inductive sequence \( (M_n, \phi_n) \) of \( C^* \)-ternary rings and the functor \( A \). Let \( (M_n, \phi_n) \) be an inductive system of \( C^* \)-ternary rings. If \( (A_{\infty}, \mu_n) \) is the inductive limit of the inductive system \( (A(M_n), A(\phi_n)) \), then from Theorem 2.11, it is known that \( (M_{\infty}, \lambda_n) \) is the inductive limit of \( (M_n, \phi_n) \) where \( M_{\infty} = \bigcup_n \lambda_n(M_n) \subset A_{\infty} \) and \( \lambda_n = \mu_n \circ i_n : M_n \to M_{\infty} \) is a homomorphism. From ([12], Proposition 6.2.4),
\[
\| \lambda_n(y) \| = \| \mu_n \circ i_n(y) \| = \lim_{m \to \infty} \| A(\phi_{m,n})(i_n(y)) \| = \lim_{m \to \infty} \| \phi_{m,n}(y) \|
\]
which proves the following.

Lemma 2.19. Let \( (M_n, \phi_n) \) be an inductive system of \( C^* \)-ternary rings with inductive limit \( (M_{\infty}, \lambda_n) \). Then
\[
\ker(\lambda_n) = \{ x \in M_n : \lim_{m \to \infty} \| \phi_{m,n}(x) \| = 0 \}.
\]

Theorem 2.20. If \( (M_n, \phi_n) \) is an inductive system of \( C^* \)-ternary rings, then
\[
\varinjlim A(M_n) = A(\varinjlim M_n)
\]
where the inductive limits are taken in the corresponding categories.
Proof. First observe that for every $n \in \mathbb{N}$, $A(\lambda_{n+1}) \circ A(\phi_n) = A(\lambda_n)$. Thus $(A(M_\infty), A(\lambda_n))$ is an inductive system which is compatible with the system $(A(M_n), A(\phi_n))$. Hence by universal property of inductive limit we get a unique $^*$-homomorphism $\mu : \varinjlim A(M_n) \to A(M)$ and the following commutative diagram:

\[
\begin{array}{ccc}
A(M_n) & \xrightarrow{\mu} & \varinjlim A(M_n) \\
\downarrow{A(\lambda_n)} & & \downarrow{\mu} \\
A(M) & & \\
\end{array}
\]

Since $A(M) = \bigcup_n A(\lambda_n)(A(M_n))$ therefore by ([12], Proposition 6.2.4(iv)), the map $\mu$ is injective. Again by ([12], Proposition 6.2.4(iv)), to show, $\mu$ is surjective we only need to show that ker$(A(\lambda_n)) \subset$ ker$(\mu_n)$. Let $x = [x_1, x_2, x_3, x_4]$ be an element of ker$(A(\lambda_n))$. Then by above lemma, $\lim_{m \to \infty} \|A(\phi_{m,n})(x_j)\| = 0$, $j = 2, 3$ and by ([12], Proposition 6.2.4(iii)),

\[
\lim_{m \to \infty} \|L(\phi_{m,n})(x_1)\| = 0, \quad \lim_{m \to \infty} \|R(\phi_{m,n})(x_4)\| = 0.
\]

Thus we get,

\[
\lim_{m \to \infty} \|A(\phi_{m,n})(x)\| = \lim_{m \to \infty} \left\| \begin{array}{cc}
L(\phi_{m,n})(x_1) & \phi_{m,n}(x_2) \\
\phi_{m,n}(x_3) & R(\phi_{m,n})(x_4)
\end{array} \right\| \\
\leq \lim_{m \to \infty} (\|L(\phi_{m,n})(x_1)\| + \|\phi_{m,n}(x_2)\| \\
+ \|\phi_{m,n}(x_3)\| + \|R(\phi_{m,n})(x_4)\|) \\
= 0
\]

which implies $x \in \ker(\mu_n)$ and therefore $\mu$ is an isomorphism. 

We shall next study the connection between ideals of $C^*$-ternary ring $M$ and $A(M)$. In [1, Proposition 4.2], it was shown that the map $I \to I'$ is a one-to-one correspondence between closed ideals of $M$ and $M'$. It is not difficult to see that the map $I \to A(I)$ is a one-to-one correspondence between closed ideals of $M$ and $A(M)$. Hence we have the following result.

**Proposition 2.21.** Let $M$ be a $C^*$-ternary ring. Then there are one-to-one correspondences between

1. closed ideals in the $C^*$-ternary ring $M$.
2. closed ideals in the $C^*$-algebra $M'$.
3. closed ideals in the $C^*$-algebra $A(M)$.

As a consequence of the above proposition and Theorem 2.15, we have the following.

**Corollary 2.22.** Every closed ideal of inductive limit $M = \varinjlim (M_n, \phi_n)$ is an inductive limit of ideals of $M_n$. 

Proof. Let $I$ be a closed ideal of $M$. Since every ideal of inductive limit of $\mathcal{C}^*$-algebras is an inductive limit of ideals of $\mathcal{C}^*$-algebras therefore $\mathcal{A}(I)$ being an ideal of the inductive limit of $(\mathcal{A}(M_n), \mathcal{A}(\phi_n))$ is an inductive limit of ideals of $\mathcal{A}(M_n)$. Now apply Theorem 2.20 to conclude the result.

Recall that a $\mathcal{C}^*$-ternary ring $M$ is called simple if $M$ has no non trivial closed ideal. Note that, $M$ is simple if and only if $\mathcal{A}(M)$ is simple $\mathcal{C}^*$-algebra.

Corollary 2.23. Let $(M_n, \phi_n)$ be an inductive system of simple $\mathcal{C}^*$-ternary rings. Then $M = \lim \mathcal{A}(M_n)$ is also simple.

Proof. In view of Theorem 2.15, it is enough to show that $\lim \mathcal{A}(M_n)$ is simple $\mathcal{C}^*$-algebra which follows from the fact that inductive limit of simple $\mathcal{C}^*$-algebras is again simple. \hfill $\Box$

As an application of Proposition 2.21, we classify closed ideals of $\mathcal{C}^*$-ternary ring of continuous functions vanishing at infinity and $M_n(M)$ space of $n \times n$ matrices with entries from $M$.

Example 2.24. Let $X$ be a locally compact Hausdorff topological space and $M$ be a $\mathcal{C}^*$-ternary ring. Let $f : X \to M$ be a continuous function. Recall that $f$ is said to vanish at infinity if for each $\epsilon > 0$, there exists a compact subset $K$ of $X$ such that $||f(x)|| < \epsilon$ whenever $x \notin K$. Denote, $C_0(X, M) := \{ f : X \to M : f$ is continuous and vanishes at infinity $\}$. Note that $C_0(X, M)$ is a $\mathcal{C}^*$-ternary ring with the ternary product defined as $[f_1, f_2, f_3][x] \to [f_1(x), f_2(x), f_3(x)]$. Note that by the map $\theta : (C_0(X, M))^r \to C_0(X, M^r)$ defined as $\theta([f], [g]) = [\cdot, f(x), g(x)]$, $(C_0(X, M))^r$ is isomorphic to $C_0(X, M^r)$ as $\mathcal{C}^*$-algebras. For each $x \in X$, let $I_x$ be an ideal of $M$. Then the set of $f \in C_0(X, M)$ satisfying $f(x) \in I_x$ is an ideal of $C_0(X, M)$. Conversely, let $I$ be an ideal of $C_0(X, M)$. In view of ([10], V.26.2.1) and Proposition 2.21, it follows that that $I^r$ is of the form $\{ f \in C_0(X, M^r) : f(x) \in I_x, \forall x \in X \}$ where for every $x \in X$, $I_x$ is a closed ideal of $M$. Since $\{ f \in C_0(X, M^r) : f(x) \in I_x, \forall x \in X \} = \{ f \in C_0(X, M) : f(x) \in I_x, \forall x \in X \}^r$ so every ideal of $C_0(X, M)$ is of the form $\{ f \in C_0(X, M) : f(x) \in I_x, \forall x \in X \}$.

Example 2.25. For a $\mathcal{C}^*$-ternary ring $M$, let $M_n(M)$ denote the space of $n \times n$ matrices with entries from $M$. Define, 

$$[A, B, C]_{ij} = \sum_{i,k=1}^{n} [A_{ik}, B_{kj}, C_{kj}].$$

$M_n(M)$ with this ternary operation is a $\mathcal{C}^*$-ternary ring. Moreover by Corollary 2.10, $\mathcal{A}(M_n(M)) = M_n(\mathcal{A}(M))$. From Proposition 2.21, it follows that closed ideals of $M_n(M)$ are of the form $M_n(I)$ where $I$ is a closed ideal of $M$. In particular, if $M$ is simple $\mathcal{C}^*$-ternary ring, then $M_n(M)$ is also simple.

Now, we consider biduals of $\mathcal{C}^*$-ternary rings and study the commutativity of biduals with inductive limits. Let $M$ be a $\mathcal{C}^*$-ternary ring. In [9], it was
proven that second dual of a $C^*$-ternary ring is again a $C^*$-ternary ring. Our aim in this section is to see if the identity $\lim \overset{\rightarrow}{M_n^{**}} = (\lim \overset{\rightarrow}{M_n})^{**}$ holds for an inductive system $(M_n, \phi_n)$ of $C^*$-ternary rings. Keeping in mind that every injective homomorphism of $C^*$-ternary rings is an isometry, the idea of the proof of next proposition is similar to the proof of ([2], Lemma 2.1), we shall sketch an outline of a proof.

**Proposition 2.26.** If $(M_n, \phi_n)$ is an inductive system of $C^*$-ternary rings with injective connecting maps, then $\lim \overset{\rightarrow}{M_n^{**}} = (\lim \overset{\rightarrow}{M_n})^{**}$.

**Proof.** Observe that for every $n \in \mathbb{N}$, the injective map $\phi_n : M_n \to M_{n+1}$ induces the canonical injective map $\phi_n^{**} : M_n^{**} \to M_{n+1}^{**}$. Thus $(M_n^{**}, \phi_n^{**})$ becomes an inductive system of $C^*$-ternary rings with injective connecting maps. Let $((M^{**})_\infty, \lambda_n)$ be the inductive limit of this inductive system and $(M_\infty, \lambda_n)$ be the inductive limit of $(M_n, \phi_n)$. Since the connecting maps $\phi_n$ are injective therefore $\lambda_n$ and $\mu_n$ are also injective. Note that $\lambda_{n+1}^{**} \circ \phi_n^{**} = \lambda_n^{**}$. Thus $(M_\infty^{**}, \lambda_n^{**})$ is an inductive system which is compatible with the system $(M_n^{**}, \phi_n^{**})$. Hence by universal property of inductive limit we get a unique homomorphism $\mu : \lim \overset{\rightarrow}{M_n^{**}} \to (\lim \overset{\rightarrow}{M_n})^{**}$ and the following commutative diagram:

$$\begin{array}{ccc}
M_n^{**} & \xrightarrow{\mu_n} & \lim \overset{\rightarrow}{M_n^{**}} \\
\downarrow{\lambda_n^{**}} & & \downarrow{\mu} \\
(\lim \overset{\rightarrow}{M_n})^{**}
\end{array}$$

Finally it is not difficult to check that $\mu$ is an isomorphism and therefore $\lim \overset{\rightarrow}{M_n^{**}} = (\lim \overset{\rightarrow}{M_n})^{**}$. □

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