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UN RINGS AND GROUP RINGS

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Abstract. A ring R is called a UN ring if every non unit of it can be written as product of a unit and a nilpotent element. We obtain results about lifting of conjugate idempotents and unit regular elements modulo an ideal I of a UN ring R. Matrix rings over UN rings are discussed and it is obtained that for a commutative ring R, a matrix ring $M_n(R)$ is UN if and only if R is UN. Lastly, UN group rings are investigated and we obtain the conditions on a group G and a field K for the group algebra KG to be UN. Then we extend the results obtained for KG to the group ring RG over a ring R (which may not necessarily be a field).

1. Introduction

In last two decades one of the active areas of research have been the rings whose elements can be written as a sum/product of units/ idempotents/ nilpotent elements. For example, *clean rings* are those in which every element of the ring R can be written as sum of a unit and an idempotent. If in place of addition, we take the multiplication, i.e., if every element of a ring R can be written as product of a unit and an idempotent, then we obtain the well known class of *unit regular* rings. Taking into consideration the unit and nilpotent elements, Călugăreanu and Lam in [2] defined a ring as *fine ring*, if every non zero element of a ring R can be written as a sum of a unit and a nilpotent element. They proved that the class of fine rings is a proper subclass of that of simple rings. Now, if in place of addition, the multiplication of unit and nilpotent elements is taken into consideration, then, Călugăreanu in [1] defined a ring R to be a *UN ring* if every non unit of R can be written as product of a unit and a nilpotent element. A non unit element $x \in R$ is called *strongly UN* if in its UN-decomposition, the unit and nilpotent commute.

In Section 2, we discuss certain properties of UN rings. Lifting of various types of elements modulo an ideal I have been studied by Khurana, Lam and Nielsen in [4]. We discuss lifting properties of UN rings modulo an ideal I. It is pertinent to mention here that the lifting properties like conjugate idempotent lifting and unit regular elements lifting are considered modulo a two sided

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ideal in contrast to the exchange rings, where idempotents lift modulo each left (right) ideal. Then we discuss the question raised by Călugăreanu [1] that "is $M_n(R)$ over a UN ring R, also UN?". We obtain that if R is commutative, then $M_n(R)$ is UN if and only if R is UN.

In Section 3, we focus on UN group rings. The group rings involving units and idempotents to represent every element as sum (clean ring)/product (unit regular ring) of these elements have been studied by many authors. So, our focus is on the group rings involving units and nilpotent elements, i.e., fine rings and UN rings. A group ring RG can never be a fine ring, because a fine ring is a simple ring and RG always has $\omega(G)$ as its proper ideal. So, we investigate the structure of UN group rings. We first take up the case of group algebra KG of a group G over a field K. We obtain the result that if $char K = 0$, then KG can be a UN ring if and only if G is trivial. If $char K = p$, then KG is a UN ring implies that the group G must be a p-group and the converse holds if G is locally finite. Next we investigate that what could be the characteristic of a UN ring R. We arrive at the conclusion that the $char R$ of a UN ring can be either 0 or p^α and particularly in case of group ring RG, the characteristic of R can not be 0. Then we investigate the structure of the group ring RG of a group G over an arbitrary ring R (which may not necessarily be a field) and obtain the result that if RG is a UN ring then R is a UN ring, G is a p-group and $p \in J(R)$; and the converse holds if G is locally finite.

We briefly summarize the basic terminologies and notations which we will use in this paper. Throughout this paper we consider R to be an associative ring with identity $1 \neq 0$. The full matrix ring over a ring R is denoted by $M_n(R)$. Let $U(R)$, $P(R)$, $J(R)$, $Z(R)$ and $N(R)$ denote the unit group, the prime radical, the Jacobson radical, the center and the set of nilpotent elements of a ring R, respectively. We denote by RG the group ring of a group G over a ring R. The *augmentation ideal* of RG, denoted by $\omega(G)$, is the kernel of the augmentation map $\omega : RG \to R$ given by $\omega(\sum_{g\in G} a_g g) = \sum a_g$. It can be seen that $\omega(G)$ is generated as an ideal of RG by the set $\{1-g : g \in G, g \neq 1\}$. If H is a subgroup of G, then we denote by $\omega(H)$ the left ideal of RG generated by the set $\{1-h : h \in H, h \neq 1\}$. If $H \lhd G$, then $\omega(H)$ is a two sided ideal of RG and $RG/\omega(H) \cong R(G/H)$. For other group ring related results and notations we refer to Connell [3] and Passman [7]. For ring theoretic results we refer to Lam [5].

2. UN rings

We list below some of the properties of UN rings in the form of the lemma, which we will require in the following sections.

Lemma 2.1. *Let R be a ring. Then the following statements hold*:

(a) *A UN ring is left-right symmetric* ([1]*, Proposition* 1(3))*.*
(b) *Homomorphic image of a UN ring is UN* ([1]*, Proposition* 3(1))*.*

(c) *Let I be a nil ideal of R. Then R is UN if and only if R/I is UN* ([9], *Proposition* 0(a)).
(d) *A UN ring has no nontrivial central idempotents* ([9], *Proposition* 0(c)).
(e) *Every left or right regular element of a UN ring R is invertible, i.e., R is its own classical ring of quotients* ([9], *Proposition* 0(f)).

The classes of local rings and UN rings are separate, some examples to this effect can be found in [1]. We prove a theorem below and using it we get a result somewhat in line with local rings.

Theorem 2.2. *Let $I \lhd R$. Then the following are equivalent*:
(1) *R/I is UN.*
(2) *R/I^n is UN for all $n \in \mathbb{N}$.*
(3) *R/I^n is UN for some $n \in \mathbb{N}$.*

Proof. (1)⇒(2) Let $I \lhd R$ and R/I be UN. It can be seen that
$$R/I \cong (R/I^n)/(I/I^n).$$
Since (I/I^n) is nilpotent in (R/I^n), the result follows by Lemma 2.1(c).

(2)⇒(3) is evident.

(3)⇒(1) Let R/I^n be UN for some $n \in \mathbb{N}$. As homomorphic image of a UN ring is UN and
$$R/I \cong (R/I^n)/(I/I^n).$$
So we get that R/I is UN. □

By using the above theorem we get a result for UN rings similar to the local rings ([5], Ex. 19.5).

Corollary 2.3. *Let $I \lhd R$ such that I is maximal as a left ideal. Then R/I^n is a UN ring for all $n \in \mathbb{N}$.*

2.1. Lifting properties

In this subsection we discuss about lifting properties of UN rings. We start with the definition of isomorphic and conjugate idempotents in R.

Definition. Two idempotents $e \in R$ and $f \in R$ are called
(1) *conjugate* (written as $e \sim f$), if $f = u^{-1}eu$ for some $u \in U(R)$.
(2) *isomorphic* (written as $e \cong f$), if $eR \cong fR$ as right R-modules.

The well known results about isomorphic and conjugate idempotents are mentioned below in the form of lemmas.

Lemma 2.4. *Let e and f be idempotents in a ring R. Then following are equivalent*:
(1) $e \sim f$.
(2) $e \cong f$ *and* $(1-e) \cong (1-f)$.

Lemma 2.5. *Let e and f be idempotents in a ring R. Then following are equivalent*:

(1) $e \cong f$.
(2) $eR \cong fR$ *as right R-modules.*
(3) $Re \cong Rf$ *as left R-modules.*
(4) $e = ab$ *and* $f = ba$ *for some* $a, b \in R$.

We observe that if e is a non trivial idempotent in a UN ring R with $e = ut$ for some $u \in U(R)$ and $t \in N(R)$, then t is unit regular with $t = tut$. And also $f = tu$ is an idempotent isomorphic as well as conjugate to e.

Let $I \lhd R$, we say that idempotents lift modulo I if for any idempotent $\bar{e} \in R/I$ there exists an idempotent $x \in R$ such that $\bar{x} = \bar{e}$. And conjugate idempotents $\bar{e}$, $\bar{f} \in R/I$ are said to lift modulo I if there exit conjugate idempotents x and y in R such that $\bar{x} = \bar{e}$ and $\bar{y} = \bar{f}$.

Theorem 2.6. *In a UN ring if idempotents lift modulo an ideal I, then conjugate idempotents lift modulo I.*

Proof. Let R be a UN ring and $I \lhd R$ such that idempotents lift modulo I. Let $\bar{e}$, $\bar{f} \in R/I$ be conjugate idempotents in R/I such that $\bar{f} = \overline{u^{-1}}\bar{e}\bar{u}$ for some unit $\bar{u} \in U(R/I)$. Since idempotents lift modulo I, there exit idempotents x and $y \in R$ such that $\bar{x} = \bar{e}$ and $\bar{y} = \bar{f}$. Let the preimage of $\bar{u}$ in R be v. As R is UN, so if $v \notin U(R)$, then $v = wt$ for some $w \in U(R)$ and $t \in N(R)$. So,

$$\bar{u} = \bar{v} = \bar{w}\bar{t} \Longrightarrow \bar{t} = \overline{w^{-1}}\bar{v} \in U(R/I)$$

which is not possible. Thus, $v \in U(R)$ and hence the preimage of $\overline{u^{-1}}\bar{e}\bar{u}$ is $v^{-1}xv = z$ (say), which is an idempotent conjugate to x and $\bar{z} = \bar{f}$. $\square$

Theorem 2.7. *Let R be a UN ring and $I \lhd R$. Then unit regular elements lift modulo I if and only if idempotents lift modulo I.*

Proof. Let R be UN and unit regular elements lift modulo I. Let $\bar{e}$ be an idempotent in R/I. As idempotents are unit regular elements, so $\bar{e}$ lifts to a unit regular element, say $x \in R$, such that $x = xvx$ for some $v \in U(R)$. By using the fact that $x = xvx$, it is a routine calculation to check that

$$(x - v(x^2 - x))^2 = x - v(x^2 - x).$$

So $y := x - v(x^2 - x)$ is an idempotent in R. Also as $\bar{x} = \bar{e}$, it can be easily seen that

$$\overline{x^2} = \overline{x.1.x} = \bar{x}\bar{1}\bar{x} = \bar{e}\bar{1}\bar{e} = \bar{e}.$$

Thus, we get $(x^2 - x) \in I$, which implies that $\bar{y} = \bar{x} = \bar{e}$. Hence $\bar{e}$ lifts to an idempotent in R.

Conversely, let idempotents lift modulo I. Let $a \in R/I$ be unit regular. It is well known that a unit regular element is a multiple of a unit and an idempotent. So $\bar{a} = \bar{u}\bar{e}$ for some $\bar{u} \in U(R/I)$ and $\bar{e}^2 = \bar{e}$. By following the proof of above theorem, we get that $\bar{u}$ lifts to some $v \in U(R)$ and by given hypothesis $\bar{e}$ lifts to some idempotent $z \in R$. Thus, $\bar{a}$ lifts to a unit regular element vz. □

2.2. UN matrix rings

In this subsection we discuss the question raised by Călugăreanu in [1] that "is $M_n(R)$ over a UN ring R, also UN?". As introduced in [8], a ring R is called a *US-ring* if every non unit element of it can be written as product of a unit and a strongly nilpotent element. An element $x \in R$ is called *strongly nilpotent*, if every sequence $x = x_0, x_1, x_2, \ldots$ such that $x_{i+1} \in x_i R x_i$ converges to zero. It is evident that every strongly nilpotent element is nilpotent but the converse may not hold good ([8, Example 1]). In the case of a commutative ring R, an element is strongly nilpotent if and only if it is nilpotent.

Theorem 2.8. *Let R be ring.*

(1) *If R is a US-ring, then $M_n(R)$ is UN.*
(2) *If $M_n(R)$ is UN, then $Z(R)$ is a US-ring.*

Proof. (1) It is well known that $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$. Since R is a US-ring, by [8, Theorem 1] we get that $R/P(R)$ is a division ring, and hence $J(R) = P(R)$. Then by [1, Corollary 7], $M_n(R/J(R)) \cong M_n(R)/J(M_n(R))$ is UN. By using the fact that $P(M_n(R)) = M_n(P(R))$, we obtain that

$$J(M_n(R)) = M_n(J(R)) = M_n(P(R)) = P(M_n(R))$$

is nil. Thus, we have obtained that $M_n(R)/J(M_n(R))$ is UN and $J(M_n(R))$ is nil. By Lemma 2.1(c), we get that $M_n(R)$ is UN.

(2) Let $M_n(R)$ be UN. The center of $M_n(R)$ is $Z(M_n(R)) = \{aI_n : a \in Z(R)\}$, i.e., the scalar matrices of the form aI_n for $a \in Z(R)$. We have $Z(M_n(R)) \cong Z(R)$ by the mapping $f : Z(R) \to Z(M_n(R))$ defined by $f(a) = aI_n, a \in Z(R)$. Now the result follows from [9, Proposition 0(b)] and the fact that in a commutative ring, the nilpotent and strongly nilpotent elements coincide. □

Corollary 2.9. *Let R be a commutative ring. Then $M_n(R)$ is UN if and only if R is UN.*

Corollary 2.10 ([10])**.** *Let R be a commutative ring. Then $M_n(R)$ is UN if and only if R is a local ring with $J(R)$ nil.*

A ring R is called 2-primal if R/I is a domain for every minimal prime ideal I of R. It is well known in literature that for 2-primal rings $P(R) = N(R)$.

Corollary 2.11. *Let R be a 2-primal UN ring. Then $M_n(R)$ is UN.*

Since a reduced UN ring is a division ring, we get the result obtained by Călugăreanu in [1] as a corollary of the above theorem.

Corollary 2.12. *A simple Artinian ring is UN.*

3. UN Group rings

3.1. Group algebras

First we take up the case of group algebra of a group G over a field K. If G is a finite group, then let us denote by $\hat{G}$, the following element of KG, $\hat{G} = \sum_{g\in G} g$.

Theorem 3.1. *Let K be a field and G be a group.*

(i) *If $char K = 0$, then KG is UN if and only if $G = \langle 1\rangle$.*

(ii) *If $char K = p$, then KG is UN implies that G is a p-group; the converse holds if G is locally finite.*

Proof. First of all we see that if KG is UN, then G is a torsion group irrespective of whether characteristic of field K is 0 or p. Let $g \in G$, then $1 - g \notin U(KG)$. So $1 - g = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. If $g \neq 1$, then $t \neq 0$ and we can choose a positive integer k such that $t^k = 0$ but $t^{(k-1)} \neq 0$. Thus, we have $(1 - g)t^{(k-1)} = ut^k = 0$. So $1 - g$ is a zero divisor in KG. Hence, the order of g is finite ([3], Proposition 6).

(i) Let $char K = 0$. If G is a finite group, then $|G|^{-1} \in K$. So there would exist a central idempotent $\frac{1}{|G|}\hat{G}$ in KG, which is a contradiction to Lemma 2.1(d). Hence, KG can not be UN for a nontrivial finite group G. Now, let us consider the case of infinite group. We observe that, again in light of Lemma 2.1(d), G can not be an Abelian group. So the only case left out is that G be a non Abelian group. In view of Lemma 2.1(e) and [7, Theorem 3.13, page 54] G must be a locally finite group. As G is locally finite, so KG is von Neumann regular ring and in particular $J(KG) = 0$. So, $\omega(G)$ is not a quasi regular ideal. Thus, there must exist an $\alpha \in \omega(G)$ such that $1 - \alpha \notin U(KG)$. Since KG is UN, we get $1 - \alpha = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. Applying augmentation map we get $\omega(1 - \alpha) = \omega(ut) \Longrightarrow 1 = \omega(u)\omega(t) \Longrightarrow \omega(t) = \omega(u)^{-1}$, which is absurd. Thus in all the above cases, for KG to be a UN ring, the group G must be trivial.

The converse part is straight forward, since every field is a UN ring.

(ii) Let $char K = p$. If G is a finite group, then $|G| \neq p'$, because if it is so, then there would exist a central idempotent $\frac{1}{|G|}\hat{G}$ and hence contradicting Lemma 2.1(d). So, let $|G| = p^k m$ with $(p, m) = 1$. By Cauchy's Theorem there exists an element $g \in G$ of order p' such that $p' \mid m$. As $(1 + g + g^2 + \cdots + g^{(p'-1)})(1 - g) = 0$, so we get $(1 + g + g^2 + \cdots + g^{(p'-1)}) = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. Applying augmentation map we get $\omega(1 + g + g^2 + \cdots + g^{(p'-1)}) = \omega(ut) \Longrightarrow p' = \omega(u)\omega(t) \Longrightarrow \omega(t) = p'\omega(u)^{-1}$, which is a contradiction, since $p' \in U(K)$. Thus, G must be a p-group. Now let us consider G to be an infinite group. If G is Abelian, then G should be p-group, because otherwise there would exist non trivial central idempotents in KG. If G is non Abelian, then following the method adopted for finite group, it can be shown that G is a p-group, as desired.

Conversely, let G be a locally finite p-group and K be a field of characteristic p. By [3, Proposition 16(ii)], $\omega(G)$ is a nil ideal. As is well known that $KG/\omega(G) \cong K$, so following Lemma 2.1(c) we get that KG is a UN ring. □

3.2. Group rings

Before taking up the group ring case, we discuss the characteristic of a UN ring and obtain the following results below.

Lemma 3.2. *Let R be a UN ring and $n \in \mathbb{Z}$, the set of integers. Then for an element $n \in R$, either $n \in U(R)$ or $n \in N(R)$.*

Proof. If $n \notin U(R)$, then $n = ut$ for some $u \in U(R)$ and $t \in N(R)$. This amounts to $u^{-1}nu = tu \Longrightarrow u^{-1}(\underbrace{1+1+\cdots+1}_{\text{n-times}})u = tu \Longrightarrow n = tu$. Thus, n is strongly UN and in particular $n \in N(R)$. □

Lemma 3.3. *Let R be a UN ring and $charR = n$. Then either $n = 0$ or $n = p^\alpha$ for some prime p and in this case $p \in J(R)$.*

Proof. If $n \neq 0$, then we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are primes. Since $n = 0$ in R, at least one of the p_i's is nilpotent in R. It can be easily seen that this is possible only when $k = 1$. Thus, $charR = p^\alpha$. Now, if $charR = p^\alpha$, then $p \in N(R)$. Since, $xp = px$ for all $x \in R$, we get $xp \in N(R)$ for all $x \in R$. So, $1 - xp \in U(R)$ for all $x \in R$. Thus, $p \in J(R)$ ([5], Lemma 4.1). □

Now let us consider the group ring of a group G over an arbitrary ring R (which may not necessarily be a field).

Theorem 3.4. *Let R be a ring and G be a non trivial group. If RG is UN, then R is UN of characteristic p^α, G is a p-group and $p \in J(R)$; the converse holds if G is locally finite.*

Proof. Let RG be UN, then by augmentation map $\omega : RG \to R$, we obtain that R is a homomorphic image of RG. So, by Lemma 2.1(b), R is UN. Going by the proof of Theorem 3.1, it can be seen that G is a torsion group. Now let if possible $charR = 0$, then by Lemma 3.2 all $n(\neq 0) \in \mathbb{Z}$ are invertible in R. Thus $|g|^{-1} \in R$ for all $g \in G$. Following the proof of Theorem 3.1, G can neither be a finite group nor an infinite Abelian group. Now, let G be an infinite non Abelian group and let the order of an element $g(\neq 1) \in G$ be m, for some positive integer m. So, we have that $(1+g+g^2+\cdots+g^{(m-1)})(1-g) = 0$ $\Longrightarrow 1+g+g^2+\cdots+g^{(m-1)} = ut$ for some $u \in U(RG)$ and $t \in N(RG) \Longrightarrow$ $\omega(1+g+g^2+\cdots+g^{(m-1)}) = \omega(ut) \Longrightarrow m = \omega(u)\omega(t) \Longrightarrow \omega(t) = m\omega(u)^{-1}$, which is not possible, since $m \in U(R)$. Thus, $charR \neq 0$. By Lemma 3.3, if $charR \neq 0$, then $charR = p^\alpha$. In this case also following the proof of Theorem 3.1, we can arrive at the result that G can not have p' elements and hence, G is a p-group. By Lemma 3.3, we get $p \in J(R)$. Conversely, let R be a UN ring of characteristic p^α, G a locally finite p-group and $p \in J(R)$. By [3, Proposition

16(ii)], $\omega(G)$ is a nil ideal. By augmentation map $\omega : RG \to R$, we observe that $RG/\omega(G) \cong R$. Because R is UN, we get RG is UN (by Lemma 2.1(c)). $\square$

Since an Abelian torsion group is locally finite, for the commutative group rings we get:

Corollary 3.5. *Let R be a commutative ring and G be a non trivial Abelian group. Then, RG is UN if and only if R is UN of characteristic p^α, G is a p-group and $p \in J(R)$.*

The above results resemble to the result obtained for local group rings by W. K. Nicholson in [6]. But we give below group ring specific examples which show that a UN group ring may not be local and a local group ring may not be UN.

Example 3.6. Let us consider the group ring RG, where $R = M_2(Z_2)$ (Z_2 be the ring of integers modulo 2) and $G = C_2$ be a cyclic group of order 2. By the mapping $\phi : M_2(Z_2)C_2 \to M_2(Z_2C_2)$ defined by

$$\phi\big(\Sigma_{i=1}^{k}(A_i g_i)\big) = (c_{ij}),$$

where $c_{ij} = \Sigma_{t=1}^{k} a_{ij}^{(m)} g_m$ and $a_{ij}^{(l)}$ is the i-th row and j-th column entry of A_t; it can be seen that $M_2(Z_2)C_2 \cong M_2(Z_2C_2)$.

Now $e = \left(\begin{smallmatrix} 1 & 1+g \\ 0 & 0 \end{smallmatrix}\right)$ is a non zero idempotent in $M_2(Z_2C_2)$. And hence, $M_2(Z_2C_2) \cong M_2(Z_2)C_2 = RG$ is not local. By [1, Corollary 7], $M_2(Z_2)$ is UN and thus by Theorem 3.4 it follows that $RG = M_2(Z_2)C_2$ is a UN ring.

Example 3.7. Let $R = Z_{(p)}$, i.e., the localization of the ring of integers at a prime ideal (p) and $G = C_p$ be a cyclic group of order p, where p is a prime. We consider the group ring RG. By [6, Theorem], RG is a local ring.

It is well known that R is a domain and hence a reduced ring; but R is not a field. Since a reduced UN ring is a division ring, we get that R is not UN. Thus, by Theorem 3.4, RG is not UN.

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References

[1] G. Călugăreanu, *UN-rings*, J. Algebra Appl. **15** (2016), no. 10, 1650182, 9 pp. `https://doi.org/10.1142/S0219498816501826`

[2] G. Călugăreanu and T. Y. Lam, *Fine rings: a new class of simple rings*, J. Algebra Appl. **15** (2016), no. 9, 1650173, 18 pp. `https://doi.org/10.1142/S0219498816501735`

[3] I. G. Connell, *On the group ring*, Canadian J. Math. **15** (1963), 650–685. `https://doi.org/10.4153/CJM-1963-067-0`

[4] D. Khurana, T. Y. Lam, and P. P. Nielsen, *An ensemble of idempotent lifting hypotheses*, J. Pure Appl. Algebra **222** (2018), no. 6, 1489–1511. `https://doi.org/10.1016/j.jpaa.2017.07.008`

[5] T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 1991. `https://doi.org/10.1007/978-1-4684-0406-7`

[6] W. K. Nicholson, *Local group rings*, Canad. Math. Bull. **15** (1972), 137–138. https://doi.org/10.4153/CMB-1972-025-1
[7] D. S. Passman, *The Algebraic Structure of Group Rings*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1977.
[8] D. Udar, *Rings whose nonunits are multiple of unit and strongly nilpotent element*, Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy, Springer, 25–30, 2022. https://doi.org/10.1007/978-981-16-5952-2_3
[9] P. Vámos, *On rings whose nonunits are a unit multiple of a nilpotent*, J. Algebra Appl. **16** (2017), no. 7, 1750140, 13 pp. https://doi.org/10.1142/S0219498817501407
[10] Y. Zhou, *A note on UN-rings*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **64(112)** (2021), no. 1, 97–100.

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