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ON CERTAIN ESTIMATES FOR ROUGH GENERALIZED PARAMETRIC MARCINKIEWICZ INTEGRALS

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Abstract. This paper is devoted to establishing certain L^p bounds for the generalized parametric Marcinkiewicz integral operators associated to surfaces generated by polynomial compound mappings with rough kernels given by $h \in \Delta_\gamma(\mathbb{R}_+)$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some γ, $\beta \in (1, \infty]$. As applications, the corresponding results for the generalized parametric Marcinkiewicz integral operators related to the Littlewood-Paley g_λ^* functions and area integrals are also presented.

1. Introduction

The main motivation of this paper is to establish some new results concerning rough generalized parametric Marcinkiewicz integrals. To be more precise, we shall establish certain L^p bounds for rough generalized parametric Marcinkiewicz integrals along polynomial compound curves under some pretty much weaker size conditions assumed on the integral kernels both on the unit sphere and in the radial directions.

Throughout this paper, let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For $y \in \mathbb{R}^n \setminus \{0\}$, we set $y' = y/|y|$. Let $\Gamma_{P,\varphi} = \{P(\varphi(|y|))y' : y \in \mathbb{R}^n\}$ be the polynomial compound curves generated by a continuous function $\varphi : [0, \infty) \to \mathbb{R}$ and a real polynomial P on $\mathbb{R}$ satisfying $P(0) = 0$. Assume that $\Omega \in L^1(\mathrm{S}^{n-1})$ is a function of homogeneous degree zero and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

Let $1 < q < \infty$ and h be a measurable function on $\mathbb{R}_+ := [0, \infty)$, the generalized parametric Marcinkiewicz integral operators along polynomial compound

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curves $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}$ are defined by

$$(2)\ \mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}f(x)=\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{|y|\le t}f(x-P(\varphi(|y|))y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^q\frac{dt}{t}\Big)^{1/q},$$

where $\rho=\varsigma+i\tau$ ($\varsigma,\tau\in\mathbb{R}$ with $\varsigma>0$) and $f\in\mathscr{S}(\mathbb{R}^n)$ (the space of Schwartz functions on $\mathbb{R}^n$).

For the sake of simplicity, we denote $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}=\mathfrak{M}^q_{\Omega,P,\varphi,\rho}$ when $h\equiv 1$. When $\varphi(t)=t$ and $P(t)=t$, we denote $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}=\mathfrak{M}^q_{h,\Omega,\rho}$ and $\mathfrak{M}^q_{\Omega,P,\varphi,\rho}=\mathfrak{M}^q_{\Omega,\rho}$. We also denote $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}=\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$, $\mathfrak{M}^q_{h,\Omega,\rho}=\mathfrak{M}_{h,\Omega,\rho}$ and $\mathfrak{M}^q_{\Omega,\rho}=\mathfrak{M}_{\Omega,\rho}$ when $q=2$.

When $\rho=1$, the operator $\mathfrak{M}_{\Omega,\rho}$ reduces to the well-known Marcinkiewicz integral operator $\mathfrak{M}_\Omega$, which was originally introduced by Stein [23] who proved that $\mathfrak{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p\le 2$ if $\Omega\in\mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for $0<\alpha\le 1$. Subsequently, Benedek et al. [5] improved the condition on rough kernel Ω to $\Omega\in\mathcal{C}^1(S^{n-1})$ and extended the above range on index p to $1<p<\infty$. Since then, a considerable amount of attentions has been given to study Marcinkiewicz integrals, successfully extending the above results to more rough kernels. For example, see [8, 9] for the case $\Omega\in H^1(\mathrm{S}^{n-1})$ (the Hardy space on S^{n-1}), [3, 4] for the case $\Omega\in L(\log L)^{1/2}(\mathrm{S}^{n-1})$, [3, 10] for the case $\Omega\in B_r^{(0,-1/2)}(\mathrm{S}^{n-1})$ (the block space generated by r-blocks), [6, 24] for the case $\Omega\in\mathcal{F}_\beta(\mathrm{S}^{n-1})$ (the Grafakos–Stefanov class). When $\rho\not\equiv 1$, the operator $\mathfrak{M}_{\Omega,\rho}$ is just the classical parametric Marcinkiewicz integral operator $\mathscr{M}_{\Omega,\rho}$. Hörmander [14] (resp., Sakamoto and Yabuta [22]) first studied the L^p bounds for $\mathfrak{M}_{\Omega,\rho}$ with real (resp., complex) number ρ. Later on, the above results were improved and generalized by many authors (see [17, 21, 24] for example).

On the other hand, the investigation on the generalized Marcinkiewicz integral operator has also attracted the attention of many authors. When $\rho=1$, we denote $\mathfrak{M}^q_{\Omega,\rho}=\mathfrak{M}^q_\Omega$. The operator $\mathfrak{M}^q_\Omega$ was first investigated by Chen, Fan and Ying [7] who obtained that $\mathfrak{M}^q_\Omega$ is bounded from the homogeneous Triebel–Lizorkin space $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1<p,\,q<\infty$ under the conditions that $\Omega\in L^s(\mathrm{S}^{n-1})$ for some $s\in(1,\infty]$. The above result was later improved by Fan and Wu [12] to the case $\Omega\in L(\log L)^{1/q}(\mathrm{S}^{n-1})$ for $q\ge 2$ and $\Omega\in L(\log L)^{1/q+\epsilon}(\mathrm{S}^{n-1})$ for $1<q<2$ and any $\epsilon>0$. Meanwhile, Al-Qassem et al. [1] established the bounds of $\mathfrak{M}^q_\Omega:\dot{F}^0_{p,q}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ for $p\in(2\beta/(2\beta-1),2\beta)$ and $q\in(2\beta/(2\beta-1),2\beta)$ provided that $\Omega\in\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>1$. Recently, Liu [18] improved and generalized the result of [1]. We now introduce the main result of [18] as follows:

Theorem A ([18])**.** *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$ and $\varphi\in\mathfrak{F}$. Here $\mathfrak{F}$ is the set of all functions ϕ satisfying the following conditions*:

(a) *ϕ is a positive increasing $\mathcal{C}^1((0,\infty))$ function such that $t^\delta\phi'(t)$ is monotonic on $\mathbb{R}_+$ for some $\delta\in\mathbb{R}$;*

(b) *there exist* C_ϕ, $c_\phi > 0$ *such that* $t\phi'(t) \geq C_\phi \phi(t)$ *and* $\phi(2t) \leq c_\phi \phi(t)$ *for all* $t > 0$.

Assume that $h \equiv 1$ *and* $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta > 1/2$ *and satisfying* (1). *Then*

$$\|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $p \in (\frac{2\beta+1}{2\beta}, 1+2\beta)$ *and* $q \in (\frac{2\beta+1}{2\beta}, 1+2\beta)$. *Here the constant* $C_p > 0$ *is independent of the coefficients of* P, *but may depend on* $p, q, n, \varphi, \rho, N$.

For the generalized Marcinkiewicz integral operator with radial kernel h, Le [15] observed that $\mathfrak{M}^q_{h,\Omega}$ is bounded from $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p, q < \infty$, provided that $h \in \Delta_{\max\{2,q'\}}(\mathbb{R}_+)$ and $\Omega \in L(\log L)(\mathrm{S}^{n-1})$. Here $\Delta_\gamma(\mathbb{R}_+)$ $(1 \leq \gamma \leq \infty)$ is the collection of all measurable functions $h : [0, \infty) \to \mathbb{R}$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} = \sup_{R>0} \Big(\frac{1}{R}\int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is easy to see that $\Delta_\gamma(\mathbb{R}_+)$ enjoys the properties that $L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+)$, $\Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+)$ for $\gamma_2 > \gamma_1 > 0$. Recently, Al-Qassem et al. [2] improved the main results of [12, 15] to more weaker size conditions on h. Very recently, Liu et al. [19] extended the main results of [2] to the generalized Marcinkiewicz integral operator along polynomial compound curves. Partial results of [19] can be formulated as follows:

Theorem B ([19])**.** *Let* P *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P(0) = 0$ *and* $\varphi \in \mathfrak{F}$. *Let* $1 < q < \infty$, $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma \in (2, \infty]$ *and* $\Omega \in L(\log L)^{1/q}(\mathrm{S}^{n-1}) \cup \big(\bigcup_{r>1} B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})\big)$ *satisfying* (1). *Then*

$$\|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $1 < p < q$ *if* $2 < \gamma < \infty$ *and* $q' \geq \gamma$, *and for* $\gamma' < p < \infty$ *if* $2 < \gamma \leq \infty$ *and* $q' < \gamma$. *Here the above constants* $C_p > 0$ *are independent of* h *and the coefficients of* P.

Remark 1.1. For the class $\mathfrak{F}$, there are some model examples such as t^α $(\alpha > 0)$, $t^\beta \ln(1+t)$ $(\beta \geq 1)$, $t \ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. Note that there exists $B_\varphi > 1$ such that $\varphi(2t) \geq B_\varphi \varphi(t)$ for any $\varphi \in \mathfrak{F}$ (see [17]).

In this paper we focus on the generalized parametric Marcinkiewicz integrals $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}$ with rough kernels $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 0$, where the function class $W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$ is the set of all $L^1(\mathrm{S}^{n-1})$ functions Ω which satisfy

$$\sup_{\xi \in \mathrm{S}^{n-1}} \iint_{\mathrm{S}^{n-1}\times\mathrm{S}^{n-1}} |\Omega(\theta)\Omega(u')| \log^\beta \frac{2e}{|(\theta - u')\cdot\xi|} d\sigma(\theta) d\sigma(u') < \infty.$$

We would like to point out that the class $W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ was originally introduced by Fan and Sato [11] in more general form. It is closely related to the Grafakos–Stefanov function class $\mathcal{F}_\beta(\mathrm{S}^{n-1})$, which was introduced in [13] and is given by

$$\mathcal{F}_\beta(\mathrm{S}^{n-1}) := \Big\{\Omega \in L^1(\mathrm{S}^{n-1}) : \sup_{\xi\in\mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} |\Omega(y')| \log^\beta \frac{2}{|\xi\cdot y'|} d\sigma(y') < \infty\Big\} \quad \text{for } \beta > 0.$$

In 2009, Fan and Sato [11] first studied the L^p boundedness for the singular integrals with rough kernels $h \in \Delta_\gamma(\mathbb{R}_+)$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$. Later on, a considerable amount of attention has been given to investigate the boundedness for various kinds of integral operators under the same conditions on rough kernels. For examples, see [16, 20] for singular integral operators, [21, 25] for Marcinkiewicz integral operators. Particularly, it was shown in [21] that:

Theorem C ([21]). *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0) = 0$ and $\varphi \in \mathfrak{F}$. Let $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,\infty]$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > \frac{1}{2}\max\{2,\gamma'\}$ satisfying* (1)*. Then*

$$\|\mathfrak{M}_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}$$

for $|\frac{1}{p}-\frac{1}{2}| < \min\{\frac{1}{\gamma'},\frac{1}{2}\} - \frac{1}{\beta+1}\min\{\frac{1}{\gamma'}+\frac{1}{2},1\}$. Here the constant $C_p > 0$ is independent of the coefficients of P.

It was shown in [11, 16] that

$$\mathcal{F}_\beta(\mathrm{S}^1) \subset W\mathcal{F}_\beta(\mathrm{S}^1) \ \text{ and } \ W\mathcal{F}_{2\beta}(\mathrm{S}^{n-1}) \setminus \mathcal{F}_\beta(\mathrm{S}^{n-1}) \ne \emptyset \ \text{ for } \beta > 0;$$

$$\bigcup_{r>1} L^r(\mathrm{S}^{n-1}) \subset \mathcal{F}_{\beta_2}(\mathrm{S}^{n-1}) \subset \mathcal{F}_{\beta_1}(\mathrm{S}^{n-1}) \ \text{ for } 0 < \beta_1 < \beta_2 < \infty;$$

$$\bigcup_{r>1} L^r(\mathrm{S}^{n-1}) \subset W\mathcal{F}_{\beta_2}(\mathrm{S}^{n-1}) \subset W\mathcal{F}_{\beta_1}(\mathrm{S}^{n-1}) \ \text{ for } 0 < \beta_1 < \beta_2 < \infty.$$

Moreover, the following inclusion relations are valid:

$$L^r(\mathrm{S}^{n-1}) \subsetneq L(\log L)^{\beta_1}(\mathrm{S}^{n-1}) \subsetneq L(\log L)^{\beta_2}(\mathrm{S}^{n-1}) \text{ for } r > 1 \text{ and } 0 < \beta_2 < \beta_1;$$

$$L(\log L)^\beta(\mathrm{S}^{n-1}) \subsetneq H^1(\mathrm{S}^{n-1}) \text{ for } \beta \ge 1;$$

$$L(\log L)^\beta(\mathrm{S}^{n-1}) \nsubseteq H^1(\mathrm{S}^{n-1}) \nsubseteq L(\log L)^\beta(\mathrm{S}^{n-1}) \text{ for } 0 < \beta < 1;$$

$$\bigcup_{q>1} L^q(\mathrm{S}^{n-1}) \subsetneq \bigcap_{\beta>1} \mathcal{F}_\beta(\mathrm{S}^{n-1}) \nsubseteq L\log L(\mathrm{S}^{n-1});$$

$$\bigcap_{\beta>1} \mathcal{F}_\beta(\mathrm{S}^{n-1}) \nsubseteq H^1(\mathrm{S}^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{F}_\beta(\mathrm{S}^{n-1});$$

$$\bigcup_{r>1} L^r(\mathrm{S}^{n-1}) \subsetneq B_q^{(0,v)}(\mathrm{S}^{n-1}) \quad \text{for } q > 1 \text{ and } v > -1;$$

$$B_q^{(0,v_2)}(\mathrm{S}^{n-1}) \subsetneq B_q^{(0,v_1)}(\mathrm{S}^{n-1}) \quad \text{for } q > 1 \text{ and } v_2 > v_1 > -1;$$

$$\bigcup_{q>1} B_q^{(0,v)}(\mathrm{S}^{n-1}) \nsubseteq \bigcup_{r>1} L^r(\mathrm{S}^{n-1}) \quad \text{for } v > -1;$$

$$B_q^{(0,v)}(\mathrm{S}^{n-1}) \subset H^1(\mathrm{S}^{n-1}) + L(\log L)^{1+v}(\mathrm{S}^{n-1}) \qquad \text{for } q>1,\ v>-1.$$

Based on the above, a question that arises naturally is the following.

Question. Is the operator $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}$ with $q \neq 2$ bounded from $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ under the same conditions h, Ω, P, φ in Theorem C?

The main motivation of this paper is to answer the above question. Our main results can be listed as follows:

Theorem 1.2. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$ and $\varphi \in \mathfrak{F}$. Assume that $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,\infty]$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > \tilde{\gamma}$ satisfying* (1). *Here $\tilde{\gamma} = \max\{2,\gamma'\}$. Then for any $\frac{2\beta}{2\beta-\tilde{\gamma}} < q < \frac{2\beta}{\tilde{\gamma}}$ and $\frac{1}{q\gamma} + \frac{\tilde{\gamma}}{2\beta\gamma'} < \frac{1}{p} < \frac{1}{q\gamma} + \frac{1}{\gamma'} - \frac{\tilde{\gamma}}{2\beta\gamma'}$, it holds that*

$$\|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

Here the constant $C_p > 0$ is independent of h and the coefficients of P, but may depend on p, d, N, φ.

Theorem 1.3. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$ and $\varphi \in \mathfrak{F}$. Assume that $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in [2,\infty]$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 2$ satisfying* (1). *Then for $(p, q) \in (\frac{\beta\gamma'}{\beta+\gamma'-2}, \beta)^2$ or $\frac{\beta}{\beta-1} < p < q \le 2$, it holds that*

$$\|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

Here the constant $C_p > 0$ is independent of h and the coefficients of P, but may depend on p, d, N, φ.

Theorem 1.4. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$ and $\varphi \in \mathfrak{F}$. Assume that $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in [2,\infty]$ and $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$ satisfying* (1). *Then*

$$\|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

provided that one of the following conditions holds:

(i) $q \in (\frac{\gamma'(\beta-1)+2}{\beta}, \gamma'(\beta-1)+2)$, $p \in (\frac{\gamma'(\beta-1)+2}{\beta}, 2]$ *and* $p < q$;

(ii) $q \in (\frac{\beta+1}{\beta}, \beta+1)$, $p \in (2, \beta+1)$ *and* $p > q$;

(iii) $q \in (\frac{\gamma'(\beta-1)+2}{\beta}, \beta+1)$ *and* $p = q$.

Here the constant $C_p > 0$ is independent of h and the coefficients of P, but may depend on p, d, N, φ.

Remark 1.5. There are some remarks as follows:

(i) By the fact that $\mathcal{F}_\beta(\mathrm{S}^1) \subset W\mathcal{F}_\beta(\mathrm{S}^1)$ for $\beta > 0$, we know that our main results also hold if the condition $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ replacing by $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ when $n = 2$.

(ii) Our main results are new even in the special case $P(t) = \varphi(t) = t$ and $\rho = 1$.

The rest of this paper will be organized as follows. Section 2 contains some preliminary notations and lemmas, which are the main ingredients of our proofs of main results. The proofs of Theorems 1.2–1.4 will be given in Section 3. Finally, we establish the L^p bounds for generalized parametric Marcinkiewicz integral operators related to Littlewood-Paley g_λ^*-functions and area integrals in Section 4. We remark that the proofs of Theorems 1.2–1.4 are motivated by [18, 19, 21].

Throughout this paper, the letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. Especially, the letter $C_{\alpha,\beta}$ denote the positive constants that depend on the parameters α, β.

2. Preliminary notations and lemmas

Let us begin with the definition of the homogeneous Triebel–Lizorkin spaces.

Definition (homogeneous Triebel–Lizorkin spaces)**.** Let $\mathcal{S}'(\mathbb{R}^n)$ be the tempered distribution class on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty\, (p \neq \infty)$, the homogeneous Triebel–Lizorkin spaces $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^\alpha_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\},$$

where $\widehat{\Psi_i}(\xi) = \phi(2^i \xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfies the conditions: $0 \leq \phi(x) \leq 1$; $\mathrm{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$; $\phi(x) \geq c > 0$ if $3/5 \leq |x| \leq 5/3$; $\sum_{j \in \mathbb{Z}} \phi(2^j \xi) = 1$ for $\xi \neq 0$.

Remark 2.1. It is well-known that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and also the following hold:

(i) $\dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$;

(ii) $(\dot{F}^\alpha_{p,q}(\mathbb{R}^n))* = \dot{F}^{-\alpha}_{p',q'}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$;

(iii) $\dot{F}^\alpha_{p,q_1}(\mathbb{R}^n) \subset \dot{F}^\alpha_{p,q_2}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ and $q_1 \leq q_2$.

Let $\{a_k\}$ be a lacunary sequence with satisfying $\inf_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} \geq a > 1$. Let $\eta_0 \in \mathcal{C}^\infty(\mathbb{R})$ be an even function satisfying $0 \leq \eta_0(t) \leq 1$, $\eta_0(0) = 1$ and $\eta_0(t) = 0$ for $|t| \geq 1$. Set $\eta(\xi) = 1$ for $|\xi| \leq 1$, $\eta(\xi) = \eta_0(\frac{|\xi|-1}{a-1})$, where $a > 1$. Then, η satisfies $\chi_{|\xi| \leq 1}(\xi) \leq \eta(\xi) \leq \chi_{|\xi| \leq a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \leq c_\alpha (a-1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^d$, where c_α is independent of a. We define functions $\{\psi_k\}_k$ on $\mathbb{R}^n$ by

$$\psi_k(\xi) = \eta(a_{k+1}^{-1}\xi) - \eta(a_k^{-1}\xi), \quad \xi \in \mathbb{R}^n.$$

It is easy to see that the function ψ_k enjoys the following properties:

$$\mathrm{supp}(\psi_k) \subset \{a_k \leq |\xi| \leq a a_{k+1}\}; \quad \mathrm{supp}(\psi_k) \cap \mathrm{supp}(\psi_j) = \emptyset \ \text{ for } |j-k| \geq 2;$$

$$\sum_{k \in \mathbb{Z}} \psi_k(\xi) = 1 \ \text{ for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

The following is a well-known characterization of homogeneous Triebel–Lizorkin spaces, which plays a key role in our proofs.

Lemma 2.2 ([25]). *Let* $\{a_k\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers with* $1 < a \le \frac{a_{k+1}}{a_k} \le b$ *for all* $k \in \mathbb{Z}$. *Let* Φ_k *be defined on* $\mathbb{R}^n$ *by* $\widehat{\Phi_k}(\xi) = \psi_k(\xi)$ *and* $\mathscr{A}_n$ *denote the set of all polynomials on* $\mathbb{R}^n$. *Let* $1 < p,\ q < \infty$ *and* $\alpha \in \mathbb{R}$. *For* $f \in \mathcal{S}(\mathbb{R}^n)/\mathscr{A}_n$, *we define the norm* $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ *by*

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)} = \Big\|\Big(\sum_{k\in\mathbb{Z}} a_k^{\alpha q}|\Phi_k * f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

Then $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ *is equivalent to* $\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$.

Let h, Ω, ρ be given as in (2) and $\Gamma : \mathbb{R}^n \to \mathbb{R}^d$ ($d \ge 1$) be a mapping. We define the family of measures $\{\sigma_{h,\Omega,\Gamma,t}\}_{t>0}$ on $\mathbb{R}^d$ as follows:

$$\widehat{\sigma_{h,\Omega,\Gamma,t}}(\xi) = \frac{1}{t^{\rho}}\int_{t/2<|y|\le t} e^{-2\pi i\xi\cdot\Gamma(y)}\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy. \tag{3}$$

The related maximal operator $\sigma^*_{h,\Omega,\Gamma}$ is defined by

$$\sigma^*_{h,\Omega,\Gamma}(f)(x) = \sup_{t>0}\big||\sigma_{h,\Omega,\Gamma,t}| * f(x)\big|,$$

where $|\sigma_{h,\Omega,\Gamma,t}|$ is defined in the same way as $\sigma_{h,\Omega,\Gamma,t}$, but with Ω and h replaced by $|\Omega|$ and $|h|$, respectively.

In what follows, for the sake of simplicity, we denote $\sigma^{*,1}_{h,\Omega,\Gamma} = \sigma^*_{h,\Omega,\Gamma}$ when $\rho = 1$ and $\sigma^{*,1}_{\Omega,\Gamma} = \sigma^{*,1}_{h,\Omega,\Gamma}$ when $h \equiv 1$. In addition, for any arbitrary functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, we define the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{g} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by $\tilde{f}(x) = f(-x)$ and $\tilde{g}(x,t) = g(-x,t)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$.

Lemma 2.3. *Let* $\varphi \in \mathfrak{F}$ *and* $\Gamma(y) = \mathcal{P}(\varphi(|y|)y')$, *where* $\mathcal{P} = (P_1, P_2, \dots, P_d)$ *with each* P_j *being a real-valued polynomial on* $\mathbb{R}^n$. *Suppose that* $\Omega \in L^1(\mathrm{S}^{n-1})$ *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $1 < \gamma \le \infty$. *Then for* $\gamma' < p \le \infty$, *there exists a constant* $C > 0$ *such that*

$$\|\sigma^*_{h,\Omega,\Gamma}(f)\|_{L^p(\mathbb{R}^d)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}. \tag{4}$$

Here the above constant $C > 0$ *is independent of* h, Ω, γ *and the coefficients of* $\{P_j\}_{j=1}^d$, *but depend on* φ *and* p.

Proof. By a change of variable and Hölder's inequality one finds

$$\begin{aligned}
&\big||\sigma_{h,\Omega,\Gamma,t}| * f(x)\big| \\
&\le \int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}|\Omega(y')||f(x-\mathcal{P}(\varphi(r)y'))|d\sigma(y')|h(r)|\frac{dr}{r} \\
&\le 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{t/2}^{t}\Big|\int_{\mathrm{S}^{n-1}}|\Omega(y')||f(x-\mathcal{P}(\varphi(r)y'))|d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}
\end{aligned}$$

$$\le 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma} \times \Big(\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}|\Omega(y')||f(x-\mathcal{P}(\varphi(r)y'))|^{\gamma'}d\sigma(y')\frac{dr}{r}\Big)^{1/\gamma'}.$$

It follows that

$$\begin{aligned}(5)\qquad &\sigma^*_{h,\Omega,\Gamma}(f)(x)\\ &\le 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma}\\ &\quad\times\Big(\int_{\mathrm{S}^{n-1}}|\Omega(y')|\Big(\sup_{t>0}\int_{t/2}^{t}|f(x-\mathcal{P}(\varphi(r)y'))|^{\gamma'}\frac{dr}{r}\Big)d\sigma(y')\Big)^{1/\gamma'}.\end{aligned}$$

It was shown in the proof of [18, Lemma 3.1] that

$$(6)\qquad \Big\|\sup_{t>0}\int_{t/2}^{t}|f(\cdot-\mathcal{P}(\varphi(r)y'))|^{\gamma'}\frac{dr}{r}\Big\|_{L^p(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)}$$

for all $\gamma' < p \le \infty$. Here the constant $C > 0$ is independent of y' and the coefficients of $\{P_j\}_{j=1}^d$, but may depend on φ, p. By (5), (6) and Minkowski's inequality it holds that

$$\|\sigma^*_{h,\Omega,\Gamma}(f)\|_{L^p(\mathbb{R}^d)}\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for $\gamma' < p \le \infty$. This proves (4). □

As several applications of Lemma 2.3, we can obtain the following lemmas, which play key roles in the proofs of main results.

Lemma 2.4. *Let $1 < q < \infty$ and Γ be given as in Lemma 2.3. Suppose that $\Omega \in L^1(\mathrm{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,\infty]$. Then for $(q'\gamma)' < p < q\gamma$, it holds that*

$$\begin{aligned}(7)\qquad &\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Here the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of $\{P_j\}_{j=1}^d$, but may depend on φ and p.

Proof. For a fixed $1 < q < \infty$ and let us consider the following cases:

Case 1: $((q'\gamma)' < p < q)$. By duality, there exists a sequence of functions $\{f_k(x,t)\}$ defined on $\mathbb{R}^d \times \mathbb{R}_+$ with $\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^d,\ell^{q'}(L^{q'}([2^k,2^{k+1}],dt/t)))} \le 1$ such that

$$\begin{aligned}&\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|_{L^{p'/q'}(\mathbb{R}^d)}\\ &=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx.\end{aligned}$$

By Hölder's inequality one finds that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}\sigma_{h,\Omega,\Gamma,t}*g_k(x)f_k(x,t)\frac{dt}{t}dx\\
&=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}g_k(x)\int_{2^k}^{2^{k+1}}\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(x,t)\frac{dt}{t}dx\\
&\le\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|^{1/q'}_{L^{p'/q'}(\mathbb{R}^d)}.
\end{aligned}$$

Noting that $p'/q'>1$, by duality again there exists a nonnegative function $u\in L^{(p'/q')'}(\mathbb{R}^d)$ with $\|u\|_{L^{(p'/q')'}(\mathbb{R}^d)}=1$ such that

$$\begin{aligned}
&\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|_{L^{p'/q'}(\mathbb{R}^d)}\\
&=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx.
\end{aligned}$$

Hence, we can get

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\le\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx\Big)^{1/q'}.
\end{aligned}\tag{8}$$

By a change of variable and Hölder's inequality, one has

$$\begin{aligned}
\int_{t/2<|y|\le t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy&=\int_{t/2}^{t}|h(r)|\frac{dr}{r}\int_{\mathrm{S}^{n-1}}|\Omega(y')|d\sigma(y')\\
&\le 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}.
\end{aligned}\tag{9}$$

Combining (9) with Hölder's inequality implies that

$$\begin{aligned}
&|\sigma_{h,\Omega,\Gamma,t}*g_k(x)|^{q'}\\
&=\Big(\int_{t/2<|y|\le t}|g_k(x-\Gamma(y))|\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\Big)^{q'}\\
&\le\Big(\int_{t/2<|y|\le t}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\Big)^{q'/q}\\
&\quad\times\int_{t/2<|y|\le t}|g_k(x-\Gamma(y))|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy
\end{aligned}\tag{10}$$

$$\leq (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q'/q}\int_{t/2<|y|\leq t}|g_k(x-\Gamma(y))|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy.$$

Then we get from (10) that

$$\begin{aligned}(11)\quad &\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*\tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx\\ &\leq (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q'-1}\\ &\quad\times\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}\int_{t/2<|y|\leq t}|\tilde{f}_k(x-\Gamma(y),t)|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t}u(x)dx.\end{aligned}$$

Noting that $\gamma'<(p'/q')'<\infty$. By Hölder's inequality and invoking Lemma 2.3, one has

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}\int_{t/2<|y|\leq t}|\tilde{f}_k(x-\Gamma(y),t)|^{q'}\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\frac{dt}{t}u(x)dx\\ &\leq \int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|f_k(z,t)|^{q'}\int_{t/2<|y|\leq t}u(\Gamma(y)-z)\frac{|h(|y|)\Omega(y)|}{|y|^n}dydz\frac{dt}{t}\\ &\leq 2\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|f_k(z,t)|^{q'}\frac{dt}{t}\Big)\sigma^{*,1}_{h,\Omega,\Gamma}(\tilde{u})(z)dz\\ &\leq 2\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|f_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|_{L^{p'/q'}(\mathbb{R}^d)}\|\sigma^{*,1}_{h,\Omega,\Gamma}(\tilde{u})\|_{L^{(p'/q')'}(\mathbb{R}^d)}\\ &\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}.\end{aligned}$$

This together with (8) and (11) implies (7) for $(q'\gamma)'<p<q$.

Case 2: $(p=q)$. By the arguments similar to those used in deriving (11), one can get

$$\begin{aligned}(12)\quad &|\sigma_{h,\Omega,\Gamma,t}*g_k(x)|^q\\ &\leq (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1}\int_{t/2<|y|\leq t}|g_k(x-\Gamma(y))|^q\frac{|h(|y|)\Omega(y)|}{|y|^n}dy.\end{aligned}$$

By (9), (12), Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)}\\ &=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k(x)|^q\frac{dt}{t}dx\\ &\leq (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1}\end{aligned}$$

$$\times \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{t/2<|y|\le t} |g_k(x-\Gamma(y))|^q \frac{|h(|y|)\Omega(y)|}{|y|^n} dy \frac{dt}{t} dx$$
$$\le (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |g_k(x)|^q dx$$
$$\times \sup_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{t/2<|y|\le t} \frac{|h(|y|)\Omega(y)|}{|y|^n} dy \frac{dt}{t}$$
$$\le 2^q(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^q \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)},$$

which gives (7) for $p = q$.

Case 3: ($q < p < q\gamma$). By duality and the fact that $p/q \in (1,\infty)$, there exists a nonnegative function f in $L^{(p/q)'}(\mathbb{R}^d)$ with $\|f\|_{L^{(p/q)'}(\mathbb{R}^d)} \le 1$ such that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t} * g_k|^q \frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)} \tag{13}$$
$$= \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t} * g_k(x)|^q \frac{dt}{t} f(x) dx.$$

By a change of variable and (12), it holds that

$$\int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t} * g_k(x)|^q \frac{dt}{t} f(x) dx \tag{14}$$
$$\le (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1}$$
$$\times \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{t/2<|y|\le t} |g_k(x-\Gamma(y))|^q \frac{|h(|y|)\Omega(y)|}{|y|^n} dy \frac{dt}{t} f(x) dx$$
$$\le (2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1}$$
$$\times \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |g_k(x)|^q \int_{2^k}^{2^{k+1}} \int_{t/2<|y|\le t} f(x+\Gamma(y)) \frac{|h(|y|)\Omega(y)|}{|y|^n} dy \frac{dt}{t} dx$$
$$\le 2(2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1} \int_{\mathbb{R}^d} \Big(\sum_{k\in\mathbb{Z}} |g_k(x)|^q\Big) \sigma^{*,1}_{h,\Omega,\Gamma}(\tilde{f})(-x) dx.$$

Since $\gamma' < (p/q)' < \infty$, then by Lemma 2.3 and Hölder's inequality, we get from (13) and (14) that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t} * g_k|^q \frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)}$$
$$\le 2(2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^{q-1} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)}$$

$$\times \|\sigma^{*,1}_{h,\Omega,\Gamma}(\tilde{f})(-\cdot)\|_{L^{(p/q)'}(\mathbb{R}^d)}$$
$$\leq C(\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})})^q \Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^d)}.$$

This proves (7) for $q < p < q\gamma$ and completes the proof. □

Lemma 2.5. *Let* $1 < q < \infty$ *and* Γ *be given as in Lemma* 2.3. *Suppose that* $\Omega \in L^1(\mathrm{S}^{n-1})$ *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma \in [2, \infty)$. *Then*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t} * g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{15}$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$

holds, provided that one of the following conditions holds:

(i) $1 < p < q \leq \gamma'$;

(ii) $\gamma' < p < \infty$ *and* $\gamma' < q < \infty$.

Here the above constant $C > 0$ *is independent of* h, Ω, γ *and the coefficients of* $\{P_j\}_{j=1}^d$.

Proof. This lemma will be proved by considering the following cases:

Case 1: $(1 < p < q \leq \gamma')$. By the arguments same as those used in deriving (8), there exist a nonnegative function $u \in L^{(p'/q')'}(\mathbb{R}^d)$ with $\|u\|_{L^{(p'/q')'}(\mathbb{R}^d)} = 1$ and functions $\{f_k(x,t)\}$ defined on $\mathbb{R}^d \times \mathbb{R}_+$ with

$$\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^d,\ell^{q'}(L^{q'}([2^k,2^{k+1}],dt/t)))} \leq 1$$

such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t} * g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{16}$$
$$\leq \Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$
$$\times\Big(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t} * \tilde{f}_k(x,t)|^{q'}\frac{dt}{t}u(x)dx\Big)^{1/q'}.$$

By some changes of variables and Hölder's inequality it holds that

$$|\sigma_{h,\Omega,\Gamma,t} * \tilde{f}_k(x,t)| \tag{17}$$
$$\leq \int_{t/2<|y|\leq t}|\tilde{f}_k(x-\Gamma(y),t)|\frac{|h(|y|)\Omega(y)|}{|y|^n}dy$$
$$= \int_{t/2}^t\int_{\mathrm{S}^{n-1}}|\tilde{f}_k(x-\Gamma(ry'),t)||\Omega(y')|d\sigma(y')|h(r)|\frac{dr}{r}$$
$$\leq C\|h\|_{\Delta_q(\mathbb{R}_+)}\|\Omega\|^{1/q}_{L^1(\mathrm{S}^{n-1})}$$

$$\times \Big(\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}} |\tilde{f}_k(x-\Gamma(ry'),t)|^{q'}|\Omega(y')|d\sigma(y')\frac{dr}{r}\Big)^{1/q'}$$
$$= C\|h\|_{\Delta_q(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/q}\Big(\int_{t/2<|y|\le t} |\tilde{f}_k(x-\Gamma(y),t)|^{q'}\frac{|\Omega(y)|}{|y|^n}dy\Big)^{1/q'}.$$

Note that $1<(p'/q')'<\infty$. Invoking Lemma 2.3 and using (17) and Hölder's inequality, one finds that

$$\begin{aligned}
&\int_{t/2<|y|\le t} |\tilde{f}_k(x-\Gamma(y),t)|^{q'}\frac{|\Omega(y)|}{|y|^n}dy\\
&\le \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^k}^{2^{k+1}}\int_{t/2<|y|\le t} |\tilde{f}_k(x-\Gamma(y),t)|^{q'}\frac{|\Omega(y)|}{|y|^n}dy\frac{dt}{t}u(x)dx\\
&\le \int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |f_k(z,t)|^{q'}\int_{t/2<|y|\le t} u(\Gamma(y)-z)\frac{|\Omega(y)|}{|y|^n}dydz\frac{dt}{t}\\
&\le \int_{\mathbb{R}^d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |f_k(z,t)|^{q'}\frac{dt}{t}\Big)\sigma^{*,1}_{\Omega,\Gamma}(\tilde{u})(z)dz\\
&\le C\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |f_k(\cdot,t)|^{q'}\frac{dt}{t}\Big\|_{L^{p'/q'}(\mathbb{R}^d)}\|\sigma^{*,1}_{\Omega,\Gamma}(\tilde{u})\|_{L^{(p'/q')'}(\mathbb{R}^d)}\\
&\le C\|\Omega\|_{L^1(\mathrm{S}^{n-1})},
\end{aligned}$$

which combining with (16) and the fact that $\|h\|_{\Delta_q(\mathbb{R}_+)}\le\|h\|_{\Delta_\gamma(\mathbb{R}_+)}$ implies that

(18)
$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

This proves (15) for $1<p<q\le\gamma'$.

Case 2: ($\gamma'<p<\infty$, $\gamma'<q<\infty$). At first we shall prove the following inequality

(19)
$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |\sigma_{h,\Omega,\Gamma,t}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^d)}\\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

for $2\le\gamma\le\infty$.

When $\gamma=\infty$. By duality, there exists a nonnegative function $f\in L^{p'}(\mathbb{R}^d)$ with $\|f\|_{L^{p'}(\mathbb{R}^d)}=1$ such that

$$\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|\frac{dt}{t}\Big\|_{L^p(\mathbb{R}^d)}=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|\frac{dt}{t}f(x)dx.$$

Invoking Lemma 2.3, we can get

$$\begin{aligned}
&\Big\|\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|\frac{dt}{t}\Big\|_{L^p(\mathbb{R}^d)}\\
&\le\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|\frac{dt}{t}f(x)dx\\
&\le\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_k(x)|\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}|*\tilde{f}(-x)\frac{dt}{t}dx\\
&\le\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_k(x)|\sigma^*_{h,\Omega,\Gamma}(\tilde{f})(-x)dx\\
&\le\Big\|\sum_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)}\|\sigma^*_{h,\Omega,\Gamma}(\tilde{f})(-\cdot)\|_{L^{p'}(\mathbb{R}^d)}\\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sum_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)},
\end{aligned}$$

which yields (19) for $\gamma=\infty$.

When $2\le\gamma<\infty$. Since $p/\gamma'>1$, then by duality we can find a nonnegative function $f\in L^{(p/\gamma')'}(\mathbb{R}^d)$ with $\|f\|_{L^{(p/\gamma')'}(\mathbb{R}^d)}\le 1$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|^{\gamma'}_{L^p(\mathbb{R}^d)}\\
&=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^{\gamma'}\frac{dt}{t}f(x)dx.
\end{aligned}\tag{20}$$

By the arguments similar to those used to derive (17),

$$\begin{aligned}
&|\sigma_{h,\Omega,\Gamma,t}*g_k(x)|\\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|^{1/\gamma}_{L^1(\mathrm{S}^{n-1})}\Big(\int_{t/2<|y|\le t}|g_k(x-\Gamma(y))|^{\gamma'}\frac{|\Omega(y)|}{|y|^n}dy\Big)^{1/\gamma'},
\end{aligned}$$

which together with (20), the change of variables, Hölder's inequality and Lemma 2.3 leads to

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|^{\gamma'}_{L^p(\mathbb{R}^d)}$$

$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}^{\gamma'}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{\gamma'-1}\int_{\mathbb{R}^d}$$
$$\times\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}\int_{t/2<|y|\leq t}|g_k(x-\Gamma(y))|^{\gamma'}\frac{|\Omega(y)|}{|y|^n}dy\frac{dt}{t}f(x)dx$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}^{\gamma'}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{\gamma'-1}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_k(x)|^{\gamma'}\sigma_{\Omega,\Gamma}^{*,1}(\tilde{f})(-x)dx$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}^{\gamma'}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{\gamma'-1}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^{p/\gamma'}(\mathbb{R}^d)}^{\gamma'}$$
$$\times\|\sigma_{\Omega,\Gamma}^{*,1}(\tilde{f})(-\cdot)\|_{L^{(p/\gamma')'}(\mathbb{R}^d)}$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}^{\gamma'}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{\gamma'}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^{p/\gamma'}(\mathbb{R}^d)}^{\gamma'}.$$

This proves (19) for $2\leq\gamma<\infty$.

By a change of variable, we have that for any $1<u,v<\infty$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|\sigma_{h,\Omega,\Gamma,t}*g_k|^u\frac{dt}{t}\Big)^{1/u}\Big\|_{L^v(\mathbb{R}^d)} \tag{21}$$
$$=\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{h,\Omega,\Gamma,2^kt}*g_k|^u\frac{dt}{t}\Big)^{1/u}\Big\|_{L^v(\mathbb{R}^d)}.$$

In light of (19) and (21) we would have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{h,\Omega,\Gamma,2^kt}*g_k|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^d)} \tag{22}$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\mathbb{R}^d)}.$$

An application of Lemma 2.3 may yield that

$$\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}|\sigma_{h,\Omega,\Gamma,2^kt}*g_k|\Big\|_{L^p(\mathbb{R}^d)} \tag{23}$$
$$\leq\Big\|\sigma_{h,\Omega,\Gamma}^*\Big(\sup_{k\in\mathbb{Z}}|g_k|\Big)\Big\|_{L^p(\mathbb{R}^d)}$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sup_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)}.$$

By the interpolation between (22) and (23) one has

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{h,\Omega,\Gamma,2^kt}*g_k|^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

This together with (21) yields (15) for $\gamma' < p < \infty$ and $\gamma' < q < \infty$. This finishes the proof of Lemma 2.5. □

Lemma 2.6. *Let $1 < q < \infty$ and Γ be given as in Lemma 2.3. Suppose that $\Omega \in L^1(\mathrm{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in [2, \infty)$. Then for $\gamma' < p < \infty$ and $t \in [1, 2]$, it holds that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |\sigma_{h,\Omega,\Gamma,2^k t} * g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \tag{24}$$

The above constant $C > 0$ is independent of h, Ω, γ, t and the coefficients of $\{P_j\}_{j=1}^d$.

Proof. By duality, there exists a nonnegative function $u \in L^{p'}(\mathbb{R}^d)$ with

$$\|h\|_{L^{p'}(\mathbb{R}^d)} = 1$$

such that

$$\Big\|\sum_{k\in\mathbb{Z}} |\sigma_{h,\Omega,\Gamma,2^k t} * g_k|\Big\|_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |\sigma_{h,\Omega,\Gamma,2^k t} * g_k(x)|u(x)dx \leq \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |g_k(x)|\sigma^*_{h,\Omega,\Gamma}(\tilde{u})(-x)dx \tag{25}$$

for all $t \in [1, 2]$ and $p \in (1, \infty)$. By Lemma 2.3 and Hölder's inequality, we get from (25) that

$$\Big\|\sum_{k\in\mathbb{Z}} |\sigma_{\Omega,\Gamma,2^k t} * g_k|\Big\|_{L^p(\mathbb{R}^d)} \leq \Big\|\sum_{k\in\mathbb{Z}} |g_k|\Big\|_{L^p(\mathbb{R}^d)} \|\sigma^*_{h,\Omega,\Gamma}(\tilde{u})(-\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sum_{k\in\mathbb{Z}} |g_k|\Big\|_{L^p(\mathbb{R}^d)} \tag{26}$$

for all $\gamma' < p < \infty$. On the other hand, by the argument similar to those used in deriving (23),

$$\Big\|\sup_{k\in\mathbb{Z}} |\sigma_{\Omega,\Gamma,2^k t} * g_k|\Big\|_{L^p(\mathbb{R}^d)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sup_{k\in\mathbb{Z}} |g_k|\Big\|_{L^p(\mathbb{R}^d)} \tag{27}$$

for $\gamma' < p < \infty$. Then (24) follows from an interpolation between (26) and (27). □

We end this section by present the following lemma, which was proved in [18].

Lemma 2.7 ([18])**.** *For each $k \in \mathbb{Z}$, define the multiplier operator S_k in $\mathbb{R}^n$ by $S_k f(x) = \Phi_k * f(x)$. Here Φ_k is defined as in Lemma 2.2. Then*

(i) *For* $1 < p < q < \infty$ *and* $1 < r < p$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leq C\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|g_{t,j,k}|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}^r\Big)^{1/r}.$$

(ii) *For* $1 < q < p < \infty$ *and* $1 < r < p'$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leq C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{t,j,k}|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}^q dt\Big)^{r/q}\Big)^{1/r}.$$

3. Proofs of Theorems 1.2–1.4

This section is devoted to presenting Theorems 1.2–1.4. Let us begin with proving Theorem 1.2.

Proof of Theorem 1.2. Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0) = 0$. We may assume without loss of generality that $P(t) = \sum_{i=1}^N b_i t^i$ with each $b_i \neq 0$. Let $P_0(t) = 0$ and $P_\lambda(t) = \sum_{i=1}^\lambda b_i t^i$ for $\lambda \in \{1, 2, \ldots, N\}$. For $0 \leq \lambda \leq N$, we define the family of measures $\{\tau_{k,t}^\lambda\}$ by

$$\widehat{\tau_{k,t}^\lambda}(\xi) = \frac{1}{(2^k t)^\rho}\int_{2^{k-1}t<|y|\leq 2^k t} e^{-2\pi i\xi\cdot P_\lambda(\varphi(|y|))y'}\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy.$$

At first, one can easily check that for any $\lambda \in \{1, 2, \ldots, N\}$, the following estimates hold:

$$\widehat{\tau_{k,t}^0}(\xi) = 0; \tag{28}$$

$$|\widehat{\tau_{k,t}^\lambda}(\xi)| \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}; \tag{29}$$

$$|\widehat{\tau_{k,t}^\lambda}(\xi) - \widehat{\tau_{k,t}^{\lambda-1}}(\xi)| \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\varphi(2^k t)^\lambda|b_\lambda\xi|. \tag{30}$$

Also, by [17, Lemma 2.2] and the arguments similar to those used in deriving [21, Lemma 2(ii)], one can get

$$|\widehat{\tau_{k,t}^\lambda}(\xi)| \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}(\log\varphi(2^k t)^\lambda|b_\lambda\xi|)^{-\frac{\beta}{\max\{2,\gamma'\}}}, \quad \text{if } \varphi(2^k t)^\lambda|b_\lambda\xi| > 1 \tag{31}$$

for any $\lambda \in \{1, 2, \ldots, N\}$. Here the above constants $C > 0$ are independent of the coefficients of P.

Now we choose a $\mathcal{C}_0^\infty(\mathbb{R})$ function ψ such that $\psi(t) \equiv 1$ for $|t| \leq 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. For $1 \leq \lambda \leq N$ and $\xi \in \mathbb{R}^n$, we define the family of measures $\{\nu_{k,t}^\lambda\}$ by

$$\widehat{\nu_{k,t}^{\lambda}}(\xi) = \widehat{\sigma_{k,t}^{\lambda}}(\xi) \prod_{j=\lambda+1}^{N} \psi(\varphi(2^k t)^j |b_j \xi|) - \widehat{\sigma_{k,t}^{\lambda-1}}(\xi) \prod_{j=\lambda}^{N} \psi(\varphi(2^k t)^j |b_j \xi|). \tag{32}$$

It follows from (29)–(31) that for any $\lambda \in \{1, 2, \ldots, N\}$,

$$|\widehat{\nu_{k,t}^{\lambda}}(\xi)| \leq C \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \min\{1, \varphi(2^k t)^\lambda |b_\lambda \xi|\}; \tag{33}$$

$$|\widehat{\nu_{k,t}^{\lambda}}(\xi)| \leq C \|h\|_{\Delta_\gamma(\mathbb{R}_+)} (\log \varphi(2^k t)^\lambda |b_\lambda \xi|)^{-\frac{\beta}{\max\{2,\gamma'\}}}, \quad \text{if } \varphi(2^k t)^\lambda |b_\lambda \xi| > 1. \tag{34}$$

Here the above constants $C > 0$ are independent of h and the coefficients of P. Moreover, we get from (28) and (32) that

$$\tau_{k,t}^N = \sum_{\lambda=1}^{N} \nu_{k,t}^\lambda. \tag{35}$$

Here we use the convention $\Pi_{j \in \emptyset} a_j = 1$. Invoking Lemma 2.4 and using the change of variable, there exists $C > 0$ such that

$$\begin{aligned} &\Big\| \Big(\sum_{k \in \mathbb{Z}} \int_1^2 |\tau_{k,t}^\lambda * g_k|^q dt \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\leq 2 \Big\| \Big(\sum_{k \in \mathbb{Z}} \int_1^2 |\tau_{0,2^k t}^\lambda * g_k|^q \frac{dt}{t} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &= 2 \Big\| \Big(\sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\tau_{0,t}^\lambda * g_k|^q \frac{dt}{t} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\leq C \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})} \Big\| \Big(\sum_{k \in \mathbb{Z}} |g_k|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \end{aligned} \tag{36}$$

for $1 < q < \infty$ and $(q'\gamma)' < p < q\gamma$, where the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of P. By (36) and the definition of $\nu_{k,t}^\lambda$, one can get

$$\begin{aligned} &\Big\| \Big(\sum_{k \in \mathbb{Z}} \int_1^2 |\nu_{k,t}^\lambda * g_k|^q dt \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\leq C \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|\Omega\|_{L^1(\mathrm{S}^{n-1})} \Big\| \Big(\sum_{k \in \mathbb{Z}} |g_k|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \end{aligned} \tag{37}$$

for $1 < q < \infty$ and $(q'\gamma)' < p < q\gamma$, where the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of P.

On the other hand, we get by Minkowski's inequality and (35) that

$$\begin{aligned}\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho}f(x) &= \Big(\int_0^\infty \Big|\sum_{k=-\infty}^{0} 2^{k\rho}\tau^N_{k,t} * f(x)\Big|^q \frac{dt}{t}\Big)^{1/q} \\ &= \Big(\int_0^\infty \Big|\sum_{k=-\infty}^{0} 2^{k\rho}\tau^N_{0,t} * f(x)\Big|^q \frac{dt}{t}\Big)^{1/q} \\ &\le \sum_{k=-\infty}^{0} 2^{k\varsigma}\Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |\tau^N_{0,t} * f(x)|^q \frac{dt}{t}\Big)^{1/q} \\ &\le \frac{1}{1-2^{-\varsigma}}\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\tau^N_{k,t} * f(x)|^q \frac{dt}{t}\Big)^{1/q} \\ &\le \frac{1}{1-2^{-\varsigma}}\sum_{\lambda=1}^{N}\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\nu^\lambda_{k,t} * f(x)|^q dt\Big)^{1/q} \\ &=: \frac{1}{1-2^{-\varsigma}}\sum_{\lambda=1}^{N} T_{\lambda,q}f(x).\end{aligned} \tag{38}$$

Therefore, to prove Theorem 1.2, it suffices to show that there exists a constant $C > 0$ independent of h, Ω, γ and the coefficients of P such that

$$\|T_{\lambda,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{39}$$

for any $\lambda \in \{1, 2, \dots, N\}$, $\frac{2\beta}{2\beta-\tilde{\gamma}} < q < \frac{2\beta}{\tilde{\gamma}}$ and $\frac{1}{q\gamma} + \frac{\tilde{\gamma}}{2\beta\gamma'} < \frac{1}{p} < \frac{1}{q\gamma} + \frac{1}{\gamma'} - \frac{\tilde{\gamma}}{2\beta\gamma'}$.

Let $\Psi_{k,\lambda}$ be defined by $\Psi_{k,\lambda}(\xi) = \Phi_k(\xi)$, where Φ_k is given as in Lemma 2.2 with $a_k = \varphi(2^{-k})^{-\lambda}|b_\lambda|^{-1}$. By the properties of φ we have

$$1 < B^\lambda_\varphi \le \frac{a_{k+1}}{a_k} \le c^\lambda_\varphi \quad \text{for} \quad k \in \mathbb{Z}.$$

This together with Lemma 2.2 yields that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_{k,\lambda} * f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{40}$$

for all $1 \le \lambda \le N$ and $1 < p, q < \infty$. By Minkowski's inequality and the definition of $\Psi_{k,\lambda}$, we have

$$\begin{aligned}T_{\lambda,q}f(x) &= \Big(\sum_{k\in\mathbb{Z}}\int_1^2 \Big|(\nu^\lambda_{k,t} * \Psi_{j-k,\lambda} * f)(x)\Big|^q \frac{dt}{t}\Big)^{1/q} \\ &\le \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_1^2 \Big|(\nu^\lambda_{k,t} * \Psi_{j-k,\lambda} * f)(x)\Big|^q \frac{dt}{t}\Big)^{1/q} \\ &=: \sum_{j\in\mathbb{Z}} I_{\lambda,j,q}f(x).\end{aligned} \tag{41}$$

By (33), (34) and Plancherel's theorem, we have

$$\begin{aligned}(42)\qquad \|I_{\lambda,j,2}f\|^2_{L^2(\mathbb{R}^n)} &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_1^2 |\nu^{\lambda}_{k,t} * \Psi_{j-k,\lambda} * f(x)|^2 dtdx \\ &\le \sum_{k\in\mathbb{Z}}\int_{E_{j-k}}\int_1^2 |\widehat{\nu^{\lambda}_{k,t}}(x)|^2 dt |\hat{f}(x)|^2 dx \\ &\le C\|h\|^2_{\Delta_\gamma(\mathbb{R}_+)} B_j^2 \sum_{k\in\mathbb{Z}}\int_{E_{j-k}} |\hat{f}(x)|^2 dx \\ &\le C\|h\|^2_{\Delta_\gamma(\mathbb{R}_+)} B_j^2 \|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

where $E_{j-k} = \{x \in \mathbb{R}^n : \varphi(2^{k-j+1})^{-\lambda} \le |b_\lambda x| \le \varphi(2^{k-j-1})^{-\lambda}\}$ and

$$B_j = |j|^{-\frac{\beta}{\max\{2,\gamma'\}}}\chi_{\{j\ge 2\}}(j) + B_\varphi^{-|j|\lambda}\chi_{\{j\le 1\}}(j).$$

Combining (42) with the fact that $\dot{F}^0_{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ implies that

$$(43)\qquad \|I_{\lambda,j,2}f\|_{L^2(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} B_j \|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.$$

By (37) and (40) we can get

$$(44)\qquad \|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $1 < q < \infty$ and $(q'\gamma)' < p < q\gamma$, where the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of P.

By interpolation between (43) and (44) we have that for $\beta > \tilde{\gamma}$, $\frac{2\beta}{2\beta-\tilde{\gamma}} < q < \frac{2\beta}{\tilde{\gamma}}$ and $\frac{1}{q\gamma} + \frac{\tilde{\gamma}}{2\beta\gamma'} < \frac{1}{p} < \frac{1}{q\gamma} + \frac{1}{\gamma'} - \frac{\tilde{\gamma}}{2\beta\gamma'}$, there exist $\theta \in [0,1]$, $q_1 \in (1,\infty)$ and $p_1 \in ((q_1'\gamma)', q_1\gamma)$ such that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_1}, \quad \frac{\theta\beta}{\tilde{\gamma}} > 1$$

and

$$(45)\qquad \|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} B_j^{\theta} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

Combining (45) with (41) leads to

$$\|T_{\lambda,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \sum_{j\in\mathbb{Z}} B_j^{\theta} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for any $\beta > \tilde{\gamma}$, $\frac{2\beta}{2\beta-\tilde{\gamma}} < q < \frac{2\beta}{\tilde{\gamma}}$ and $\frac{1}{q\gamma} + \frac{\tilde{\gamma}}{2\beta\gamma'} < \frac{1}{p} < \frac{1}{q\gamma} + \frac{1}{\gamma'} - \frac{\tilde{\gamma}}{2\beta\gamma'}$. This gives (39). □

Proof of Theorem 1.3. By (38), to prove Theorem 1.3, it suffices to show that there exists a constant $C > 0$ independent of h, Ω, γ and the coefficients of P such that

$$(46)\qquad \|T_{\lambda,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $(p, q) \in (\frac{\beta\gamma'}{\beta+\gamma'-2}, \beta)^2$ or $\frac{\beta}{\beta-1} < p < q \le 2$.

By Lemma 2.5 and the arguments similar to those used to derive (36), there exists $C > 0$ independent of h, Ω, γ and the coefficients of P such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\tau_{k,t}^{\lambda} * g_k|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{47}$$

for $1 < p < q \leq \gamma'$ or $\gamma' < p,\, q < \infty$. By (47) and the definition of $\nu_{k,t}^{\lambda}$, it holds that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\nu_{k,t}^{\lambda} * g_k|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{48}$$

for $1 < p < q \leq \gamma'$ or $\gamma' < p,\, q < \infty$. Here the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of P. By (48) and (40) we can get

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{49}$$

for $\gamma' < p,\, q < \infty$ or $1 < p < q \leq \gamma'$, where the above constant $C > 0$ is independent of h, Ω, γ and the coefficients of P.

By interpolation between (43) and (49) we have that for any $p \in (\frac{\beta\gamma'}{\beta+\gamma'-2}, \beta)$ and $q \in (\frac{\beta\gamma'}{\beta+\gamma'-2}, \beta)$, there exist $\theta \in (\frac{2}{\beta}, 1]$ and $p_1 \in (\gamma', \infty)$ and $q_1 \in (\gamma', \infty)$ such that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_1}, \quad \frac{\theta\beta}{2} > 1$$

and

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}B_j^{\theta}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}. \tag{50}$$

Combining (50) with (41) leads to (46) for $(p,\, q) \in (\frac{\beta\gamma'}{\beta+\gamma'-2}, \beta)^2$.

By interpolation between (43) and (49) again we have that for any pair (p, q) with $\frac{\beta}{\beta-1} < p < q \leq 2$ and $\beta > 2$, there exist $\theta \in (\frac{2}{\beta}, 1]$ and $1 < p_1 < q_1 \leq \gamma'$ such that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_1}, \quad \frac{\theta\beta}{2} > 1$$

and

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}B_j^{\theta}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}. \tag{51}$$

Inequality (51) together with (41) leads to (46) for $\frac{\beta}{\beta-1} < p < q \leq 2$ and $\beta > 2$. □

Proof of Theorem 1.4. By (38), to prove Theorem 1.4, it suffices to show that there exists a constant $C>0$, which is independent of h and the coefficients of P, such that

$$\|T_{\lambda,q}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{52}$$

for any $\lambda\in\{1,2,\dots,N\}$, provided that one of the following conditions holds:

(i) $q\in(\frac{\gamma'(\beta-1)+2}{\beta},\gamma'(\beta-1)+2)$, $p\in(\frac{\gamma'(\beta-1)+2}{\beta},2]$ and $p<q$;

(ii) $q\in(\frac{\beta+1}{\beta},\beta+1)$, $p\in(2,\beta+1)$ and $p>q$;

(iii) $q\in(\frac{\gamma'(\beta-1)+2}{\beta},\beta+1)$ and $p=q$.

We now prove (52) by considering the following cases:

Case 1: Proof of (52) for the case (i). For $1\le\lambda\le N$, we define the multiplier operator $S_{k,\lambda}$ in $\mathbb{R}^n$ by

$$S_{k,\lambda}f(x)=\Psi_{k,\lambda}*f(x),$$

where the functions $\Psi_{k,\lambda}$ are given as in the proof of Theorem 1.2.

By the definition of $\Psi_{k,\lambda}$, we can write

$$T_{\lambda,q}f(x)=\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k,\lambda}(\nu_{k,t}^\lambda*\Psi_{j-k,\lambda}*f)(x)\Big|^q\frac{dt}{t}\Big)^{1/q}. \tag{53}$$

By Lemma 2.7(i) and (53) we can get

$$\begin{aligned}
&\|T_{\lambda,q}f\|_{L^p(\mathbb{R}^n)} \\
&\le C\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu_{k,t}^\lambda*\Psi_{j-k,\lambda}*f(x)|^q dt\Big)^{1/q}\Big\|^r_{L^p(\mathbb{R}^n)}\Big)^{1/r} \\
&\le C\Big(\sum_{j\in\mathbb{Z}}\|I_{\lambda,j,q}f\|^r_{L^p(\mathbb{R}^n)}\Big)^{1/r}
\end{aligned} \tag{54}$$

for $1<r<p<q<\infty$, where $I_{\lambda,j,q}$ is given as in (41).

By interpolation between (43) and (49), for $q\in(\frac{\gamma'(\beta-1)+2}{\beta},\gamma'(\beta-1)+2)$ and $p\in(\frac{\gamma'(\beta-1)+2}{\beta},2]$, there exist $p_1\in(\gamma',\infty)$, $q_1\in(\gamma',\infty)$ and $\theta\in(\frac{2}{\gamma'(\beta-1)+2},1]$ such that

$$\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{p_1},\quad \frac{1}{q}=\frac{\theta}{2}+\frac{1-\theta}{q_1}$$

and

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)}\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}B_j^\theta\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}. \tag{55}$$

Fix $q\in(\frac{\gamma'(\beta-1)+2}{\beta},\gamma'(\beta-1)+2)$ and $p\in(\frac{\gamma'(\beta-1)+2}{\beta},2]$ such that $p<q$, we can choose $1<r<p$ such that $\frac{r\theta\beta}{2}>1$. Therefore, it follows from (55) that

$$\|T_\lambda f\|_{L^p(\mathbb{R}^n)}\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\sum_{j\in\mathbb{Z}}B_j^{\theta r}\Big)^{1/r}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}\le C\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $q \in (\frac{\gamma'(\beta-1)+2}{\beta}, \gamma'(\beta-1)+2)$ and $p \in (\frac{\gamma'(\beta-1)+2}{\beta}, 2]$ such that $p < q$. This proves (52) for the case (i).

Case 2: Proof of (52) for the case (ii). By Lemma 2.7(ii) and (53), we have

$$\begin{aligned}(56)\qquad &\|T_\lambda f\|_{L^p(\mathbb{R}^n)} \\ &\le C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu^\lambda_{k,t}*\Psi_{j-k,\lambda}*f(x)|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}dt\Big)^{r/q}\Big)^{1/r} \\ &\le C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2\|J_{\lambda,j,q,t}f\|^q_{L^p(\mathbb{R}^n)}dt\Big)^{r/q}\Big)^{1/r} \\ &\le C\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{t\in[1,2]}\|J_{\lambda,j,q,t}f\|_{L^p(\mathbb{R}^n)}\Big)^r\Big)^{1/r}\end{aligned}$$

for any $r \in (1, p')$ and $p > q$, where

$$J_{\lambda,j,q,t}f(x) = \Big(\sum_{k\in\mathbb{Z}}|\nu^\lambda_{k,t}*\Psi_{j-k,\lambda}*f(x)|^q\Big)^{1/q}.$$

Fix $t \in [1,2]$. By Lemma 2.6 and the definition of $\nu_{k,t}$, it holds that

$$\begin{aligned}(57)\qquad &\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu^\lambda_{k,t}*g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

for $\gamma' < p < \infty$ and $1 < q < \infty$. In light of (57) and (40) we would have

$$(58)\qquad \|J_{\lambda,j,q,t}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $\gamma' < p < \infty$ and $1 < q < \infty$. Similar arguments to those used in getting (43) may yield

$$(59)\qquad \|J_{\lambda,j,2,t}f\|_{L^2(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}B_j\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.$$

By interpolating between (58) and (59), for fixed $p \in (2, \beta+1)$ and $q \in (\frac{\beta+1}{\beta}, \beta+1)$, we can choose $r \in (1, p')$, $p_1 \in (\gamma', \infty)$, $q_1 \in (1, \infty)$ and $\delta \in (\frac{2}{\beta+1}, 1)$ such that $\frac{r\delta\beta}{2} > 1$, $\frac{1}{p} = \frac{\delta}{2} + \frac{1-\delta}{p_1}$, $\frac{1}{q} = \frac{\delta}{2} + \frac{1-\delta}{q_1}$ and

$$(60)\qquad \|J_{\lambda,j,q,t}f\|_{L^p(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}B_j^\delta\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

Here $C > 0$ is independent of t and the coefficients of P. It follows from (56) and (60) that

$$\begin{aligned}(61)\qquad \|T_\lambda f\|_{L^p(\mathbb{R}^n)} &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\sum_{j\in\mathbb{Z}}B_j^{\delta r}\Big)^{1/r}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

This proves (52) for the case (ii).

Case 3: Proof of (52) for the case (iii). Notice that $\gamma'(\beta-1)+2>\beta+1$. Then the case $p=q$ follows easily from the interpolation between Case 1 and Case 2. This completes the proof of Theorem 1.4. $\square$

4. Addition results

In this section we shall present some new results for the parametric Marcinkiewicz integral operators related to the Littlewood-Paley g_λ^*-function and the area integral S, which are respectively defined by

$$\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f(x) := \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} \times\Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-P(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},$$

where $\lambda>0$ and $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times(0,\infty)$;

$$\mathscr{M}^{q}_{h,\Omega,P_N,\varphi,\rho,S}f(x) := \Big(\iint_{\Gamma(x)}\Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-P(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},$$

where $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\}$ and h, Ω, P, φ, ρ be given as in (2).

Before presenting our main results, let us introduce the following lemma, which follows from [19].

Lemma 4.1. *Let $\lambda>1$ and $1<q<\infty$. Then there exists a constant $C_{n,\lambda}>0$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}(\mathscr{M}^{\lambda,q,*}_{h,\Omega,P_N,\varphi,\rho}f(x))^q g(x)dx \le C_{n,\lambda}\int_{\mathbb{R}^n}(\mathfrak{M}^{q}_{h,\Omega,P_N,\varphi,\rho}f(x))^q M(g)(x)dx,$$

where M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

As applications of Theorems 1.2–1.4, we can get:

Theorem 4.2. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$ and $\varphi\in\mathfrak{F}$. Let Ω satisfy (1) and $1<q<\infty$.*

(i) *If $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma\in(1,\infty]$ and $\Omega\in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>\tilde\gamma$. Then for any $\frac{2\beta}{2\beta-\tilde\gamma}<q<\frac{2\beta}{\tilde\gamma}$ and $\frac{1}{q\gamma}+\frac{\tilde\gamma}{2\beta\gamma'}<\frac1p\le\frac1q$, we have*

$$\|\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\le C_p\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}.$$

(ii) *If $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma\in[2,\infty]$ and $\Omega\in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>2$. Then for $\frac{\beta\gamma'}{\beta+\gamma'-2}<q\le p<\beta$, we have*

$$\|\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)}\le C_p\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|f\|_{\dot F^0_{p,q}(\mathbb{R}^n)}.$$

(iii) *If* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma \in [2, \infty]$ *and* $\Omega \in W\mathcal{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta > 1$. *Then*

$$\|\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^n)}$$

provided that one of the following conditions holds:

(a) $q \in (\frac{\beta+1}{\beta}, \beta+1)$, $p \in (2, \beta+1)$ *and* $p > q$;

(b) $q \in (\frac{\gamma'(\beta-1)+2}{\beta}, \beta+1)$ *and* $p = q$.

Here the above constants $C_p > 0$ *are independent of* h *and the coefficients of* P, *but may depend on* $p, q, n, \lambda, \varphi, \rho, N$. *The same results hold for* $\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho,S}$.

Proof. Fix $1 < q \le p < \infty$. By the duality, L^p bounds for M, Hölder's inequality and Lemma 4.1, one has

$$\begin{aligned}
&\|\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho} f\|^q_{L^p(\mathbb{R}^n)} \\
&= \sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} (\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho} f(x))^q g(x)dx \\
&\le C_{n,\lambda} \sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} (\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f(x))^q M(g)(x)dx \\
&\le C_{n,\lambda,p,q} \|\mathfrak{M}^q_{h,\Omega,P,\varphi,\rho} f\|^q_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Combining this with Theorems 1.2–1.4 implies the conclusions of Theorem 4.2 for $\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho}$.

On the other hand, one can easily check that

$$\mathscr{M}^q_{h,\Omega,P,\varphi,\rho,S} f(x) \le 2^{n\lambda/q} \mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho} f(x),$$

which together with the bounds for $\mathscr{M}^{\lambda,q,*}_{h,\Omega,P,\varphi,\rho}$ implies the bounds for $\mathscr{M}^q_{h,\Omega,P,\varphi,\rho,S}$. □

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