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S-CURVATURE AND GEODESIC ORBIT PROPERTY OF INVARIANT (α_1, α_2)-METRICS ON SPHERES

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Abstract. Geodesic orbit spaces are homogeneous Finsler spaces whose geodesics are all orbits of one-parameter subgroups of isometries. Such Finsler spaces have vanishing S-curvature and hold the Bishop-Gromov volume comparison theorem. In this paper, we obtain a complete description of invariant (α_1, α_2)-metrics on spheres with vanishing S-curvature. Also, we give a description of invariant geodesic orbit (α_1, α_2)-metrics on spheres. We mainly show that a $\mathrm{Sp}(n+1)$-invariant (α_1, α_2)-metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is geodesic orbit with respect to $\mathrm{Sp}(n+1)$ if and only if it is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant. As an interesting consequence, we find infinitely many Finsler spheres with vanishing S-curvature which are not geodesic orbit spaces.

1. Introduction

The notion of S-curvature was introduced by Z. Shen in [20] in his study of volume comparison in Finsler geometry. S-curvature is an important non-Riemannian quantity in Finsler geometry, or in other words, any Riemannian manifold has vanishing S-curvature. Z. Shen showed that the Bishop-Gromov volume comparison theorem holds for Finsler spaces with vanishing S-curvature. Therefore, it is significant to characterize Finsler spaces with vanishing S-curvature.

Fortunately, there is a special class of Finsler spaces called geodesic orbit space in homogeneous Finsler geometry satisfying this property. Geodesic orbit space is a Finsler space whose geodesics are all orbits of one-parameter subgroups of isometries. More exactly, we call a Riemannian (Finsler) metric F on a homogeneous space G/H a geodesic orbit metric with respect to G if every geodesic of $(G/H, F)$ is an orbit of a one-parameter subgroup of G. This terminology was first introduced by O. Kowalski and L. Vanhecke [15], who initiated

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a systematic study of such spaces. There is extensive literature on Riemannian geodesic orbit spaces and we refer the readers to [1, 3, 13] and the references therein. In 2014, Z. Yan and S. Deng [23] generalized some geometric results on Riemannian geodesic orbit spaces to the Finslerian setting. Moreover, they obtained a sufficient and necessary condition for a Randers space to be a geodesic orbit space. Later, in [22], Z. Yan studied geodesic orbit (α, β)-spaces and gave many non-Riemannian geodesic orbit spaces. Recently, in [24], L. Zhang and M. Xu studied standard geodesic orbit (α_1, α_2)-spaces and found some new examples of non-Riemannian geodesic orbit Finsler spaces which are not weakly symmetric. Meanwhile, we should mention that, for some very special Finsler spaces, such as symmetric spaces and weakly symmetric spaces [11] are all geodesic orbit spaces.

The goal of this paper is to study the S-curvature and geodesic orbit property of invariant (α_1, α_2)-metrics on spheres. Spheres can be viewed as a special class of homogeneous spaces. At first, A. Borel in [6] and D. Montgomery and H. Samelson in [17] classified the compact connected Lie groups that admit an effective transitive action on spheres. Recently, Y. Nikonorov in [18] studied the geodesic orbit Riemannian metrics on spheres. In this paper, he gave a complete classification of geodesic orbit metrics on Riemannian spheres and constructed some explicit geodesic vectors. He mainly showed a $\mathrm{Sp}(n+1)$-invariant Riemannian metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is geodesic orbit with respect to $\mathrm{Sp}(n+1)$ if and only if it is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant. Later, in independent works, M. Xu in [21] and S. Zhang and Z. Yan in [25] generalized Y. Nikonorov's classification of geodesic orbit Riemannian metrics on spheres to Finslerian setting. In [21], using a geometric method, the author proved a homogeneous Finsler metric on a sphere S^n with $n > 1$ is a geodesic orbit metric if and only if its connected isometry group is not isomorphic to $\mathrm{Sp}(k)$ for any $k \geq 1$. By this result, he gave a classification of geodesic orbit metrics on Finsler spheres. In [25], the authors classified geodesic orbit Randers spheres with a more algebraic method, and obtained some explicit metrics and geodesic vectors. By now, it is still unknown whether there exists a non-Riemannian Finsler geodesic orbit metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ with geodesic vectors all in $\mathfrak{sp}(n+1)$ or not, here $\mathfrak{sp}(n+1)$ denotes the Lie algebra of $\mathrm{Sp}(n+1)$. In our paper, we first obtain a complete description of invariant (α_1, α_2)-metrics on spheres with vanishing S-curvature. Based on this result, we find there are infinitely many non-Riemannian geodesic orbit (α_1, α_2)-metrics on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ with its geodesic vectors all in $\mathfrak{sp}(n+1)$ by showing the following

Theorem 1.1. *A* $\mathrm{Sp}(n+1)$*-invariant* (α_1, α_2)*-metric on* $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ *is geodesic orbit with respect to* $\mathrm{Sp}(n+1)$ *if and only if it is* $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$*-invariant.*

Remark 1.2. We should mention that, the "only if" part of Theorem 1.1 is a special case of Corollary 4.3 in [21].

In Sections 2, 3 and 4, we present some facts about Finsler spaces, S-curvature, geodesic orbit spaces and weakly symmetric spaces, for more details we refer the readers to [2] and [8]. In Section 5, we obtain a complete description of invariant (α_1, α_2)-metrics on spheres with vanishing S-curvature. In Section 6, we mainly prove Theorem 1.1.

2. Finsler spaces and S-curvature

Recall a Finsler metric on a smooth manifold M is a real continuous function $F : TM \to [0, +\infty)$ such that

1. F is C^∞ on the slit tangent bundle $TM \setminus \{0\}$;
2. The restriction of F to any T_xM, $x \in M$, is a Minkowski norm. Namely,
   (1) $F(u) \geq 0, \forall u \in T_xM$;
   (2) $F(\lambda u) = \lambda F(u), \forall \lambda > 0$;
   (3) For any basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of T_xM, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = \sum_{i=1}^n y^i \varepsilon_i$. The Hessian matrix
   $$(g_{ij}(y)) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$
   is positive definite at any point of $T_xM \setminus \{0\}$.

Definition 2.1 ([12], Definition 3.8)**.** For a Riemannian metric α on a manifold M, an α-orthogonal decomposition $TM = \mathcal{V}_1 \oplus \mathcal{V}_2$ and $\alpha_i = \alpha|_{\mathcal{V}_i}$, a Finsler metric is called an (α_1, α_2)-metric if it can be presented as

$$F(x, y) = f(\alpha(x, y_1), \alpha(x, y_2)),\ \forall y = y_1 + y_2 \in T_xM \text{ with } y_i \in \mathcal{V}_i,\ i = 1, 2,$$

for some positive smooth function $f(\cdot, \cdot)$.

It is easily seen that an (α_1, α_2)-metric F is Riemannian if and only if it is defined by $f(s,t) = \sqrt{as^2 + bt^2}$ for some positive constants a and b. There are lots of non-Riemannian (α_1, α_2)-metrics. For example, one can take

$$(*) \qquad f(s,t) = \sqrt{s^2 + t^2 + \varepsilon(s^{2k} + t^{2k})^{\frac{1}{k}}},$$

where ε is a positive number and $k \geq 2$ is a positive integer (see [23], Section 5).

An (α_1, α_2)-metric can also be expressed by $F = \alpha\varphi(\alpha_2/\alpha)$, where $\varphi(\theta)$ is positive and smooth for $\theta \in (0, 1)$. If $F = \alpha\varphi(\alpha_2/\alpha)$ is a regular (α_1, α_2)-metric, then $\varphi(\theta)$ satisfies

$$\varphi(\theta) - \theta\varphi'(\theta) > 0,\ \varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta) > 0$$

for $\theta \in (0, 1)$. See Lemma 2.4 in [24] for a proof.

We now recall the notion of S-curvature of a Finsler space. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{\varepsilon_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(\sum_i y^i \varepsilon_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{\varepsilon_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V \setminus \{0\}$$

is independent of the choice of the basis. The quantity $\tau = \tau(y)$ is called the distortion of (V, F).

Definition 2.2 ([20])**.** Let (M, F) be an n-dimensional Finsler space and let $\tau(x, y)$ be the distortion of the Minkowski norm F_x on T_xM. For any $y \in T_xM \setminus \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0}$$

is called the S-curvature of the Finsler space (M, F).

3. Geodesic orbit spaces

Let (M, F) be a connected Finsler space. Then the full group of isometries of (M, F), denoted by $I(M, F)$, is a Lie transformation group on M with respect to the compact-open topology [9]. The space (M, F) is called homogeneous if the action of $I(M, F)$ on M is transitive. In this case, M can be written as a coset space G/H, where G is a Lie subgroup of $I(M, F)$ acting transitively on M and H is the isotropy subgroup of G at a point $p \in M$. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H, respectively. The reductive decomposition of the Lie algebras is $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$. The subspace $\mathfrak{m}$ can be identified with the tangent space $T_p(G/H)$ at $p = eH$ via the mapping $X \mapsto \frac{d}{dt}|_{t=0} \exp(tX) \cdot p$. The G-invariant Finsler metric F on $M = G/H$ is one-to-one correspondence to the $\mathrm{Ad}(H)$-invariant Minkowski norm (still denoted by F) on $\mathfrak{m}$ [10].

Definition 3.1 ([23])**.** Let (M, F) be a Finsler space and G be a subgroup of $I(M, F)$ the full group of isometries. The Finsler space (M, F) is called a Finsler geodesic orbit space with respect to G if every geodesic of (M, F) is an orbit of a one-parameter subgroup of G. That is, if γ is a geodesic through $p \in M$, then there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX) \cdot p$. In this case, X is called a geodesic vector.

For simplify, if $G = I(M, F)$, we call (M, F) a Finsler geodesic orbit space without saying G.

S. Deng [11] and D. Latifi [16] proved the following:

Theorem 3.2 ([11, 16])**.** *Let (M, F) be a Finsler geodesic orbit space. Then the S-curvature of (M, F) vanishes.*

From Definition 3.1, one easily sees that a homogeneous Finsler space $(G/H, F)$ is a Finsler geodesic orbit space with respect to G if and only if the projection of all geodesic vectors cover the set $T_pM \setminus \{0\}$ if and only if for any non-zero $X \in \mathfrak{m}$, there exists $A \in \mathfrak{h}$ such that $X + A$ is a geodesic vector.

Concerning geodesic vectors, one has the following:

Lemma 3.3 ([15,16]). *A vector $X \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector of $(G/H, F)$ if and only if*

$$g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = 0, \forall Z \in \mathfrak{m},$$

where the subscript $\mathfrak{m}$ means the corresponding projection, and g is the fundamental tensor of F on $\mathfrak{m}$ defined by

$$g_y(u, v) = \sum_{i,j} g_{ij}(y) u^i v^j,\ \forall y \neq 0,\ u = \sum_i u^i \varepsilon_i,\ v = \sum_j v^j \varepsilon_j \in \mathfrak{m},$$

$\{\varepsilon_i\}$ is a basis of $\mathfrak{m}$.

Now we consider invariant (α_1, α_2)-metrics on the homogeneous space G/H. According to Corollary 1.2 in [10], such metrics can be stated as follows. Let $|\cdot|^2 = \langle\cdot,\cdot\rangle$ be an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$, $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$ be an $\mathrm{Ad}(H)$-invariant $\langle\cdot,\cdot\rangle$-orthogonal decomposition of $\mathfrak{m}$. Then the (α_1, α_2)-norm

$$F(X) = |X|\varphi(|X_\mathfrak{q}|/|X|),\ \forall X = X_\mathfrak{p} + X_\mathfrak{q} \in \mathfrak{m} \setminus \{0\},\ X_\mathfrak{p} \in \mathfrak{p},\ X_\mathfrak{q} \in \mathfrak{q}$$

defines a G-invariant (α_1, α_2)-metric on G/H. Conversely, the $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$ induced by a G-invariant (α_1, α_2)-metric can be expressed by this form for some $\langle\cdot,\cdot\rangle$, $\mathfrak{p}$, $\mathfrak{q}$ and φ.

Theorem 3.4 ([12], Theorem 4.3). *Let F be an invariant non-Riemannian (α_1, α_2)-metric on G/H. Then F has vanishing S-curvature if and only if*

$$\langle [X, Y]_\mathfrak{m}, X\rangle = \langle [X, Y]_\mathfrak{m}, Y\rangle = 0,\ \forall X \in \mathfrak{p},\ Y \in \mathfrak{q}.$$

Remark 3.5. From Theorem 3.4, we can see that the vanishing of S-curvature of an invariant non-Riemannian (α_1, α_2)-metric on a homogeneous space does not depend on the (α_1, α_2)-metric, it depends only on the underlying Riemannian metric and the decomposition $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$. Recall that a naturally reductive metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ satisfies

$$\langle [X, Y]_\mathfrak{m}, Z\rangle + \langle Y, [X, Z]_\mathfrak{m}\rangle = 0,\ \forall X, Y, Z \in \mathfrak{m}.$$

So every invariant (α_1, α_2)-metric induced by a naturally reductive metric must have vanishing S-curvature.

From Lemma 3.3, the geodesic vectors of an invariant (α_1, α_2)-metric on a homogeneous space G/H can be equivalently described by the following theorem.

Theorem 3.6 ([24]). *Assume $X = X_\mathfrak{h} + X_\mathfrak{m} = X_\mathfrak{h} + X_\mathfrak{p} + X_\mathfrak{q} \in \mathfrak{g}$ is according to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = \mathfrak{h} + \mathfrak{p} + \mathfrak{q}$, we have:*

(1) *When* $X_{\mathfrak{m}} \in (\mathfrak{p} \cup \mathfrak{q}) \setminus \{0\}$, *then* X *is a geodesic vector if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \, \forall Z \in \mathfrak{m}.$$

(2) *When* $X_m \in \mathfrak{m} \setminus (\mathfrak{p} \cup \mathfrak{q})$, *then* X *is a geodesic vector if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, (\varphi(\theta) - \theta\varphi'(\theta))X_{\mathfrak{p}} + (\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta))X_{\mathfrak{q}} \rangle = 0, \, \forall Z \in \mathfrak{m},$$

where $\theta = |X_{\mathfrak{q}}|/|X_{\mathfrak{m}}| \in (0, 1)$.

4. Weakly symmetric metrics on spheres

The original definition of weakly symmetric Riemannian manifold was given by A. Selberg in [19]. A connected Riemannian manifold (M, Q) is called weakly symmetric if there exists a subgroup G of the full group $I(M, Q)$ of isometries such that G acts transitively on M and there exists an isometry f of (M, Q) with $f^2 \in G$ and $fGf^{-1} = G$ such that for every two points $p, q \in M$, there exists an isometry g of (M, Q) satisfying $g(p) = f(q)$ and $g(q) = f(p)$. Later, in [5], J. Berndt and L. Vanhecke obtained a simple geometrical characterization of weakly symmetric Riemannian manifolds. Namely, a connected Riemannian manifold (M, Q) is weakly symmetric if and only if for any two points $p, q \in M$ there exists an isometry f of (M, Q) such that $f(p) = q$ and $f(q) = p$. This notion of weakly symmetric Riemannian manifold has been generalized to the Finslerian setting by S. Deng.

Definition 4.1 ([11])**.** Let (M, F) be a connected Finsler space and $I(M, F)$ be the full group of isometries. Then (M, F) is called weakly symmetric if for every two points p, q in M there exists an isometry $\sigma \in I(M, F)$ such that $\sigma(p) = q$ and $\sigma(q) = p$.

Weakly symmetric Finsler spaces were deeply studied by S. Deng, who proved the following important results.

Theorem 4.2 ([7])**.** *Let* (M, F) *be a connected weakly symmetric Finsler space and* G *be the full group of isometries of* (M, F). *Then any* G*-invariant Finsler metric on* M *must be weakly symmetric.*

Theorem 4.3 ([4, 11])**.** *A weakly symmetric Finsler space must be a Finsler geodesic orbit space with respect to the full group of isometries.*

We now study invariant (α_1, α_2)-metrics on spheres. The spheres can be expressed by homogeneous spaces. In Table 1 we list all homogeneous spheres G/H, where G is a compact connected Lie group with an effective action on G/H. We also give the isotropy representations of the homogeneous spaces.

• Cases 1, 2 and 3. Since the isotropy representation is irreducible, every (α_1, α_2)-metric on these three kinds of homogeneous spaces is Riemannian.

• Cases 4, 5 and 8. All invariant Finsler metrics are geodesic orbit metrics and have vanishing S-curvature [21]. Notice that the isotropy representation is not irreducible, so there exist invariant non-Riemannian (α_1, α_2)-metrics.

TABLE 1. Invariant (α_1, α_2)-metrics on spheres.

	G	H	$\dim G/H$	isotropy representation	(α_1, α_2)-metric	Cond.
1	$\mathrm{SO}(n+1)$	$\mathrm{SO}(n)$	n	irreducible	Riemannian	$n \geq 1$
2	G_2	$\mathrm{SU}(3)$	6	irreducible	Riemannian	
3	$\mathrm{Spin}(7)$	G_2	7	irreducible	Riemannian	
4	$\mathrm{SU}(n+1)$	$\mathrm{SU}(n)$	$2n+1$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	geodesic orbit	$n \geq 2$
5	$\mathrm{U}(n+1)$	$\mathrm{U}(n)$	$2n+1$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	geodesic orbit	$n \geq 1$
6	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$	15	$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$	weakly symmetric	
7	$\mathrm{Sp}(n+1)\mathrm{Sp}(1)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$	weakly symmetric	$n \geq 1$
8	$\mathrm{Sp}(n+1)\mathrm{U}(1)$	$\mathrm{Sp}(n)\mathrm{U}(1)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2$	geodesic orbit	$n \geq 1$
9	$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$		$n \geq 0$

• Cases 6 and 7. They are weakly symmetric ([21], Section 3), hence all invariant Finsler metrics are geodesic orbit metrics and have vanishing S-curvature. Notice that the isotropy representation is not irreducible, so there exist invariant non-Riemannian (α_1, α_2)-metrics.

• Case 9. This is a unique case that we should study in details. We deal with this case in Sections 5 and 6. The isotropy representation is not irreducible, there exist invariant non-Riemannian (α_1, α_2)-metrics. The Lie group pair $(\mathrm{Sp}(n+1), \mathrm{Sp}(n))$ is not a weakly symmetric pair, that is, there exists an invariant Riemannian (Finsler) metric F on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ such that $(\mathrm{Sp}(n+1)/\mathrm{Sp}(n), F)$ is not a weakly symmetric Riemannian (Finsler) space. When $n = 0$, $\mathrm{Sp}(1)/\{e\} = \mathrm{S}^3$, it follows from Lemma 3.3 that, all $\mathrm{Sp}(1)$-invariant Finsler metrics on S^3 which are geodesic orbit with respect to $\mathrm{Sp}(1)$, should be bi-invariant (see also Theorem 2.3 in [10]). Note also that the family of $\mathrm{Sp}(n+1)$-invariant Riemannian metrics on $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ $(n \geq 1)$ is a 7-dimensional space. Y. Nikonorov [18] proved that a $\mathrm{Sp}(n+1)$-invariant Riemannian metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is a geodesic orbit metric with respect to $\mathrm{Sp}(n+1)$ if and only if it is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant.

5. Invariant (α_1, α_2)-metrics on spheres with vanishing S-curvature

In this section, we deal with the S-curvature of $\mathrm{Sp}(n+1)$-invariant (α_1, α_2)-metrics on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$, $n \geq 0$.

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ be the field of quaternions, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the quaternionic units in $\mathbb{H}$. That is, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$. For $u = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, define $\mathrm{Re}(u) = x_0$ and $\bar{u} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. In Lie algebra $\mathfrak{sp}(n+1)$, denote by G_i the matrix with $\sqrt{2}$ in the (i,i)-th entry, and zeros elsewhere, $1 \leq i \leq n+1$. The reductive decomposition of $\mathfrak{sp}(n+1)$ is $\mathfrak{sp}(n+1) = \mathfrak{sp}(n) + \mathfrak{m} = \mathfrak{sp}(n) + \mathfrak{m}_0 + \mathfrak{m}_1$, here $\mathfrak{m}_0 = \mathbb{R}\mathbf{i}G_1 + \mathbb{R}\mathbf{j}G_1 + \mathbb{R}\mathbf{k}G_1$,

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha}' & 0 \end{pmatrix} \mid \alpha = (u_1, u_2, \ldots, u_n) \in \mathbb{H}^n \right\}, \ \bar{\alpha} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n).$$

Then, up to a positive multiple, any $\mathrm{Sp}(n+1)$-invariant Riemannian metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ can be written as

$$g_{(t_1,t_2,t_3)}(X,Y) = \mathrm{Re}(\xi\bar{\eta}') + t_1x_1y_1 + t_2x_2y_2 + t_3x_3y_3,$$

where $t_1, t_2, t_3 > 0$, $X, Y \in \mathfrak{m}_0 + \mathfrak{m}_1$ and

$$X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 + \begin{pmatrix} 0 & \xi \\ -\bar{\xi}' & 0 \end{pmatrix},$$

$$Y = y_1\mathbf{i}G_1 + y_2\mathbf{j}G_1 + y_3\mathbf{k}G_1 + \begin{pmatrix} 0 & \eta \\ -\bar{\eta}' & 0 \end{pmatrix}.$$

Remark 5.1. Note that when $t_1 = t_2 = t_3 = 1$, the underlying Riemannian metric $g_{(1,1,1)}$ is the standard Riemannian metric on S^{4n+3} with constant curvature and is invariant under $\mathrm{SO}(4(n+1))$. If $t_1 = t_2 = t_3 = t \neq 1$, then the Riemannian metrics $g_{(t,t,t)}$ are naturally reductive and invariant under $\mathrm{Sp}(n)\mathrm{Sp}(1)$. If $t_i = t_j = 1 \neq t_k$ with (i,j,k) a cyclic permutation of $(1,2,3)$, the Riemannian metrics $g_{(t_1,t_2,t_3)}$ are invariant under $\mathrm{SU}(2n+2)$ (and $\mathrm{U}(2n+2)$). If $t_i = t_j \neq 1$ and $t_i \neq t_k$, then the Riemannian metrics $g_{(t_1,t_2,t_3)}$ are invariant under $\mathrm{Sp}(n)\mathrm{U}(1)$, but not invariant under $\mathrm{U}(2n+2)$.

By a direct calculation (see also [14], p. 998), we have:

Proposition 5.2. *For all* $X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1, Y = y_1\mathbf{i}G_1 + y_2\mathbf{j}G_1 + y_3\mathbf{k}G_1 \in \mathfrak{m}_0$*,* $Z \in \mathfrak{m}_1$*,*

$$\begin{aligned} & g_{(t_1,t_2,t_3)}([X, Y+Z]_{\mathfrak{m}}, Y+Z) \\ = & -2\sqrt{2}[x_1y_2y_3(t_2-t_3) - x_2y_1y_3(t_1-t_3) + x_3y_1y_2(t_1-t_2)]. \end{aligned}$$

Now we start to classify $\mathrm{Sp}(n+1)$-invariant (α_1,α_2)-metrics on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ with vanishing S-curvature. When $n \geq 1$, by the symmetry of $\mathfrak{p}$ and $\mathfrak{q}$, we can assume $\mathfrak{m}_1 \subseteq \mathfrak{q}$ and $\mathfrak{p} \subseteq \mathfrak{m}_0$ without losing generality, since the action of $\mathfrak{sp}(n)$ on $\mathfrak{m}_1$ is irreducible. Let $\mathfrak{p}_0$ be the orthogonal complement of $\mathfrak{p}$ in $\mathfrak{m}_0$ with respect to the Riemannian metric $g_{(t_1,t_2,t_3)}$, we have $\mathfrak{m}_0 = \mathfrak{p} + \mathfrak{p}_0$ and $\mathfrak{q} = \mathfrak{m}_1 + \mathfrak{p}_0$. Notice that the dimension of $\mathfrak{p}$ is equal to $1, 2, 3$, we will have the following five cases.

(I) $\dim\mathfrak{p} = 1$, $\mathfrak{p} \in \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$.

(II) $\dim\mathfrak{p} = 1$, $\mathfrak{p} \notin \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$.

(III) $\dim\mathfrak{p} = 2$, $\dim\mathfrak{p}_0 = 1$, $\mathfrak{p}_0 \in \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$.

(IV) $\dim\mathfrak{p} = 2$, $\dim\mathfrak{p}_0 = 1$, $\mathfrak{p}_0 \notin \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$.

(V) $\mathfrak{p} = \mathfrak{m}_0, \mathfrak{q} = \mathfrak{m}_1, n \geq 1$.

When $n = 0$, we have $\mathfrak{m}_1 = \{0\}$. In this case, the decomposition $\mathfrak{m}_0 = \mathfrak{p} + \mathfrak{q}$ can also be divided into the above cases (I)-(IV).

For case (I), we consider $\mathfrak{p} = \mathbb{R}\mathbf{i}G_1$, the other cases are similar. Let $X_0 = \mathbf{i}G_1 \in \mathfrak{p}$, by Theorem 3.4 and Proposition 5.2, for all $Y = y_2\mathbf{j}G_1 + y_3\mathbf{k}G_1 \in \mathfrak{q}$ we can see

$$0 = g_{(t_1,t_2,t_3)}([X_0, Y]_{\mathfrak{m}}, Y) = -2\sqrt{2}y_2y_3(t_2 - t_3).$$

This shows $t_2 = t_3$ and hence $g_{(t_1,t_2,t_3)}|_{\mathfrak{p}_0} = g_{(t,t,t)}|_{\mathfrak{p}_0}$ for some $t \in \mathbb{R}$. In this case, the Riemannian metric $g_{(t_1,t_2,t_3)}$ is $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant and the (α_1, α_2)-metric F induced by this Riemannian metric is also $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant. Hence F is geodesic orbit and has vanishing S-curvature.

For case (II). Let $X_0 = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 \in \mathfrak{p}$. Then at least two of x_1, x_2, x_3 are non-zero. By Theorem 3.4 and Proposition 5.2 again, for all $Y = y_1\mathbf{i}G_1 + y_2\mathbf{j}G_1 + y_3\mathbf{k}G_1 \in \mathfrak{m}_0$, we have

$$\begin{aligned} 0 &= g_{(t_1,t_2,t_3)}([X_0, Y]_{\mathfrak{m}}, Y) \\ &= -2\sqrt{2}[x_1y_2y_3(t_2 - t_3) - x_2y_1y_3(t_1 - t_3) + x_3y_1y_2(t_1 - t_2)]. \end{aligned}$$

This shows $x_1(t_2 - t_3) = x_2(t_1 - t_3) = x_3(t_1 - t_2) = 0$, thus $t_1 = t_2 = t_3$. In this case, the Riemannian metric $g_{(t_1,t_2,t_3)}$ is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant and the (α_1, α_2)-metric F induced by this Riemannian metric is $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant. Hence F is geodesic orbit and has vanishing S-curvature.

For case (III), by the same argument as in case (I), we easily obtain that F has vanishing S-curvature if and only if $g_{(t_1,t_2,t_3)}|_{\mathfrak{p}} = g_{(t,t,t)}|_{\mathfrak{p}}$ for some $t \in \mathbb{R}$. In this case, both $g_{(t_1,t_2,t_3)}$ and F are $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant and geodesic orbit.

For case (IV), by the same argument as in case (II), we easily obtain that F has vanishing S-curvature if and only if $g_{(t_1,t_2,t_3)} = g_{(t,t,t)}$ for some $t \in \mathbb{R}$. In this case, $g_{(t_1,t_2,t_3)} = g_{(t,t,t)}$ is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant and weakly symmetric, F is $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant and geodesic orbit.

For case (V). Let $X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 \in \mathfrak{m}_0$, $Y = \begin{pmatrix} 0 & \eta \\ -\bar{\eta}' & 0 \end{pmatrix} \in \mathfrak{m}_1$. Then we have

$$[X, Y] = \begin{pmatrix} 0 & \sqrt{2}(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})\eta \\ \sqrt{2}\bar{\eta}'(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) & 0 \end{pmatrix} \in \mathfrak{m}_1.$$

As a result,

$$g_{(t_1,t_2,t_3)}([X, Y]_{\mathfrak{m}}, Y) = 0, \quad g_{(t_1,t_2,t_3)}([X, Y]_{\mathfrak{m}}, X) = 0.$$

According to Theorem 3.4, for any $t_1, t_2, t_3 > 0$, the (α_1, α_2)-metrics induced by $g_{(t_1,t_2,t_3)}$ on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ have vanishing S-curvature.

Summarize the above results we have the following conclusion.

Theorem 5.3. *Let F be an $\mathrm{Sp}(n+1)$-invariant non-Riemannian (α_1, α_2)-metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ constructed by the Riemannian metric $g_{(t_1,t_2,t_3)}$ and decomposition $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$ with $\mathfrak{p} \subseteq \mathfrak{m}_0$, $\mathfrak{m}_1 \subseteq \mathfrak{q}$, we have:*

(1) *When $\dim \mathfrak{p} = 1$, $\mathfrak{p} \in \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$. F has vanishing S-curvature if and only if $g_{(t_1,t_2,t_3)}|_{\mathfrak{p}_0} = g_{(t,t,t)}|_{\mathfrak{p}_0}$ for some $t \in \mathbb{R}$. In this case, both $g_{(t_1,t_2,t_3)}$ and F are $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant and geodesic orbit.*

(2) *When $\dim \mathfrak{p} = 1$, $\mathfrak{p} \notin \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$. F has vanishing S-curvature if and only if $g_{(t_1,t_2,t_3)} = g_{(t,t,t)}$ for some $t \in \mathbb{R}$. In this case, F is $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant and geodesic orbit.*

(3) *When* $\dim \mathfrak{p} = 2$, $\dim \mathfrak{p}_0 = 1$, $\mathfrak{p}_0 \in \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$. *$F$ has vanishing S-curvature if and only if* $g_{(t_1,t_2,t_3)}|_{\mathfrak{p}} = g_{(t,t,t)}|_{\mathfrak{p}}$ *for some* $t \in \mathbb{R}$. *In this case, both* $g_{(t_1,t_2,t_3)}$ *and F are* $\mathrm{Sp}(n+1)\mathrm{U}(1)$*-invariant and geodesic orbit.*

(4) *When* $\dim \mathfrak{p} = 2$, $\dim \mathfrak{p}_0 = 1$, $\mathfrak{p}_0 \notin \{\mathbb{R}\mathbf{i}G_1, \mathbb{R}\mathbf{j}G_1, \mathbb{R}\mathbf{k}G_1\}$, $\mathfrak{q} = \mathfrak{p}^\perp = \mathfrak{m}_1 + \mathfrak{p}_0$. *$F$ has vanishing S-curvature if and only if* $g_{(t_1,t_2,t_3)} = g_{(t,t,t)}$ *for some* $t \in \mathbb{R}$. *In this case, F is* $\mathrm{Sp}(n+1)\mathrm{U}(1)$*-invariant and geodesic orbit.*

(5) *When* $\mathfrak{p} = \mathfrak{m}_0, \mathfrak{q} = \mathfrak{m}_1, n \geq 1$. *$F$ has vanishing S-curvature for all* $t_1, t_2, t_3 > 0$.

As a conclusion, we obtain a complete description of invariant (α_1, α_2)-metrics on spheres with vanishing S-curvature.

Theorem 5.4. *Let F be an invariant non-Riemannian* (α_1, α_2)*-metric on a sphere* S^n *with* $n > 1$. *Then we have*:

(1) *When the connected isometry group* $I_0(\mathrm{S}^n, F)$ *is not isomorphic to* $\mathrm{Sp}(k)$ *for any* $k \geq 1$, *F has vanishing S-curvature.*

(2) *When the connected isometry group* $I_0(\mathrm{S}^n, F)$ *is isomorphic to* $\mathrm{Sp}(k)$ *for some* $k \geq 1$, *F has vanishing S-curvature can only be occurred in the case* (5) *in Theorem 5.3.*

6. Proof of Theorem 1.1

To prove Theorem 1.1, we need following results.

Proposition 6.1. *Assume an* $\mathrm{Sp}(n+1)$*-invariant* (α_1, α_2)*-metric F on* $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ *is a geodesic orbit metric with respect to* $\mathrm{Sp}(n+1)$. *Then F can be expressed as an* (α_1, α_2)*-metric induced by an invariant Riemannian metric* $g_{(t_1,t_2,t_3)}$ *with* $t_1 = t_2 = t_3$.

Proof. By the discussion in the above section, we have already proved this statement for the cases (I)-(IV).

For the case (V), $\mathfrak{p} = \mathfrak{m}_0, \mathfrak{q} = \mathfrak{m}_1$. For every $X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 \in \mathfrak{m}_0$, there exists an $A \in \mathfrak{sp}(n)$ such that $X + A$ is a geodesic vector. By Theorem 3.6, for all $Y = y_1\mathbf{i}G_1 + y_2\mathbf{j}G_1 + y_3\mathbf{k}G_1 \in \mathfrak{m}_0$, we have

$$\begin{aligned}
0 &= g_{(t_1,t_2,t_3)}([X + A, Y]_{\mathfrak{m}}, X) \\
&= g_{(t_1,t_2,t_3)}([X, Y]_{\mathfrak{m}}, X) \\
&= 2\sqrt{2}[y_1x_2x_3(t_2 - t_3) - y_2x_1x_3(t_1 - t_3) + y_3x_1x_2(t_1 - t_2)].
\end{aligned}$$

This shows that

$$x_2x_3(t_2 - t_3) = x_1x_3(t_1 - t_3) = x_1x_2(t_1 - t_2) = 0,$$

hence $t_1 = t_2 = t_3$. This completes the proof of this conclusion. □

The following result is in fact Lemma 3 in [18].

Lemma 6.2. *For any* $X \in \mathfrak{m}_0, Y \in \mathfrak{m}_1$, *there exists* $A \in \mathfrak{sp}(n)$ *such that* $[A, Y] = [X, Y]$.

Proof. In Lie algebra $\mathfrak{sp}(n+1)$, we write E_{ij} for the skew-symmetric matrix with 1 in the (i,j)-th entry and -1 in the (j,i)-th entry, and zeros elsewhere. We also denote by F_{ij} the symmetric matrix with 1 in both the (i,j)-th and the (j,i)-th entry, and zeros elsewhere. It is easy to check that the matrices $\mathbf{i}G_i$, $\mathbf{j}G_i$, $\mathbf{k}G_i$, E_{ij}, $\mathbf{i}F_{ij}$, $\mathbf{j}F_{ij}$, $\mathbf{k}F_{ij}$, where $1 \leq i,j \leq n+1$ and $i<j$, constitute a basis of $\mathfrak{sp}(n+1)$.

Suppose $X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 \in \mathfrak{m}_0$, $Y = Y_2 + Y_3 + \cdots + Y_{n+1} \in \mathfrak{m}_1$, where $Y_s = y_{s0}E_{1s} + y_{s1}\mathbf{i}F_{1s} + y_{s2}\mathbf{j}F_{1s} + y_{s3}\mathbf{k}F_{1s}$, $2 \leq s \leq n+1$. Let $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $y_s = y_{s0} + y_{s1}\mathbf{i} + y_{s2}\mathbf{j} + y_{s3}\mathbf{k}$. For any $2 \leq s \leq n+1$ satisfying $y_s = 0$, we choose a vector $A_s \in \mathrm{span}\{\mathbf{i}G_s, \mathbf{j}G_s, \mathbf{k}G_s\} \subseteq \mathfrak{sp}(n)$, it is obvious that $[A_s, Y_s] = [X, Y_s]$. For the others s satisfying $y_s \neq 0$, we let $a_s = y_s^{-1}xy_s$ and $A_s = a_sG_s \in \mathfrak{sp}(n)$ (see the proof of Lemma 3 in [18]). As $y_sa_s = xy_s$, one has $[A_s, Y_s] = [X, Y_s]$. Now let $A = \sum_{s=2}^{n+1} A_s \in \mathfrak{sp}(n)$, then

$$[A, Y] = \sum_{s=2}^{n+1}[A_s, Y] = \sum_{s=2}^{n+1}[A_s, Y_s] = \sum_{s=2}^{n+1}[X, Y_s] = [X, \sum_{s=2}^{n+1} Y_s] = [X, Y],$$

which completes the proof. □

Now we can state the proof of Theorem 1.1.

Proof of Theorem 1.1. Keep notation as above.

The "only if" part. Following Proposition 6.1, we assume the geodesic orbit (α_1, α_2)-metric F on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is induced by an $\mathrm{Sp}(n+1)$-invariant Riemannian metric $g_{(t,t,t)}$. For the case (V), F is clearly $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant.

For the cases (I)-(IV): $\{\mathfrak{p}, \mathfrak{q}\} \neq \{\mathfrak{m}_0, \mathfrak{m}_1\}$. Notice that $[\mathfrak{p} \cap \mathfrak{m}_0, \mathfrak{q} \cap \mathfrak{m}_0]_{\mathfrak{m}_0} \neq 0$, we can choose unit vectors $X_\mathfrak{p} \in \mathfrak{p} \cap \mathfrak{m}_0$, $X_\mathfrak{q} \in \mathfrak{q} \cap \mathfrak{m}_0$ such that $[X_\mathfrak{p}, X_\mathfrak{q}]_{\mathfrak{m}_0} \neq 0$. Let $X_\theta = \sqrt{1-\theta^2}X_\mathfrak{p} + \theta X_\mathfrak{q} \subseteq \mathfrak{m} \setminus (\mathfrak{p} \cup \mathfrak{q})$ be a family of unit vectors in $\mathfrak{m}_0$, $\theta \in (0,1)$.

By assumption, there exist $A_\theta \in \mathfrak{sp}(n)$ such that $X_\theta + A_\theta$ are geodesic vectors. According to Theorem 3.6, for all $Y \in \mathfrak{m}_0$,

$$\begin{aligned} 0 = g_{(t,t,t)}\Bigg(& [X_\theta + A_\theta, Y]_\mathfrak{m}, (\varphi(\theta) - \theta\varphi'(\theta))\sqrt{1-\theta^2}X_\mathfrak{p} \\ & + \Big(\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta)\Big)\theta X_\mathfrak{q}\Bigg) \\ = {} & g_{(t,t,t)}\left([X_\theta, Y]_\mathfrak{m}, (\varphi(\theta) - \theta\varphi'(\theta))X_\theta + \varphi'(\theta)X_\mathfrak{q}\right) \\ = {} & -\varphi'(\theta)g_{(t,t,t)}([X_\theta, X_\mathfrak{q}]_\mathfrak{m}, Y) \\ = {} & -\sqrt{1-\theta^2}\varphi'(\theta)g_{(t,t,t)}([X_\mathfrak{p}, X_\mathfrak{q}]_\mathfrak{m}, Y). \end{aligned}$$

We have $\sqrt{1-\theta^2}\varphi'(\theta) = 0$, that is, $\varphi(\theta)$ is a constant function. This implies that F is a Riemannian metric and hence $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant by Theorem 1 in [18].

The "if" part. According to Theorem 1 in [18], we assume F is an $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant non-Riemmanian (α_1, α_2)-metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$. In this case, F can also be expressed as a $\mathrm{Sp}(n+1)$-invariant non-Riemmanian (α_1, α_2)-metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ induced by a Riemannian metric $g_{(t,t,t)}$ for some $t \in \mathbb{R}$ and the decomposition $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$ with $\mathfrak{p} = \mathfrak{m}_0$, $\mathfrak{q} = \mathfrak{m}_1$. Notice that $g_{(t,t,t)}$ is naturally reductive, by Theorem 3.6, every vector $X \in \mathfrak{m}_0 \cup \mathfrak{m}_1$ is a geodesic vector. Now for any $X \in \mathfrak{m} \setminus (\mathfrak{m}_0 \cup \mathfrak{m}_1)$, write $X = X_0 + X_1$, $X_0 \in \mathfrak{m}_0$, $X_1 \in \mathfrak{m}_1$. By Lemma 6.2, there exists $A \in \mathfrak{sp}(n)$ such that

$$\begin{aligned} &\left[\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta)\right][A, X_1] + \theta^{-1}\varphi'(\theta)[X_0, X_1] = 0, \\ &\theta = \sqrt{\frac{g_{(t,t,t)}(X_1, X_1)}{g_{(t,t,t)}(X, X)}} \in (0,1). \end{aligned}$$

Now for all $Y \in \mathfrak{m}$,

$$\begin{aligned} &g_{(t,t,t)}\left([X+A, Y]_{\mathfrak{m}}, (\varphi(\theta) - \theta\varphi'(\theta))X_0 + \left(\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta)\right)X_1\right) \\ = {}& g_{(t,t,t)}\left([X, Y]_{\mathfrak{m}}, (\varphi(\theta) - \theta\varphi'(\theta))X + \theta^{-1}\varphi'(\theta)X_1\right) \\ &+ g_{(t,t,t)}\left([A, Y]_{\mathfrak{m}}, (\varphi(\theta) - \theta\varphi'(\theta))X_0 + \left(\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta)\right)X_1\right) \\ = {}& g_{(t,t,t)}\left(-\theta^{-1}\varphi'(\theta)[X_0, X_1]_{\mathfrak{m}}, Y\right) \\ &+ g_{(t,t,t)}\left(-\left[\varphi(\theta) - (\theta - \theta^{-1})\varphi'(\theta)\right][A, X_1]_{\mathfrak{m}}, Y\right) \\ = {}& 0. \end{aligned}$$

This implies that $X + A$ is a geodesic vector. So F is a geodesic orbit metric with respect to $\mathrm{Sp}(n+1)$. This completes the proof of Theorem 1.1. □

Combining Theorem 1 in [18] and Theorem 1.1 in [21], we have the following theorem.

Theorem 6.3. *Let F be an invariant Finsler metric on a sphere S^n with $n > 1$, we have*:

(1) *F is a geodesic orbit metric if and only if $I_0(\mathrm{S}^n, F)$ is not isomorphic to $\mathrm{Sp}(k)$ for any $k \geq 1$.*

(2) *When the connected isometry group $I_0(\mathrm{S}^n, F)$ is isomorphic to $\mathrm{Sp}(k+1)\mathrm{Sp}(1)$ for some $k \geq 0$, F is a geodesic orbit (α_1, α_2)-metric with respect to $\mathrm{Sp}(k+1)$.*

Finally, using the function f described as in $(*)$ in Section 2, we can present infinitely many non-Riemannian (α_1, α_2)-metrics on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ $(n \geq 0)$ with vanishing S-curvature which are not geodesic orbit, as follows.

$$\begin{aligned} &F_{(t_1,t_2,t_3)}(X) \\ = {}& \sqrt{\mathrm{Re}(\xi\bar{\xi}') + t_1x_1^2 + t_2x_2^2 + t_3x_3^2 + \varepsilon\left((\mathrm{Re}(\xi\bar{\xi}'))^k + (t_1x_1^2 + t_2x_2^2 + t_3x_3^2)^k\right)^{\frac{1}{k}}} \end{aligned}$$

for all $X = x_1\mathbf{i}G_1 + x_2\mathbf{j}G_1 + x_3\mathbf{k}G_1 + \begin{pmatrix} 0 & \xi \\ -\bar{\xi}' & 0 \end{pmatrix} \in \mathfrak{m}_0 + \mathfrak{m}_1$, where $t_1 \neq t_2$, $t_1 \neq t_3$, $t_2 \neq t_3$, ε is a positive number and $k \geq 2$ is a positive integer.

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