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HOMOLOGY AND SERRE CLASS IN D(R)

Zhicheng Wang

Abstract. Let $\mathcal{S}$ be a Serre class in the category of modules and $\mathfrak{a}$ an ideal of a commutative Noetherian ring R. We study the containment of Tor modules, Koszul homology and local homology in $\mathcal{S}$ from below. With these results at our disposal, by specializing the Serre class to be Noetherian or zero, a handful of conclusions on Noetherianness and vanishing of the foregoing homology theories are obtained. We also determine when $\mathrm{Tor}^R_{s+t}(R/\mathfrak{a}, X) \cong \mathrm{Tor}^R_s(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_t(X))$.

Introduction

Throughout this paper, R is a commutative Noetherian ring with identity and $\mathfrak{a}$ is an ideal of R. The proofs of some results concerning local homology and cohomology modules indicate that these proofs apply to certain subcategories of R-modules that are closed under taking extensions, submodules and quotients. These subcategories are called Serre classes.

An excursion among the results [5, Proposition 1 and Corollary 1], [10, Lemma 4.2] and [12, Theorem 2.1] revealed a connection between local homology, local cohomology, Ext modules, Tor modules and Koszul (co)homology in terms of their containment in a Serre class of modules. Aghapournahr, Melkersson and Tousi [1, 2] approached the study of local cohomology modules by means of the Serre subcategories, and it is noteworthy that their approach enables us to deal with several important problems on local cohomology modules comprehensively. Faridian [7, 8] brought the local homology into play and uncovered the connection between all these homology and cohomology theories, and enhanced the aforementioned results.

The aim of this paper is to extend the ideas of Faridian to complexes of modules. We study the containment of Tor modules, Koszul homology and local homology in $\mathcal{S}$ from below. This connection has provided a common language for expressing some results regarding the usual width of modules and complexes that have appeared in different papers. With these results at our disposal, a handful of conclusions on Noetherianness and vanishing of the

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foregoing homology theories are obtained. For an R-complex X in $\mathrm{D}_{\sqsupset}(R)$, we also determine when the R-modules $\mathrm{Tor}^R_{s+t}(R/\mathfrak{a}, X)$ and $\mathrm{Tor}^R_s(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_t(X))$ are isomorphic.

1. Preliminaries

This section collects some notions of complexes and Serre classes for use throughout this paper. For terminology we shall follow [2], [3], [4], [9] and [11].

Complexes. By an R-complex X we mean a sequence of R-modules

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

The derived category $\mathrm{D}(R)$ is defined as the localization of the homotopy category $\mathrm{K}(R)$ with respect to the multiplicative system of quasi-isomorphisms. An R-complexes X is called bounded above if $\mathrm{H}_n(X) = 0$ for $n \gg 0$, bounded below if $\mathrm{H}_n(X) = 0$ for $n \ll 0$, and bounded if it is both bounded above and bounded below. The full triangulated subcategories consisting of bounded above, bounded below and bounded R-complexes are denoted by $\mathrm{D}_{\sqsubset}(R), \mathrm{D}_{\sqsupset}(R)$ and $\mathrm{D}_{\square}(R)$. We also denote by $\mathrm{D}^{\mathrm{f}}(R)$ the full triangulated subcategory of $\mathrm{D}(R)$ consisting of R-complexes X such that $\mathrm{H}_i(X)$ are finitely generated R-modules for all $i \in \mathbb{Z}$. For an R-complex $X \in \mathrm{D}(R)$, set

$$\inf X := \inf\{n \in \mathbb{Z} \,|\, \mathrm{H}_n(X) \neq 0\}, \quad \sup X := \sup\{n \in \mathbb{Z} \,|\, \mathrm{H}_n(X) \neq 0\}.$$

Let X and Y be two R-complexes. For every $i \in \mathbb{Z}$, let

$$\mathrm{Tor}^R_i(X, Y) := \mathrm{H}_i(X \otimes^{\mathrm{L}}_R Y).$$

For an element x in R, denote by $K(x)$ the complex $0 \to R \xrightarrow{x} R \to 0$ concentrated in degrees 1 and 0. The Koszul complex on a sequence $\boldsymbol{x} = \{x_1, \dots, x_n\}$ is the complex

$$K(\boldsymbol{x}) = K(x_1) \otimes_R \cdots \otimes_R K(x_n).$$

We write $\mathrm{Spec}R$ for the set of prime ideals of R and $\mathrm{Max}R$ for the set of maximal ideals of R, and set

$$\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}R \,|\, \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The support and annihilators for $X \in \mathrm{D}(R)$ are defined by uniting/intersecting the corresponding sets for the homology modules

$$\mathrm{Supp}_R X := \bigcup_{\ell \in \mathbb{Z}} \mathrm{Supp}_R \mathrm{H}_\ell(X) = \{\mathfrak{p} \in \mathrm{Spec}R \,|\, X_{\mathfrak{p}} \not\simeq 0\},$$

$$\mathrm{Ann}_R X := \bigcap_{\ell \in \mathbb{Z}} \mathrm{Ann}_R \mathrm{H}_\ell(X) = \{r \in R \,|\, r\mathrm{H}(X) = 0\}.$$

$\mathfrak{a}$-adic completion. Let M be an R-module. The $\mathfrak{a}$-adic completion of M is

$$\Lambda^{\mathfrak{a}}(M) = \varprojlim M/\mathfrak{a}^t M$$

with inverse system $\{M/\mathfrak{a}^{t+1}M \twoheadrightarrow M/\mathfrak{a}^tM\}_{t>0}$. The map $M \to \Lambda^{\mathfrak{a}}(M)$ extends to an additive functor on the category of complexes of R-modules. This functor admits a left derived functor that we denote $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$ following [11], that can be computed by $\mathrm{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(F)$, where $F \xrightarrow{\simeq} X$ is a semi-flat resolution of X. For each R-complex X and integer i, the ith derived completion of X with respect to $\mathfrak{a}$ is the R-module

$$\mathrm{H}_i^{\mathfrak{a}}(X) := \mathrm{H}_i(\mathrm{L}\Lambda^{\mathfrak{a}}(X)).$$

A class $\mathcal{S}$ of R-modules is said to be a Serre class if for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, one has $M \in \mathcal{S}$ if and only if $M', M'' \in \mathcal{S}$. A property $\mathcal{P}$ concerning modules is said to be a Serre property if

$$\mathcal{S}_{\mathcal{P}}(R) := \{M \in \mathrm{Mod}R \,|\, M \text{ satisfies the property } \mathcal{P}\}$$

is a Serre class for every ring R.

Example 1.1 ([7, Example 2.2] or [8, Example 2.2.2])**.** Let $\mathfrak{a}$ be an ideal of R. The following classes of modules are Serre classes:

(1) The zero class.
(2) The class of all Noetherian R-modules.
(3) The class of all artinian R-modules.
(4) The class of all minimax R-modules.
(5) The class of all minimax and $\mathfrak{a}$-cofinite R-modules.
(6) The class of all weakly Laskerian R-modules.
(7) The class of all Matlis reflexive R-modules.
(8) The class of all semi-discrete linearly compact R-modules.

2. Containment in Serre properties

For an arbitrary X in $\mathrm{D}_{\sqsupset}(R)$, we study the containment of homology $-\otimes_R^{\mathrm{L}} X$, Koszul homology and local homology $\mathrm{H}_i^{\mathfrak{a}}(X)$ in $\mathcal{S}$ from below, and determine when the R-module $\mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X)$ is isomorphic to the R-module $\mathrm{Tor}_s^R(R/\mathfrak{a}, \mathrm{H}_t^{\mathfrak{a}}(X))$. The crucial step to achieve these is to recruit the technique of spectral sequences.

Let X be in $\mathrm{D}_{\sqsupset}(R)$ and N an R-module with $\mathrm{Supp}_R N \subseteq \mathrm{V}(\mathfrak{a})$. Then there exists a Künneth spectral sequences

$$(*) \qquad E_{p,q}^2 = \mathrm{Tor}_p^R(N, \mathrm{H}_q^{\mathfrak{a}}(X)) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^R(N, X).$$

The next lemma is one of the principal results in this work.

Lemma 2.1. *Let X be in $\mathrm{D}_{\sqsupset}(R)$ and $s \geqslant 0, t \geqslant \inf X$ such that*

(1) $\mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$;
(2) $\mathrm{Tor}_{s+1+i}^R(R/\mathfrak{a}, \mathrm{H}_{t-i}^{\mathfrak{a}}(X))$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $1 \leqslant i \leqslant t - \inf X$;
(3) $\mathrm{Tor}_{s-1-i}^R(R/\mathfrak{a}, \mathrm{H}_{t+i}^{\mathfrak{a}}(X))$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $1 \leqslant i \leqslant s - 1$.

Then the R-module $\mathrm{Tor}_s^R(R/\mathfrak{a}, \mathrm{H}_t^{\mathfrak{a}}(X))$ *belongs to* $\mathcal{S}_{\mathcal{P}}(R)$.

Proof. Consider the spectral sequence $(*)$. If $s=0$, there exists a finite filtration

$$0=U^{-1}\subseteq U^0\subseteq\cdots\subseteq U^{t-\inf X}=\mathrm{Tor}_t^R(R/\mathfrak{a},X)$$

such that $U^p/U^{p-1}\cong E^\infty_{p,t-p}$ for $t\geqslant p+\inf X$. Let $r\geqslant 2$. Consider the differential

$$E^r_{r,t-r+1}\xrightarrow{d^r_{r,t-r+1}}E^r_{0,t}\xrightarrow{d^r_{0,t}}E^r_{-r,t+r-1}=0.$$

We obtain the following short exact sequence

$$0\to \mathrm{Im}d^r_{r,t-r+1}\to E^r_{0,t}\to E^{r+1}_{0,t}\to 0.$$

Since $t-r+1\leqslant t-1$ and $E^r_{r,t-r+1}$ is a subquotient of $E^2_{r,t-r+1}$ and $E^2_{r,t-r+1}\in\mathcal{S}_\mathcal{P}(R)$ by the condition (2), it follows that $\mathrm{Im}d^r_{r,t-r+1}\in\mathcal{S}_\mathcal{P}(R)$ for $r\geqslant 2$. By the condition (1), $E^r_{0,t}\cong E^\infty_{0,t}\cong U^0/U^{-1}\in\mathcal{S}_\mathcal{P}(R)$ for $r\gg 0$. By using the above sequence inductively, one has $R/\mathfrak{a}\otimes\mathrm{H}^\mathfrak{a}_t(X)\cong E^2_{0,t}\in\mathcal{S}_\mathcal{P}(R)$. The proof of the case $s=1$ is similar to $s=0$. Assume $s\geqslant 2$. Consider the following filtration

$$0=U^{-1}\subseteq U^0\subseteq\cdots\subseteq U^{s+t-\inf X}=\mathrm{Tor}_{s+t}^R(R/\mathfrak{a},X),$$

where $U^p/U^{p-1}\cong E^\infty_{p,s+t-p}$ for $s+t\geqslant p+\inf X$. Let $r\geqslant 2$. Consider the differential

$$E^r_{s+r,t-r+1}\xrightarrow{d^r_{s+r,t-r+1}}E^r_{s,t}\xrightarrow{d^r_{s,t}}E^r_{s-r,t+r-1}.$$

Since $E^r_{s-r,t+r-1}=0$ for $r\geqslant s+1$ and $E^r_{s+r,t-r+1}=0$ for $r\geqslant t-\inf X+2$, it follows from the conditions that $\mathrm{Im}d^r_{s+r,t-r+1}$ and $\mathrm{Im}d^r_{s,t}$ are in $\mathcal{S}_\mathcal{P}(R)$ for $r\geqslant 2$. Let $r\geqslant s+1$. Then $E^r_{s-r,t+r-1}=0$. So we have a short exact sequence

$$0\to\mathrm{Im}d^r_{s+r,t-r+1}\to E^r_{s,t}\to E^{r+1}_{s,t}\to 0.$$

Since $E^r_{s,t}\cong E^\infty_{s,t}\cong U^s/U^{s-1}\in\mathcal{S}_\mathcal{P}(R)$ for $r\gg 0$, it follows from the above sequence that $E^{s+1}_{s,t}\in\mathcal{S}_\mathcal{P}(R)$. Thus the following exact sequence

$$0\to\mathrm{Im}d^s_{2s,t-s+1}\to\mathrm{Ker}d^s_{s,t}\to E^{s+1}_{s,t}\to 0$$

implies that $\mathrm{Ker}d^s_{s,t}\in\mathcal{S}_\mathcal{P}(R)$, and the next exact sequence

$$0\to\mathrm{Ker}d^s_{s,t}\to E^s_{s,t}\to\mathrm{Im}d^s_{s,t}\to 0$$

implies that $E^s_{s,t}\in\mathcal{S}_\mathcal{P}(R)$. By repeating this process, we obtain that

$$\mathrm{Tor}_s^R(R/\mathfrak{a},\mathrm{H}^\mathfrak{a}_t(X))\cong E^2_{s,t}\in\mathcal{S}_\mathcal{P}(R).$$

The proof is complete. □

Corollary 2.2. *Let X be in $\mathrm{D}_\sqsupset(R)$ and $t\geqslant\inf X$.*

(1) *If $\mathrm{Tor}_t^R(R/\mathfrak{a},X)\in\mathcal{S}_\mathcal{P}(R)$ and $\mathrm{Tor}^R_{t+1-i}(R/\mathfrak{a},\mathrm{H}^\mathfrak{a}_i(X))\in\mathcal{S}_\mathcal{P}(R)$ for all $i<t$, then $R/\mathfrak{a}\otimes\mathrm{H}^\mathfrak{a}_t(X)$ is in $\mathcal{S}_\mathcal{P}(R)$.*

(2) *If $\mathrm{Tor}_{t+1}^R(R/\mathfrak{a},X)\in\mathcal{S}_\mathcal{P}(R)$ and $\mathrm{Tor}^R_{t+2-i}(R/\mathfrak{a},\mathrm{H}^\mathfrak{a}_i(X))\in\mathcal{S}_\mathcal{P}(R)$ for all $i<t$, then $\mathrm{Tor}_1^R(R/\mathfrak{a},\mathrm{H}^\mathfrak{a}_t(X))$ is in $\mathcal{S}_\mathcal{P}(R)$.*

(3) *If* $\mathrm{Tor}^R_{t+2}(R/\mathfrak{a}, X) \in \mathcal{S}_{\mathcal{P}}(R)$ *and* $\mathrm{Tor}^R_{t+3-i}(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_i(X)) \in \mathcal{S}_{\mathcal{P}}(R)$ *for all* $i < t$*, then* $R/\mathfrak{a} \otimes_R \mathrm{H}^{\mathfrak{a}}_{t+1}(X) \in \mathcal{S}_{\mathcal{P}}(R)$ *if and only if* $\mathrm{Tor}^R_2(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_t(X)) \in \mathcal{S}_{\mathcal{P}}(R)$.

An R-module L is cocyclic if L is a submodule of $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Max}R$. The set $\mathrm{Coass}_R M$ of coassociated prime of M is the set of prime ideals $\mathfrak{p}$ of R such that there exists a cocyclic homomorphic image L of M with $\mathfrak{p} = \mathrm{Ann}_R L$.

Corollary 2.3. *Let t be an integer and $X \in \mathrm{D}_{\sqsupset}(R)$ such that $\mathrm{Tor}^R_i(R/\mathfrak{a}, X)$ is artinian for all i. If $\mathrm{H}^{\mathfrak{a}}_j(X)$ is artinian for all $j < t$, then $R/\mathfrak{a} \otimes \mathrm{H}^{\mathfrak{a}}_t(X)$ and $\mathrm{Tor}^R_1(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_t(X))$ are artinian. In particular, $\mathrm{V}(\mathfrak{a}) \cap \mathrm{Coass}_R \mathrm{H}^{\mathfrak{a}}_t(X)$ is finite.*

Lemma 2.4. *Let M be a finitely generated R-module, s an integer and $X \in \mathrm{D}_{\sqsupset}(R)$ such that $\mathrm{H}_i(X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$. Then $\mathrm{Tor}^R_i(M, X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$.*

Proof. There exists a Künneth spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^R_p(M, \mathrm{H}_q(X)) \underset{p}{\Rightarrow} \mathrm{Tor}^R_{p+q}(M, X).$$

We may assume that $i \geqslant \mathrm{inf}X$. There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{i-\mathrm{inf}X} = \mathrm{Tor}^R_i(M, X)$$

such that $U^p/U^{p-1} \cong E^{\infty}_{p,i-p}$ for $i - p \geqslant \mathrm{inf}X$. Since $E^{\infty}_{p,i-p}$ is a subquotient of $E^2_{p,i-p}$, it follows from [2, Lemma 2.1] that $E^{\infty}_{p,i-p} \in \mathcal{S}_{\mathcal{P}}(R)$ for all $\mathrm{inf}X \leqslant i - p \leqslant s$. A successive use of the short exact sequence

$$0 \to U^{p-1} \to U^p \to U^p/U^{p-1} \to 0,$$

implies that $\mathrm{Tor}^R_i(M, X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$. □

Let $\mathcal{P}$ be a Serre property and $\mathfrak{a}$ an ideal of R. Following [7, Definition 2.5], we say that $\mathcal{P}$ satisfies the condition $D_{\mathfrak{a}}$ if the following statements hold:

(i) If R is $\mathfrak{a}$-adically complete (i.e., $R \cong \Lambda^{\mathfrak{a}}(R) := \hat{R}^{\mathfrak{a}}$) and $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(R)$ for some R-module M, then $\mathrm{H}^{\mathfrak{a}}_0(M) \in \mathcal{S}_{\mathcal{P}}(R)$.

(ii) For any $\mathfrak{a}$-torsion R-module M, we have $M \in \mathcal{S}_{\mathcal{P}}(R)$ if and only if $M \in \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$.

The following theorem that is one of our main results of this paper, yields a characterization of (local) homology modules that are in $\mathcal{S}_{\mathcal{P}}(R)$.

Theorem 2.5. *Let K be the Koszul complex on a sequence of n generators for $\mathfrak{a}$ and s an integer. For each R-complex X in $\mathrm{D}_{\sqsupset}(R)$, the following are equivalent*:

(1) $\mathrm{Tor}^R_i(R/\mathfrak{a}, X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $i \leqslant s$ *(for all i)*;

(2) $\mathrm{Tor}^R_i(R/\mathfrak{b}, X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $i \leqslant s$ *(for all i) and all ideals* $\mathfrak{b} \supseteq \mathfrak{a}$;

(3) $\mathrm{Tor}^R_i(R/\mathfrak{p}, X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $i \leqslant s$ *(for all i) and all* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$;

(4) $\mathrm{Tor}^R_i(R/\sqrt{\mathfrak{a}}, X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $i \leqslant s$ *(for all i)*;

(5) $\mathrm{Tor}_i^R(L,X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for each finitely generated R-module L with* $\mathrm{Supp}_R L \subseteq \mathrm{V}(\mathfrak{a})$ *and all* $i \leqslant s$ *(for all* i*);*

(6) $\mathrm{Tor}_i^R(Y,X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for each R-complex* $Y \in \mathrm{D}_{\square}^{\mathrm{f}}(R)$ *with* $\mathrm{Supp}_R Y \subseteq \mathrm{V}(\mathfrak{a})$ *and all* $i \leqslant s + \mathrm{inf} Y$ *(for all* i*);*

(7) $\mathrm{Tor}_i^R(K,X)$ *is in* $\mathcal{S}_{\mathcal{P}}(R)$ *for all* $i \leqslant s$ *(for all* i*).*

If in addition, $\mathcal{P}$ *satisfies the condition* $D_{\mathfrak{a}}$*, then above conditions are equivalent to*

(8) $\mathrm{H}_i^{\mathfrak{a}}(X)$ *is in* $\mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$ *for all* $i \leqslant s$.

Proof. (1) $\Rightarrow$ (2) Fix $\mathfrak{b} \supseteq \mathfrak{a}$. Since $R/\mathfrak{b} \otimes_R^{\mathrm{L}} X \simeq R/\mathfrak{b} \otimes_{R/\mathfrak{a}}^{\mathrm{L}} R/\mathfrak{a} \otimes_R^{\mathrm{L}} X$, it follows from Lemma 2.4 that $\mathrm{Tor}_i^R(R/\mathfrak{b},X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$ (for all i).

(2) $\Rightarrow$ (3), (5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (3) This follows from the implication (1) $\Rightarrow$ (3) since $\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\sqrt{\mathfrak{a}})$.

(3) $\Rightarrow$ (5) Assume that L is finitely generated with $\mathrm{Supp}_R L \subseteq \mathrm{V}(\mathfrak{a})$. Then there is a prime filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = L$ such that $L_j/L_{j-1} \cong R/\mathfrak{p}_j$ and $\mathfrak{p}_j \in \mathrm{Supp}_R L$ for $j = 1, \ldots, t$. We argue by induction on t. If $t = 1$, then $L = L_1/L_0 \cong R/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. By assumption $\mathrm{Tor}_i^R(L,X) \cong \mathrm{Tor}_i^R(R/\mathfrak{p},X) \in \mathcal{S}_{\mathcal{P}}(R)$. Assume that $\mathrm{Tor}_i^R(L,X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all finitely generated R-modules L with $\mathrm{Supp}_R L \subseteq \mathrm{V}(\mathfrak{a})$ having a prime filtration of length $t-1$. Let L have a prime filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = L$. Consider the short exact sequence $0 \to L_{t-1} \to L \to L/L_{t-1} \to 0$. We obtain the following exact sequence

$$\mathrm{Tor}_i^R(L_{t-1},X) \to \mathrm{Tor}_i^R(L,X) \to \mathrm{Tor}_i^R(L/L_{t-1},X).$$

By the induction hypothesis, one has $\mathrm{Tor}_i^R(L,X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$ (for all i).

(5) $\Rightarrow$ (6) Let $Y \in \mathrm{D}_{\square}^{\mathrm{f}}(R)$ with $\mathrm{Supp}_R Y \subseteq \mathrm{V}(\mathfrak{a})$. We use induction on $\mathrm{sup}Y - \mathrm{inf}Y$. If $\mathrm{inf}Y = \mathrm{sup}Y = r$, then $Y \simeq \Sigma^r \mathrm{H}_r(Y)$ and $\mathrm{Tor}_{i+r}^R(Y,X) \cong \mathrm{Tor}_i^R(\mathrm{H}_r(Y),X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$. Now assume that $\mathrm{sup}Y - \mathrm{inf}Y > 0$. Set $Y' = 0 \to Y_{\mathrm{sup}Y}/\mathrm{Ker}d_{\mathrm{sup}Y} \xrightarrow{\bar{d}_{\mathrm{sup}Y}} Y_{\mathrm{sup}Y-1} \xrightarrow{d_{\mathrm{sup}Y-1}} \cdots$. One obtain an exact triangle $\Sigma^{\mathrm{sup}Y}\mathrm{H}_{\mathrm{sup}Y}(Y) \to Y \to Y' \rightsquigarrow$ in $\mathrm{D}(R)$, which induces the following exact sequence

$$\mathrm{Tor}_i^R(\Sigma^{\mathrm{sup}Y}\mathrm{H}_{\mathrm{sup}Y}(Y),X) \to \mathrm{Tor}_i^R(Y,X) \to \mathrm{Tor}_i^R(Y',X).$$

Therefore, $\mathrm{Tor}_i^R(Y,X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s + \mathrm{inf}Y$ by the induction.

(6) $\Rightarrow$ (7) This follows from $K \in \mathrm{D}_{\square}^{\mathrm{f}}(R)$ and $\mathrm{Supp}_R K \subseteq \mathrm{V}(\mathfrak{a})$.

(7) $\Rightarrow$ (1) One has the following isomorphism

$$R/\mathfrak{a} \otimes_R^{\mathrm{L}} K \otimes_R^{\mathrm{L}} X \simeq (\coprod_{j \geqslant 0} \Sigma^j (R/\mathfrak{a})^{\binom{n}{j}}) \otimes_R^{\mathrm{L}} X.$$

Since $K \otimes_R X \in \mathrm{D}_{\sqsupset}(R)$, it follows from Lemma 2.4 that $\mathrm{Tor}_i^R(R/\mathfrak{a}, K \otimes_R X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$ (for all i). Consequently, $\mathrm{Tor}_i^R(R/\mathfrak{a},X)$ is in $\mathcal{S}_{\mathcal{P}}(R)$ for all $i \leqslant s$ (for all i).

Now assume that $\mathcal{P}$ satisfies the condition $D_{\mathfrak{a}}$. Since $\mathrm{Tor}_i^R(R/\mathfrak{a},X)$ is $\mathfrak{a}$-torsion,

$$\mathrm{Tor}_i^R(R/\mathfrak{a},X)\cong \mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}\hat{R}^{\mathfrak{a}},\hat{R}^{\mathfrak{a}}\otimes_R X)$$

for all i both as R-modules and $\hat{R}^{\mathfrak{a}}$-modules. By [15, Lemma 2.3], one has

$$\mathrm{H}_i^{\mathfrak{a}}(X)\cong \mathrm{H}_i^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}\otimes_R X)$$

for all i both as R-modules and $\hat{R}^{\mathfrak{a}}$-modules.

(8) $\Rightarrow$ (1) First suppose that R is $\mathfrak{a}$-adically complete. By Lemma 2.4, $\mathrm{Tor}_i^R(R/\mathfrak{a},X)\cong \mathrm{Tor}_i^R(R/\mathfrak{a},\mathrm{L}\Lambda^{\mathfrak{a}}(X))\in\mathcal{S}_{\mathcal{P}}(R)$ for all $i\leqslant t$. Now, consider the general case. Since $\mathrm{H}_i^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}\otimes_R X)\cong \mathrm{H}_i^{\mathfrak{a}}(X)\in\mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$ for all $i\leqslant t$, $\mathrm{Tor}_i^R(R/\mathfrak{a},X)\cong \mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}\hat{R}^{\mathfrak{a}},\hat{R}^{\mathfrak{a}}\otimes_R X)\in\mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$ for all $i\leqslant t$ by the preceding proof. But $\mathrm{Tor}_i^R(R/\mathfrak{a},X)$ is $\mathfrak{a}$-torsion, so the condition $D_{\mathfrak{a}}$ implies that $\mathrm{Tor}_i^R(R/\mathfrak{a},X)\in\mathcal{S}_{\mathcal{P}}(R)$ for all $i\leqslant t$.

(1) $\Rightarrow$ (8) First suppose that R is $\mathfrak{a}$-adically complete. We argue by induction on s. If $s=\inf X$, then $R/\mathfrak{a}\otimes_R \mathrm{H}_{\inf X}(X)\cong\mathrm{Tor}^R_{\inf X}(R/\mathfrak{a},X)\in\mathcal{S}_{\mathcal{P}}(R)$. Hence $\mathrm{H}_0^{\mathfrak{a}}(\mathrm{H}_{\inf X}(X))\in\mathcal{S}_{\mathcal{P}}(R)$ by the condition $D_{\mathfrak{a}}$. Let $F\xrightarrow{\simeq}X$ be a semi-flat resolution of X such that $F_i=0$ for $i<\inf X$. Then $\mathrm{H}_0^{\mathfrak{a}}(\mathrm{H}_{\inf X}(X))\cong \mathrm{H}_0(\mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{H}_{\inf X}(X)))$ by definition and

$$\begin{aligned}\Lambda^{\mathfrak{a}}(\mathrm{H}_{\inf X}(F))&=\varprojlim \mathrm{H}_{\inf X}(F)/\mathfrak{a}^t\mathrm{H}_{\inf X}(F)\\&\cong\varprojlim \mathrm{H}_{\inf X}(R/\mathfrak{a}^t\otimes_R F)\\&\cong \mathrm{H}_{\inf X}(\Lambda^{\mathfrak{a}}F)=\mathrm{H}^{\mathfrak{a}}_{\inf X}(X)\end{aligned}$$

by the isomorphism $\mathrm{H}_{\inf X}(F)/\mathfrak{a}^t\mathrm{H}_{\inf X}(F)\cong \mathrm{H}_{\inf X}(R/\mathfrak{a}^t\otimes_R F)$ and the fact that $\{R/\mathfrak{a}^t\otimes_R F\}_{t\geqslant 1}$ is an inverse system of epimorphisms. Since

$$\mathrm{H}_0(\mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{H}_{\inf X}(X)))\to\Lambda^{\mathfrak{a}}(\mathrm{H}_{\inf X}(F))$$

is an epimorphism, it follows that $\mathrm{H}^{\mathfrak{a}}_{\inf X}(X)\in\mathcal{S}_{\mathcal{P}}(R)$. Suppose that $s>\inf X$ and that the result has been proved for smaller values of s. By Corollary 2.2, one obtains that $R/\mathfrak{a}\otimes_R \mathrm{H}_i^{\mathfrak{a}}(X)\in\mathcal{S}(R)$ for all $i\leqslant s$. Hence $\mathrm{H}_i^{\mathfrak{a}}(X)\cong \mathrm{H}_0^{\mathfrak{a}}(\mathrm{H}_i^{\mathfrak{a}}(X))\in\mathcal{S}(R)$ for all $i\leqslant s$. Now, consider the general case. Since $\mathrm{Tor}_i^R(R/\mathfrak{a},X)$ is $\mathfrak{a}$-torsion and $\mathrm{Tor}_i^R(R/\mathfrak{a},X)\in\mathcal{S}_{\mathcal{P}}(R)$ for all $i\leqslant s$, it follows that $\mathrm{Tor}_i^{\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}/\mathfrak{a}\hat{R}^{\mathfrak{a}},\hat{R}^{\mathfrak{a}}\otimes_R X)\in\mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$ for all $i\leqslant s$. So the special case yields that $\mathrm{H}_i^{\mathfrak{a}}(X)\cong \mathrm{H}_i^{\mathfrak{a}\hat{R}^{\mathfrak{a}}}(\hat{R}^{\mathfrak{a}}\otimes_R X)\in\mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})$ for all $i\leqslant s$. $\square$

Corollary 2.6 ([14])**.** *Let $\mathfrak{a}$ be an ideal of R and $X\in\mathrm{D}_{\sqsupset}(R)$. Suppose that R is $\mathfrak{a}$-adically complete. Then the following are equivalent*:

(1) $R/\mathfrak{a}\otimes^{\mathrm{L}}_R X\in\mathrm{D}^{\mathrm{f}}(R)$;

(2) $Y\otimes^{\mathrm{L}}_R X\in\mathrm{D}^{\mathrm{f}}(R)$ *for all* $Y\in\mathrm{D}^{\mathrm{f}}_{\square}(R)$ *with* $\mathrm{Supp}_R Y\subseteq \mathrm{V}(\mathfrak{a})$;

(3) $K(\boldsymbol{x})\otimes^{\mathrm{L}}_R X\in\mathrm{D}^{\mathrm{f}}(R)$ *for some* (*equivalently, for every*) *generating sequence* $\boldsymbol{x}$ *of* $\mathfrak{a}$;

(4) $\mathrm{L}\Lambda^{\mathfrak{a}}(X)\in\mathrm{D}^{\mathrm{f}}(\hat{R}^{\mathfrak{a}})$.

Proof. This follows from the fact that the Serre property of being finitely generated satisfies the condition $D_{\mathfrak{a}}$ by [6, Lemma 2.5]. $\square$

Proposition 2.7. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R, s be an integer and $X \in \mathrm{D}_{\sqsupset}(R)$ such that $\mathrm{Tor}_j^R(R/\mathfrak{b}, \mathrm{H}_i^{\mathfrak{a}}(X)) \in \mathcal{S}_{\mathcal{P}}(R)$ for all i and all j (resp. for $i \leqslant s$ and all j). Then $\mathrm{Tor}_i^R(R/\mathfrak{b}, X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all i (resp. for $i \leqslant s$).*

Proof. We may assume that $s \geqslant \inf X$. Consider the spectral sequence $(*)$. The hypothesis implies that $E^2_{p,q} \in \mathcal{S}_{\mathcal{P}}(R)$ for all p and all q (resp. for $\inf X \leqslant q \leqslant s$ and all p). For all i (resp. for $\inf X \leqslant i \leqslant s$), there is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{i-\inf X} = \mathrm{Tor}_i^R(R/\mathfrak{b}, X)$$

such that $U^p/U^{p-1} \cong E^{\infty}_{p,i-p}$ for $i \geqslant p + \inf X$. Since $E^{\infty}_{p,i-p}$ is a subquotient of $E^2_{p,i-p}$ for all $i-p$ (resp. for $\inf X \leqslant i-p \leqslant s$), it follows that $E^{\infty}_{p,i-p} \in \mathcal{S}$ for $i \geqslant p + \inf X$. A successive use of the short exact sequence

$$0 \to U^{p-1} \to U^p \to U^p/U^{p-1} \to 0$$

implies that $\mathrm{Tor}_i^R(R/\mathfrak{b}, X) \in \mathcal{S}_{\mathcal{P}}(R)$. $\square$

Corollary 2.8. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R and $X \in \mathrm{D}_{\sqsupset}(R)$ such that $\mathrm{H}_j^{\mathfrak{b}}(\mathrm{H}_i^{\mathfrak{a}}(X)) \in \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{b}})$ for all i and j. If $\mathcal{P}$ satisfies the condition $D_{\mathfrak{b}}$, then $\mathrm{H}_i^{\mathfrak{b}}(X) \in \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{b}})$ for all i.*

Proof. Since $\mathrm{H}_j^{\mathfrak{b}}(\mathrm{H}_i^{\mathfrak{a}}(X)) \in \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{b}})$ for all j, $\mathrm{Tor}_j^R(R/\mathfrak{b}, \mathrm{H}_i^{\mathfrak{a}}(X)) \in \mathcal{S}_{\mathcal{P}}(R)$ by Theorem 2.5 for all j. Hence Proposition 2.7 implies that $\mathrm{Tor}_i^R(R/\mathfrak{b}, X) \in \mathcal{S}_{\mathcal{P}}(R)$ for all i. But $\mathcal{P}$ satisfies the condition $D_{\mathfrak{b}}$, so $\mathrm{H}_i^{\mathfrak{b}}(X) \in \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{b}})$ for all i by Theorem 2.5 again. $\square$

Proposition 2.9. *Let K be the Koszul complex on a sequence of n generators for $\mathfrak{a}$ and X an arbitrary R-complex. If the Serre property $\mathcal{P}$ is closed under taking inverse limits, then one has*

$$\begin{aligned} &\inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell^{\mathfrak{a}}(X) \textit{ is not in } \mathcal{S}_{\mathcal{P}}(\hat{R}^{\mathfrak{a}})\} \\ &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{Tor}_\ell^R(R/\mathfrak{a}, X) \textit{ is not in } \mathcal{S}_{\mathcal{P}}(R)\} \\ &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{Tor}_\ell^R(K, X) \textit{ is not in } \mathcal{S}_{\mathcal{P}}(R)\}. \end{aligned}$$

Proof. Let the first quantity be r, the second one be s, and the third one be t.

Set $X_{\geqslant n} : \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \to 0$. Then $\{X_{\geqslant n+1} \twoheadrightarrow X_{\geqslant n}\}_{n \leqslant 0}$ is an inverse system of split epimorphism and $X = \varprojlim X_{\geqslant n}$. So the inverse system obtained from applying any additive covariant functor to $\{X_{\geqslant n+1} \twoheadrightarrow X_{\geqslant n}\}_{n \leqslant 0}$ satisfies the Mittag-Leffler condition, which implies that $\mathrm{H}_i^{\mathfrak{a}}(X) \cong \varprojlim \mathrm{H}_i^{\mathfrak{a}}(X_{\geqslant n})$, $\mathrm{Tor}_i^R(R/\mathfrak{a}, X) \cong \varprojlim \mathrm{Tor}_i^R(R/\mathfrak{a}, X_{\geqslant n})$ and $\mathrm{Tor}_i^R(K, X) \cong \varprojlim \mathrm{Tor}_i^R(K, X_{\geqslant n})$. Consequently, $r = s = t$ by Theorem 2.5, as claimed. $\square$

Corollary 2.10. *Let K be the Koszul complex on a sequence of n generators for $\mathfrak{a}$. For each R-complex X in $\mathrm{D}(R)$, the following are equivalent:*
(1) $\mathrm{Tor}_i^R(K, X) = 0$ *for all* $\inf X \leqslant i \leqslant n + \sup X$;
(2) $\mathrm{Ext}_R^i(R/\mathfrak{a}, X) = 0$ *for all* $-\sup X \leqslant i \leqslant n - \inf X$;
(3) $\mathrm{Tor}_i^R(R/\mathfrak{a}, X) = 0$ *for all* $\inf X \leqslant i \leqslant n + \sup X$;
(4) $\mathrm{H}_i^{\mathfrak{a}}(X) = 0$ *for all* $i \in \mathbb{Z}$;
(5) $\mathrm{H}_{\mathfrak{a}}^i(X) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. Note that $\mathrm{H}_{\mathfrak{a}}^i(X) = 0$ for all $i < -\sup X$ and $i > n - \inf X$ by [13, Corollary 4.28] and $\mathrm{H}_i^{\mathfrak{a}}(X) = 0$ for all $i < \inf X$ and $i > n + \sup X$ by [13, Corollary 5.27]. □

Corollary 2.11. *Let X be an R-complex and $Y \in \mathrm{D}_\square^{\mathrm{f}}(R)$. Then*
(1) $R/\mathrm{Ann}_R Y \otimes_R^{\mathrm{L}} X \simeq 0$ *if and only if* $Y \otimes_R^{\mathrm{L}} X \simeq 0$.
(2) $\mathrm{RHom}_R(R/\mathrm{Ann}_R Y, X) \simeq 0$ *if and only if* $\mathrm{RHom}_R(Y, X) \simeq 0$.

The following theorem find some sufficient conditions for validity of the isomorphism $\mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X) \cong \mathrm{Tor}_s^R(R/\mathfrak{a}, \mathrm{H}_t^{\mathfrak{a}}(X))$.

Theorem 2.12. *Let X be in $\mathrm{D}_\sqsupset(R)$ and $s \geqslant 0, t \geqslant \inf X$ such that*
(1) $\mathrm{Tor}_{s+t-i}^R(R/\mathfrak{a}, X) = 0$ *for all* $\inf X \leqslant i < t$ *or* $t + 1 \leqslant i \leqslant s + t$;
(2) $\mathrm{Tor}_{s+1+i}^R(R/\mathfrak{a}, \mathrm{H}_{t-i}^{\mathfrak{a}}(X)) = 0$ *for all* $0 \leqslant i \leqslant t - \inf X$;
(3) $\mathrm{Tor}_{s-1-i}^R(R/\mathfrak{a}, \mathrm{H}_{t+i}^{\mathfrak{a}}(X)) = 0$ *for all* $0 \leqslant i \leqslant s - 1$.
Then $\mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X) \cong \mathrm{Tor}_s^R(R/\mathfrak{a}, \mathrm{H}_t^{\mathfrak{a}}(X))$.

Proof. Consider the spectral sequence $(*)$. There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{s+t-\inf X} = \mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X)$$

such that $U^p/U^{p-1} \cong E^\infty_{p,s+t-p}$ for $s + t \geqslant p + \inf X$. Let $r \geqslant 2$. Consider the differential

$$E^r_{s+r,t-r+1} \xrightarrow{d^r_{s+r,t-r+1}} E^r_{s,t} \xrightarrow{d^r_{s,t}} E^r_{s-r,t+r-1}.$$

By conditions (2) and (3), we have $E^r_{s+r,t-r+1} = 0 = E^r_{s-r,t+r-1}$ for $r \geqslant 2$. As $E^\infty_{p,s+t-p}$ is a subquotient of $E^2_{p,s+t-p} = \mathrm{Tor}_p^R(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_{s+t-p}(X))$, we have $U^p/U^{p-1} \cong E^\infty_{p,s+t-p} = 0$ for $0 \leqslant p \leqslant s - 1$ by conditions (1) and (3), and $U^s/U^{s-1} \cong E^\infty_{s,t}$ and $U^p/U^{p-1} \cong E^\infty_{p,s+t-p} = 0$ for $s + 1 \leqslant p \leqslant s + t - \inf X$ by conditions (1) and (2), it follows that $0 = U^{-1} = \cdots = U^{s-1}$ and $U^s = \cdots = U^{s+t-\inf X} = \mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X)$. So

$$\mathrm{Tor}_s^R(R/\mathfrak{a}, \mathrm{H}_t^{\mathfrak{a}}(X)) = E^2_{s,t} \cong E^\infty_{s,t} \cong U^s = \mathrm{Tor}_{s+t}^R(R/\mathfrak{a}, X).$$

We get the isomorphism we seek. □

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Zhicheng Wang
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: wangzhch@nwnu.edu.cn