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ON CHOWLA'S HYPOTHESIS IMPLYING THAT $L(s,\chi)>0$ FOR $s>0$ FOR REAL CHARACTERS χ

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Abstract. Let $L(s,\chi)$ be the Dirichlet L-series associated with an f-periodic complex function χ. Let $P(X)\in\mathbb{C}[X]$. We give an expression for $\sum_{n=1}^{f}\chi(n)P(n)$ as a linear combination of the $L(-n,\chi)$'s for $0\leq n<\deg P(X)$. We deduce some consequences pertaining to the Chowla hypothesis implying that $L(s,\chi)>0$ for $s>0$ for real Dirichlet characters χ. To date no extended numerical computation on this hypothesis is available. In fact by a result of R. C. Baker and H. L. Montgomery we know that it does not hold for almost all fundamental discriminants. Our present numerical computation shows that surprisingly it holds true for at least 65% of the real, even and primitive Dirichlet characters of conductors less than 10^6. We also show that a generalized Chowla hypothesis holds true for at least 72% of the real, even and primitive Dirichlet characters of conductors less than 10^6. Since checking this generalized Chowla's hypothesis is easy to program and relies only on exact computation with rational integers, we do think that it should be part of any numerical computation verifying that $L(s,\chi)>0$ for $s>0$ for real Dirichlet characters χ. To date, this verification for real, even and primitive Dirichlet characters has been done only for conductors less than $2\cdot 10^5$.

1. Introduction

It is conjectured that $L(s,\chi)>0$ for $s>0$ for all non-principal real Dirichlet characters χ. By (3), we may assume that χ is primitive. By [11], this conjecture holds true for at least 20% of the odd characters modulo $8d$ associated with the imaginary quadratic fields $\mathbb{Q}(\sqrt{-2d})$ of discriminants $-8d$, where $d>0$ is odd and square-free. Moreover, numerical computations for testing this conjecture have been carried out for odd characters (see [20] for conductors $f\leq 593000$, and [26] for conductors $f\leq 3\cdot 10^8$). In fact, one proves that $\zeta_K(s)<0$ for $0<s<1$, where K is the imaginary quadratic number field associated with such a character. In contrast, for even characters numerical

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computations have been carried out only up to much smaller bounds: for conductors less than or equal to 227 in [22] and [23], less than $2 \cdot 10^5$ in [10] and less than $4 \cdot 10^5$ in [21].

1.1. Chowla's method and the Fekete polynomial

Let $L(s,\chi)$ be the meromorphic continuation to the complex plane of the L-series $L(s,\chi) := \sum_{n\geq 1}\chi(n)n^{-s}$, $\Re(s) > 1$, attached to an f-periodic real-valued function χ. It has only one pole, at $s = 1$, a simple pole of residue $\chi_1(f) := \sum_{n=1}^{f}\chi(n)$ (e.g. see the proof of Theorem 4.1 below). We will often assume that $\chi_1(f) = 0$, which is the case whenever χ is a real non-principal Dirichlet character modulo $f \geq 3$. In that situation, $L(s,\chi)$ is therefore an entire function. Set $\chi_0 = \chi$ and for $r \geq 0$, define inductively χ_r by

$$\chi_{r+1}(n) = \sum_{k=1}^{n}\chi_r(k) \qquad (n \geq 1)$$

and set

$$m(\chi) := \min\{r \geq 1 : \chi_r(n) \geq 0 \text{ for all } n \geq 1\} \tag{1}$$

with the convention $\min(\emptyset) = \infty$.

Since

$$F(t,\chi) := \sum_{n\geq 1}\chi(n)t^n = (1-t)^N\Big(\sum_{n\geq 1}\chi_N(n)t^n\Big) \qquad (0 \leq t < 1)$$

and

$$\Gamma(s)L(s,\chi) = \int_0^\infty F(e^{-t},\chi)t^{s-1}\mathrm{d}t \qquad (\Re(s) > 0),$$

we obtain that

$$m(\chi) < \infty \text{ implies } L(s,\chi) > 0 \text{ for } s > 0 \tag{2}$$

(see [4]). Since χ is f-periodic, we have

$$F(t,\chi) = \frac{P(t,\chi)}{1-t^f}, \text{ where } P(t,\chi) := \sum_{n=1}^{f}\chi(n)t^n.$$

Then,

$$m(\chi) = \infty \text{ if and only if } P(t,\chi) = 0 \text{ for some } t \in (0,1),$$

by [3, Lemma 6]. By [2] and [15], the *Fekete polynomial* $P(t,\chi)$ has a large number of zeros in $(0,1)$ for almost all real primitive characters. But as our computation after Proposition 1.1 show, at least 65% of the Fekete polynomials $P(t,\chi)$'s have no zero in $(0,1)$ for χ real, even, primitive and of conductor $\leq 10^6$, as $m(\chi) \leq 20$. Using Sturm's algorithm with Maple on a micro-computer, we obtained in 30 minutes of computation that $m(\chi) = \infty$ for exactly 9 out of the 153 real, even, non-principal and primitive Dirichlet characters of conductors $f \leq 500$, and 27 out of the 153 real, odd non-principal and primitive characters of conductors $f \leq 500$. We also checked in 3 hours of computation that $m(\chi) <$

∞ for the 2 real, even and primitive Dirichlet characters modulo 1277 and 1973 considered in [3, Section 6]. However, the degree of $P(t,\chi)$ becomes rapidly too large to use Sturm's algorithm to test whether $m(\chi)=\infty$. Indeed, it took us 8 hours and 30 minutes to obtain that $m(\chi)=\infty$ for exactly 28 out of the 302 real, even, non-principal and primitive Dirichlet characters of conductors $f\leq 1000$, the ones given in Table 2.

Let χ be a real character modulo $f\equiv 0 \pmod 4$. Then, $n\equiv(1+\frac{f}{2})(n+\frac{f}{2}) \pmod f$ for $n\in\mathbb{Z}$ odd. Hence $\chi(n+\frac{f}{2})=\varepsilon\chi(n)$ for $n\in\mathbb{Z}$, where $\varepsilon:=\chi(1+\frac{f}{2})$. Since $(1+f/2)^2\equiv 1 \pmod f$, we have $\varepsilon\in\{\pm 1\}$, and $\varepsilon=-1$ whenever χ is primitive. Hence $P(t,\chi)=(1+\varepsilon t^{f/2})tQ(t^2,\chi)$ and $m(\chi)=\infty$ if and only if $Q(t,\chi)=0$ for some $t\in(0,1)$, where $Q(t,\chi)=\sum_{n=0}^{f/4-1}\chi(2n+1)t^n$ is of degree four times smaller than the one of $P(t,\chi)$. We obtained in less than five minutes of computation that $m(\chi)=\infty$ for exactly 12 out of the 103 real, even, non-principal and primitive Dirichlet characters of even conductors less than or equal to 1000, and 22 out of the 101 real, odd and primitive Dirichlet characters of even conductors less than or equal to 1000.

Let χ denote a real non-principal character. It is known that $m(\chi)=\infty$ for infinitely many χ's, see [14]. In fact, if $\chi(2)=\chi(3)=\chi(5)=\chi(7)=\chi(11)=-1$, then $m(\chi)=\infty$, see [3]. Hence, $m(\chi)=\infty$ for at least 3.125% of the χ's. It is also known that $m(\chi)=\infty$ whenever $L(1,\chi)$ is small enough, e.g. if $L(1,\chi)\leq 1-\log 2=0.306852\dots$ (see [17], [18] and [19]). By [7], there is a positive proportion of χ's for which this holds true. For example, let χ be the real odd character modulo 163 associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-163})$ of class number equal to 1. Then $L(1,\chi)=\pi/\sqrt{163}$ is less than $1-\log(2)$ and $\chi(p)=-1$ for $2\leq p\leq 37$ and p prime. Hence, Chowla's method does not apply for proving that $L(s,\chi)>0$ for this character and B. Rosser had a hard time in [23] to prove that $L(s,\chi)>0$ for this character.

Appart from Sturm's algorithm, we do not know another algorithm for testing whether $m(\chi)<\infty$ nor one for computing $m(\chi)$. Point 2 of Proposition 1.1 to be proved in Section 3 provides us with a simple procedure to decide whether $\chi_N\geq 0$. Let χ run over the $N(B)$ real, even and primitive Dirichlet characters of conductors $f\leq B$. Hence, χ is well determined by its conductor. Using UBASIC on a PC Optiplex 780 with Intel Core 2 Duo E7500, 2.93 Ghz, we computed the number $N_{20}(B)$ of χ's for which $\chi_k(f)\geq 0$ for $1\leq k<20$ and $\chi_{20}(n)\geq 0$ for $1\leq n\leq f$, in which case $L(s,\chi)>0$ for $s>0$, by Proposition 1.1. The time needed to complete these computations are denoted by T_{20}:

B	10^2	10^3	10^4	10^5	10^6
$N(B)$	30	302	3043	30394	303957
$N_{20}(B)$	30	273	2451	21886	197899
$\rho_{20}(B)=100N_{20}(B)/N(B)$	100	$90.397\dots$	$80.545\dots$	$72.007\dots$	$65.107\dots$
T_{20}	$0sec$	$0sec$	$7sec$	$11mn13sec$	$17h56mn07sec$

Proposition 1.1. *Let χ be a real valued f-periodic function.*

(1) *χ_N is f-periodic if and only if $\chi_k(f)=0$ for $1\leq k\leq N$.*

(2) *If* $\chi_k(f) \geq 0$ *for* $1 \leq k < N$, *then* $\chi_N \geq 0$ *if and only if* $\chi_N(n) \geq 0$ *for* $1 \leq n \leq f$. *Notice that if this holds true for some* $N_0 \geq 1$, *then it holds true for all* $N \geq N_0$.

(3) *If* (i) $\chi_k(f) = 0$ *for* $1 \leq k < N_0$, (ii) $\chi_{N_0}(f) \neq 0$ *and* $m(\chi) < \infty$, *then* $\chi_{N_0}(f) > 0$.

Corollary 1.2. *Let* χ *be a real and non-principal Dirichlet character modulo* f. *Then,* $\chi_1(f) = 0$ *whenever* χ *is odd and* $\chi_1(f) = \chi_2(f) = 0$ *whenever* χ *is even. Hence,* χ_1 *is* f*-periodic,* $\chi_1 \geq 0$ *if and only if* $\chi_1(n) \geq 0$ *for* $1 \leq n \leq (f-1)/2$ *and* $\chi_2 \geq 0$ *if and only if* $\chi_2(n) \geq 0$ *for* $1 \leq n \leq f$. *Now, assume that* χ *is even. Then,* (i) *we cannot have* $\chi_1 \geq 0$, (ii) χ_2 *is* f*-periodic,* (iii) $\chi_2 \geq 0$ *if and only if* $\chi_2(n) \geq 0$ *for* $1 \leq n \leq (f-2)/2$, *and* (iv) $\chi_3 \geq 0$ *if and only if* $\chi_3(n) \geq 0$ *for* $1 \leq n \leq f$.

Proof. For the well known first assertion, see Corollary 4.2. Now, assume that χ is even. Since $\chi_1(f) = 0$, we have

$$\chi_1(f-1-n) = \chi_1(f) - \sum_{k=f-n}^{f} \chi(k) = -\sum_{k=0}^{n} \chi(f-k) = -\chi_1(n)$$

for $0 \leq n \leq f-1$. Hence $\chi_1(f-2) = -1$. Since $\chi_1(f) = \chi_2(f) = 0$, we have $\chi_2(f-1) = \chi_2(f) - \chi_1(f) = 0$, χ_2 is f-periodic (Proposition 1.1), $\chi_2 \geq 0$ if and only if $\chi_2(n) \geq 0$ for $0 \leq n \leq f-2$. Finally,

$$\chi_2(f-n-2) = \chi_2(f-1) - \sum_{k=f-1-n}^{f-1} \chi_1(k) = -\sum_{k=0}^{n} \chi_1(f-1-k) = \chi_2(n)$$

for $0 \leq n \leq f-2$. □

1.2. Using induced characters

See [5,6,8,9,16,23]. Let χ' be a non-principal real character modulo $f' = df$ induced by a character χ modulo f dividing f'. Then

$$L(s,\chi') = L(s,\chi) \times \prod_{p \text{ prime and } p \mid d} \left(1 - \frac{\chi(p)}{p^s}\right). \tag{3}$$

Hence, $L(s,\chi') > 0$ for $s > 0$ if and only if $L(s,\chi) > 0$ for $s > 0$. In particular, if $m(\chi') < \infty$ for some induced character, then $L(s,\chi) > 0$ for $s > 0$. The non-principal primitive real characters are of conductors $f = |D| > 1$, where $D \in \mathbb{Z}$ is a *fundamental discriminant*, i.e., either $D \equiv 1 \pmod 4$ is square-free or $D \equiv 8, 12 \pmod{16}$ and $D/4$ is square-free. For such a D there is exactly one such primitive real Dirichlet character. It is even if $D > 0$ and odd if $D < 0$. For example, let χ be the real, even and primitive Dirichlet character modulo $f = 173$. Then $m(\chi) = \infty$. However, $m(\chi'') = 82 < \infty$ for the induced character χ'' modulo $f'' = 10f = 1730$. In Table 2 in Section 8, for each of the 28 real, even and primitive Dirichlet characters χ of conductors $f \leq 1000$ for which $m(\chi) = \infty$ we give an induced character χ'' for which $m(\chi'') < \infty$. It follows that $L(s,\chi) > 0$ for $s > 0$ for all the positive discriminants less than 1000.

Lemma 1.3 (See [19, Lemma 6.1]). *Let χ be a non-principal real Dirichlet character modulo f. Let χ' be the character modulo $f' = pf$ induced by χ, where $p \geq 2$ is prime.*

(i) *If $\chi(p) = -1$, then $m(\chi) < \infty$ implies $m(\chi') < \infty$.*
(ii) *If $\chi(p) = 0$, then $m(\chi) = m(\chi')$.*
(iii) *If $\chi(p) = +1$, then $m(\chi) = \infty$ implies $m(\chi') = \infty$.*

Proof. (i) If $\chi(p) = 0$, then $\chi'(n) = \chi(n)$ for $n \in \mathbb{Z}$, hence $F(t,\chi') = F(t,\chi)$. Now assume that $\chi(p) \neq 0$. Then $\chi'(n) = 0$ if p divides n and $\chi'(n) = \chi(n)$ otherwise. Therefore,

$$F(t,\chi') = \sum_{n\geq 1} \chi'(n)t^n = \sum_{n\geq 1} \chi(n)t^n - \sum_{\substack{n\geq 1 \\ p\mid n}} \chi(n)t^n = F(t,\chi) - \chi(p)F(t^p,\chi).$$

(ii) If $\chi(p) = -1$ and $m(\chi) < \infty$, then $F(t,\chi) > 0$ for $t \in (0,1)$ and $F(t,\chi') = F(t,\chi) + F(t^p,\chi) > 0$ for $t \in (0,1)$ and $m(\chi') < \infty$.

(iii) Finally, assume that $\chi(p) = +1$ and $m(\chi) = \infty$. Let $t_\chi \in (0,1)$ be the smallest zero in $(0,1)$ of $F(t,\chi)$. We have $F(t,\chi) > 0$ in $(0,t_\chi)$, hence $F(t_\chi,\chi') = -F(t_\chi^p,\chi) < 0$ and $m(\chi') = \infty$. □

According to this lemma we will restrict ourselves to **d**-*induced characters.* A non-principal real Dirichlet character χ modulo $f > 1$ is called d induced, or d-induced by ψ, if (i) d divides f, (ii) χ is induced by some primitive Dirichlet character ψ modulo f/d and (iii) $d \in E(\psi) := \{d \geq 1 : d \text{ is square-free and } \psi(p) = -1 \text{ for any prime } p \geq 2 \text{ dividing } d\}$.

Conjecture 1.4 (Generalized Chowla Hypothesis). *Let ψ be a real primitive Dirichlet character modulo $f_\psi > 1$. There exists some d-induced character χ modulo $f = df_\psi$ and some $N \geq 1$ such that $\chi_r(f) \geq 0$ for $1 \leq r < N$ and $\chi_N(n) \geq 0$ for $1 \leq n \leq f$.*

The Generalized Chowla Hypothesis holds true for all the real, even and primitive Dirichlet characters of conductors less than 10^3 and for at least 73% of those of conductors less than 10^6, see Section 8. For the real primitive characters ψ of conductor 1277 for which $m(\psi) = 766$ or 1973 for which $m(\psi) = 567$ considered in [3], their 2-induced characters χ satisfy the Generalized Chowla Hypothesis with $N = 8$, and $m(\chi) = 8$ in both cases.

2. Statements of the results to be proved

Theorem 2.1. *Let χ be a real odd character modulo f. Then, $\chi_1(f) = 0$, $\chi_2(f) \geq 0$ and $\chi_3(f) \geq 0$. Hence, if $1 \leq N \leq 4$, then $\chi_N(n) \geq 0$ for all $n \geq 0$ if and only if $\chi_N(n) \geq 0$ for $0 \leq n \leq f$. Moreover, if χ is primitive, then $\chi_2(f) > 0$ and $\chi_3(f) > 0$, but there are infinitely many real, odd and primitive* *characters χ's of prime conductors $f = p \equiv 3 \pmod 4$ for which $\chi_4(f) < 0$ and infinitely many of them for which $\chi_4(f) > 0$.*

Theorem 2.2. *Let χ be a real, even and non-principal character modulo f. Then, $\chi_1(f) = \chi_2(f) = 0$. Moreover, $\chi_3(f) > 0$ whenever χ is primitive. However, if ψ is a real primitive Dirichlet character modulo $f_\psi > 1$ and p is a prime such that $\psi(p) = +1$, then $\chi_3(f) < 0$ for the character χ modulo $f = pf_\psi$ induced by ψ (and $m(\chi) = \infty$, by Proposition 1.1).*

Theorem 2.3. *Let χ be a real and even Dirichlet character modulo $f > 2$. Assume that χ is primitive or that χ d-induced by a primitive character. Then, $\chi_1(f) = \chi_2(f) = 0$ and $\chi_N(f) > 0$ for $3 \leq N \leq 13$. Hence, if $2 \leq N \leq 14$, then $\chi_N(n) \geq 0$ for all $n \geq 0$ if and only if $\chi_N(n) \geq 0$ for $0 \leq n \leq f$. However, there are infinitely many real, even and primitive Dirichlet characters χ's of prime conductors $f = p \equiv 1 \pmod 4$ for which $\chi_{14}(f) < 0$ and infinitely many real, even and primitive Dirichlet characters χ's of prime conductors $f = p \equiv 1 \pmod 4$ for which $\chi_{14}(f) > 0$.*

3. Proof of Proposition 1.1

Lemma 3.1. *Set $P_0(X) = 1$ and $P_k(X) = X(X+1)\cdots(X+k-1)/k!$ for $k \geq 1$. Let $\chi : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be f-periodic. Then*

$$\chi_N(n+f) = \chi_N(n) + \sum_{k=0}^{N-1} P_k(n)\chi_{N-k}(f) \quad (\textit{for } N \geq 1 \textit{ and } n \geq 1). \tag{4}$$

Consequently, χ_N is f-periodic if and only if $\chi_k(f) = 0$ for $1 \leq k \leq N$.

Proof. For $N = 1$ and $n \geq 1$, we do have

$$\chi_1(n+f) = \chi_1(f) + \sum_{k=1}^{n} \chi(k+f) = \chi_1(f) + \chi_1(n) = \chi_1(n) + P_0(n)\chi_1(f).$$

By induction on $N \geq 1$ we then obtain

$$\begin{aligned} \chi_{N+1}(n+f) &= \chi_{N+1}(f) + \sum_{l=1}^{n} \chi_N(l+f) \\ &= \chi_{N+1}(f) + \sum_{l=1}^{n} \left(\chi_N(l) + \sum_{k=0}^{N-1} P_k(l)\chi_{N-l}(f) \right) \\ &= \chi_{N+1}(f) + \chi_{N+1}(n) + \sum_{k=0}^{N-1} \left(\sum_{l=1}^{n} P_k(l) \right) \chi_{N-l}(f). \end{aligned}$$

Clearly, $P_k(l) = P_{k+1}(l) - P_{k+1}(l-1)$ for $k \geq 0$ and $l \geq 1$. Hence, $\sum_{l=1}^{n} P_k(l) = P_{k+1}(n)$ for $n \geq 1$ and $k \geq 0$. Therefore,

$$\chi_{N+1}(n+f) = \chi_{N+1}(f) + P_0(n)\chi_{N+1}(n) + \sum_{k=0}^{N-1} P_{k+1}(n)\chi_{N+1-(k+1)}(f)$$

$$= \chi_{N+1}(f) + \sum_{k=0}^{N} P_k(n)\chi_{N+1-k}(f)$$

and (4) follows. Consequently, χ_N is f-periodic if and only if $P(n)=0$ for $n\geq 1$, hence if and only if $P(X)=0$, where $P(X) := \sum_{k=0}^{N} P_k(X)\chi_{N+1-k}(f)$. Since $\deg P_k(X)=k$, the $P_k(X)$'s form a $\mathbb{Q}$-basis of $\mathbb{Q}[X]$ and $P(X)=0$ if and only if $\chi_{N+1-k}(f)=0$ for $0\leq k\leq N-1$. □

The first and second points of Proposition 1.1 follow. Let us prove its third point. If $\chi_N\geq 0$, then $\chi_{N'}\geq 0$ for $N'\geq N$. Hence, if $m(\chi)<\infty$ there exists $N\geq N_0$ such that $\chi_N\geq 0$. Using (4), by induction on $m\geq 1$, we have

$$\chi_N(mf) = \sum_{k=0}^{N-N_0}\left(\sum_{l=1}^{m-1} P_k(lf)\right)\chi_{N-k}(f) \qquad (m\geq 1).$$

Since

$$\sum_{l=1}^{m-1} l^n = \frac{m^{n+1}}{n+1} + \cdots$$

is a polynomial in m of degree $n+1$, it follows that

$$\sum_{l=1}^{m-1} P_k(lf) = \frac{f^k}{(k+1)!}m^{k+1} + \cdots$$

is a polynomial in m of degree $k+1$. Therefore,

$$P(m) := \chi_N(mf) = \chi_{N_0}(f)\frac{f^{N-N_0}}{(N-N_0+1)!}m^{N-N_0+1} + \cdots$$

is a polynomial in m of degree $N-N_0+1$. If $\chi_N\geq 0$, then $P(m)\geq 0$ for $m\geq 1$ and this leading coefficient must me nonnegative, which yields the desired third point.

Finally, let us prove its fourth point. Set $X(1)=1$ and $X(n)=-1$ for $n\geq 2$. Since $\chi\geq X$, we have $\chi_r\geq X_r$. Since

$$X_r(n) = \frac{r+1-n}{r-1+n}\binom{n+r-1}{r}$$

(see [18, (2.1)]), we have $X_{f-1}(n)\geq 0$ for $1\leq n\leq f$.

4. A formula for $\sum_{n=1}^{f}\chi(n)P(n)$

Theorem 4.1. *Let $0\neq P(X)\in\mathbb{C}[X]$. Let $f\geq 1$ be an integer. Let $L(s,\chi)$ be the meromorphic continuation of the L-series $L(s,\chi) := \sum_{n\geq 1}\chi(n)n^{-s}$ attached to an f-periodic function $\chi:\mathbb{Z}_{n\geq 1}\to\mathbb{C}$. Then,*

$$S(P(X),\chi) := \sum_{n=1}^{f} P(n)\chi(n)$$

$$= \frac{\chi_1(f)}{f}\int_0^f P(t)\mathrm{d}t - \sum_{n\geq 0}\frac{P^{(n)}(f)-P^{(n)}(0)}{n!}L(-n,\chi)$$

(*this sum is finite, as we can disregard the indices* $n \geq \deg P(X)$). *In particular, for* χ *a non-principal Dirichlet character modulo* f *and for* $0 \neq P(X) \in \mathbb{C}[X]$ *we have*

$$S(P(X),\chi) = -\sum_{\substack{0\leq n<\deg P(X)\\ n\equiv\delta_\chi \pmod 2}}\frac{P^{(n)}(f)-P^{(n)}(0)}{n!}L(-n,\chi),$$

where $\delta_\chi = \frac{1+\chi(-1)}{2} \in \{0,1\}$.

Proof. Both sides being linear in $P(X)$, it suffices to prove this identity for $P(X) = 1$, which is clear, and for $P(X) = X^N$ with $N \geq 1$. Let $N \geq 1$ be a positive integer. Let us prove that

$$S(X^N,\chi) := \sum_{n=1}^{f}\chi(n)n^N = \frac{f^N}{N+1}\chi_1(f) - f^N\sum_{k=0}^{N-1}\binom{N}{k}f^{-k}L(-k,\chi).$$

Let $B_n(X)$, $n \geq 0$, be the Bernoulli polynomials defined by

$$\frac{te^{Xt}}{e^t-1} = \sum_{n\geq 0}B_n(X)\frac{t^n}{n!}. \tag{5}$$

Let

$$\zeta(s,b) = \sum_{n\geq 0}\frac{1}{(n+b)^s} \qquad (\Re(s)>1,\ 0<b\leq 1)$$

be the Hurwitz zeta function. We have (see [25, Theorem 4.2]):

$$\zeta(1-m,b) = -B_m(b)/m \qquad (m\geq 1). \tag{6}$$

Now, by f-periodicity of χ, we have

$$L(s,\chi) = f^{-s}\sum_{n=1}^{f}\chi(n)\zeta(s,n/f) \qquad (\Re(s)>1). \tag{7}$$

Hence, $L(s,\chi)$ admits a meromorphic continuation to the complex plane, with only one pole, at $s = 1$, a simple pole of residue $\chi_1(f)$. By (6) and (7), it follows that

$$\begin{aligned}
-f^N\sum_{k=0}^{N-1}\binom{N}{k}f^{-k}L(-k,\chi) &= -f^N\sum_{k=0}^{N-1}\binom{N}{k}\sum_{n=1}^{f}\chi(n)\zeta(-k,n/f)\\
&= -f^N\sum_{n=1}^{f}\chi(n)\sum_{k=0}^{N-1}\binom{N}{k}\frac{-B_{k+1}(n/f)}{k+1}\\
&= f^N\sum_{n=1}^{f}\frac{\chi(n)}{N+1}\sum_{k=1}^{N}\binom{N+1}{k}B_k(n/f).
\end{aligned}$$

However, by equating the coefficients of X^N in the identity

$$\sum_{n\geq 0} X^n\frac{t^{n+1}}{n!} = te^{Xt} = \Big(\sum_{n\geq 0} B_n(X)\frac{t^n}{n!}\Big)(e^t-1) = \Big(\sum_{n\geq 0} B_n(X)\frac{t^n}{n!}\Big)\Big(\sum_{n\geq 1}\frac{t^n}{n!}\Big),$$

for $N\geq 1$ we obtain

$$X^N = \frac{1}{N+1}\sum_{k=0}^{N}\binom{N+1}{k}B_k(X) = \frac{1}{N+1}+\frac{1}{N+1}\sum_{k=1}^{N}\binom{N+1}{k}B_k(X).$$

This completes the proof of the first assertion.

For $n\geq 0$, we have $B_n(1-X)=(-1)^nB_n(X)$, by changing t into $-t$ in (5). Hence, by (6) and (7), for $n\geq 0$ we have

$$\begin{aligned} L(-n,\chi) &= -\frac{f^n}{n+1}\sum_{m=1}^{f-1}\chi(m)B_{n+1}\left(\frac{m}{f}\right) \\ &= -\frac{f^n}{n+1}\sum_{m=1}^{f-1}\chi(f-m)B_{n+1}\left(\frac{f-m}{f}\right) \\ &= -\chi(-1)(-1)^{n+1}L(-n,\chi) \end{aligned}$$

and $L(-n,\chi)=0$ for $(-1)^n\neq\chi(-1)$, i.e., for $n\not\equiv\delta_\chi \pmod 2$. The second assertion follows, noticing that $\chi_1(f)=0$. $\square$

Corollary 4.2. *For $1\leq n\leq N-1$, we set $Q_N(X)=(X+1)(X+2)\cdots(X+N)$ and*

$$R_{N,n}(X) := \frac{Q_N^{(n)}(X)-Q_N^{(n)}(0)}{\frac{N!}{(N-n)!}}\in\mathbb{Q}[X],$$

a monic polynomial of degree $N-n\geq 1$ with constant term equal to 0.

For $N\geq 2$ and χ a non-principal Dirichlet character modulo f, we have

$$\begin{aligned} \chi_N(f) &= \frac{\chi(-1)}{(N-1)!}S(Q_{N-1}(X),\chi) \\ &= -\chi(-1)\sum_{\substack{0\leq n\leq N-2\\ n\equiv\delta_\chi \pmod 2}}\frac{R_{N-1,n}(f)}{(N-1-n)!}\frac{L(-n,\chi)}{n!}. \end{aligned} \tag{8}$$

Consequently, if χ is induced by a real and primitive Dirichlet character ψ modulo f_ψ dividing f, then

$$\chi_N(f) = \frac{f^{N-1}\sqrt{f/d}}{\pi}\sum_{\substack{0\leq n\leq N-2\\ n\equiv\delta_\chi \pmod 2}}\frac{(-1)^{\frac{n-\delta_\chi}{2}}}{(N-1-n)!}a_n(\chi)\frac{R_{N-1,n}(f)}{f^{N-1-n}}\frac{L(n+1,\psi)}{(2\pi)^n}, \tag{9}$$

where $d:=f/f_\psi$ and

$$a_n(\chi) := \frac{1}{d^n}\prod_{p\mid d}(1-\psi(p)p^n)$$

(hence, $d = 1$ and $a_n(\chi) = 1$ if χ is primitive). Finally, for $n \leq N-1$ it holds that $R_{N,n}(X)$ is monic and divisible in $\mathbb{Q}[X]$ by $X+N+1$ whenever $n \equiv N \pmod 2$.

Proof. By induction on $N \geq 0$, we have

$$\sum_{n\geq 1}\chi(n)x^n = (1-x)^N \sum_{n\geq 1}\chi_N(n)x^n.$$

Hence,

$$\begin{aligned}\sum_{n\geq 1}\chi_N(n)x^n &= \left\{\frac{1}{(N-1)!}\frac{d^{N-1}}{dx^{N-1}}\left(\frac{1}{1-x}\right)\right\}\sum_{n\geq 1}\chi(n)x^n\\ &= \frac{1}{(N-1)!}\big(\sum_{n\geq 0}(n+1)(n+2)\cdots(n+N-1)x^n\big)\big(\sum_{n\geq 1}\chi(n)x^n\big).\end{aligned}$$

Identifying the coefficients of x^n of both sides, we get

$$\chi_N(n) = \frac{1}{(N-1)!}\sum_{i=0}^{n-1}Q_{N-1}(i)\chi(n-i) \qquad (N\geq 1,\ n\geq 1),$$

and (8) follows by taking $n = f$.

Since ψ is a real and primitive Dirichlet character, using the functional equation of the L-function $L(s,\psi)$ (e.g., see [25, Chapter 4]), noticing that its root number W_ψ is equal to $+1$ and setting $\delta_\psi = \frac{1+\psi(-1)}{2} \in \{0,1\}$ we have:

$$L(-n,\psi) = n!\cdot\cos\Big(\pi\frac{n+\delta_\psi}{2}\Big)\frac{\sqrt{f_\psi}}{\pi}\left(\frac{f_\psi}{2\pi}\right)^n L(n+1,\psi).$$

By (3), we have $L(-n,\chi) = d^n a_n(\chi)L(-n,\psi)$. Since $\delta_\psi = \delta_\chi$, we obtain

$$\frac{L(-n,\chi)}{n!} = \left\{\prod_{p\mid d}(1-\psi(p)p^n)\right\}\cos\Big(\pi\frac{n+\delta_\chi}{2}\Big)\frac{\sqrt{f_\psi}}{\pi}\left(\frac{f_\psi}{2\pi}\right)^n L(n+1,\psi) \tag{10}$$

(which implies $L(-n,\chi) = 0$ if $n \geq 0$ and $n \not\equiv \delta_\chi \pmod{2}$). Hence, we have

$$\frac{L(-n,\chi)}{n!} = (-1)^{\frac{n+\delta_\chi}{2}}a_n(\chi)\frac{\sqrt{f/d}}{\pi}\left(\frac{f}{2\pi}\right)^n L(n+1,\psi) \text{ for } n\equiv\delta_\chi \pmod{2},$$

and (9) follows. Finally, $Q_N(X-N-1) = (-1)^N Q_N(-X)$ gives $R_{N,n}(-N-1) = \frac{(N-n)!}{N!}((-1)^{N-n}-1)Q_N^{(n)}(0)$, and the last result follows. □

In particular, for χ an odd and primitive Dirichlet character modulo f, by Theorem 4.1 and (10) we have

$$S(X^k,\chi) := \sum_{n=1}^{f}n^k\chi(n) = -\sum_{\substack{n=0\\ n\text{ even}}}^{k-1}\binom{k}{n}f^{k-n}L(-n,\chi)$$

$$= -\frac{f^k\sqrt{f}}{\pi}\sum_{\substack{n=0\\ n\text{ even}}}^{k-1}(-1)^{n/2}\frac{k!}{(k-n)!}\frac{L(n+1,\chi)}{(2\pi)^n},$$

and we recover [24, formula for $S(k)$, p. 65] (let us take this opportunity to mention a misprint in the formula for $S(3)$ at the bottom of [1, p. 153]. This misprint is repeated in [24, (1-3)]).

If χ is a real, odd and primitive Dirichlet character modulo f, we have

$$\chi_4(f) = \frac{f^3\sqrt{f}}{6\pi}\left\{\left(1+\frac{6}{f}+\frac{11}{f^2}\right)L(1,\chi) - \frac{3}{2\pi^2}L(3,\chi)\right\}. \tag{11}$$

Notice also that $L(n+1,\psi) > 0$ for $n \geq 0$ and ψ real and primitive.

5. Proof of Theorem 2.1

Let χ be an odd Dirichlet character modulo f induced by a primitive character ψ of conductor f_ψ. We have $\delta_\chi = (1+\chi(-1))/2 = 0$. By Corollary 4.2, we have

$$\chi_2(f) = fL(0,\chi) \text{ and } \chi_3(f) = \frac{f^2+3f}{2}L(0,\chi) = \frac{f+3}{2}\chi_2(f).$$

By (10), we have

$$L(0,\chi) = \frac{\sqrt{f_\psi}}{\pi}L(1,\psi)\prod_{p\mid f}(1-\psi(p)) \geq 0.$$

Hence, $\chi_1(f) = 0$, $\chi_2(f) \geq 0$ and $\chi_3(f) \geq 0$. Moreover, $\chi_2(f) = \chi_3(f) = 0$ if and only if χ is not primitive and $\psi(p) = +1$ for some prime p dividing f. If χ is primitive modulo f, using (11) and arguing as in [1] or [24] (see also [13]), we obtain the last assertion of Theorem 2.1.

6. Proof of Theorem 2.2

Let χ be a real, non-principal and even Dirichlet character modulo f induced by a primitive Dirichlet character ψ of conductor f_ψ. Set $d = f/f_\psi$. We have $\delta_\chi = (1+\chi(-1))/2 = 1$. By Corollary 4.2, we have $\chi_1(f) = \chi_2(f) = 0$,

$$\chi_3(f) = -fL(-1,\chi) \text{ and } \chi_4(f) = -\frac{f^2+4f}{2}L(-1,\chi) = \frac{f+4}{2}\chi_3(f).$$

By (10) we have

$$-L(-1,\chi) = \frac{f_\psi^{3/2}}{2\pi^2}L(2,\psi)\prod_{p\mid f}(1-p\psi(p)) \neq 0.$$

The second point of Theorem 2.2 follows.

7. Proof of Theorem 2.3

7.1. Proof of the first assertion of Theorem 2.3

Proposition 7.1. *For a given integer N in the range $3 \leq N \leq 13$, there exists an explicit constant $C_N > 0$ such that for any real, even and non-principal d-induced Dirichlet character χ modulo f we have*

$$\chi_N(f) \geq C_N \frac{f^{N-1}\sqrt{f/d}}{30 \cdot (N-2)!} > 0.$$

Proof. We use (9) with $\delta_\chi = 1$ and notice that

$$a_n(\chi) = a_n(d) := \prod_{p|d}(1 + \frac{1}{p^n}) = 1 + \sum_{\substack{\delta|d\\ \delta>1}} \frac{1}{\delta^n}.$$

We obtain

$$\chi_N(f) = \frac{f^{N-1}\sqrt{f/d}}{2\pi^2 \cdot (N-2)!}\frac{\zeta(4)}{\zeta(2)} C_N(f,d,\psi) = \frac{f^{N-1}\sqrt{f/d}}{30 \cdot (N-2)!} C_N(f,d,\psi),$$

where

$$C_N(f,d,\psi) = \sum_{1\leq n<N/2} (-1)^{n-1} a_{2n-1}(d) \frac{(N-2)!}{(N-2n)!} P_{N,n}(f) \frac{\zeta(2)L(2n,\psi)}{(4\pi^2)^{n-1}\zeta(4)} \tag{12}$$

with

$$P_{N,n}(f) = \frac{R_{N-1,2n-1}(f)}{f^{N-2n}} = 1 + O(\frac{1}{f})$$

(recall that $R_{N,n}(X)$ is monic of degree $N-n$). We have to prove that for each $N \in \{3, \dots, 13\}$ we can find some $C_N > 0$ such that $C_N(f,d,\psi) \geq C_N$ for any real, even and non-principal Dirichlet character χ modulo f as in the third point of Theorem 2.2.

Notice that the coefficients of $R_{N,k}(X) \in \mathbb{Q}[X]$ are non-negative and that

$$R_{N,k+1}(X) = \frac{R'_{N,k}(X) - R'_{N,k}(0)}{N-k} \qquad (0 \leq k \leq N-2).$$

It follows that the coefficient of X^{i-1} of $R_{N,k+1}(X)$ is less than or equal to the coefficient of X^i of $R_{N,k}(X)$. Therefore, we have $R_{N,k+1}(x)/x^{N-k-1} \leq R_{N,k}(x)/x^{N-k}$ for $x \geq 1$ and

$$1 \leq P_{N,n+1}(f) \leq P_{N,n}(f) \qquad (1 \leq n < N/2 - 1).$$

Throughout the proof we set

$$\mathbb{Q} \ni r_n := \frac{1}{(4\pi^2)^{n-1}} \times \begin{cases} \frac{\zeta(2)\zeta(4n)}{\zeta(4)\zeta(2n)} & \text{if } n \text{ is odd,} \\ \frac{\zeta(2)\zeta(2n)}{\zeta(4)} & \text{if } n \text{ is even.} \end{cases}$$

1. The case $3 \leq N \leq 8$. Set

$$\mathbb{Q} \ni \kappa_N := \sum_{1\leq n<N/2} (-1)^{n-1} \frac{(N-2)!}{(N-2n)!} r_n.$$

Hence, $\kappa_3=1$, $\kappa_4=1$, $\kappa_5=3/4$, $\kappa_6=1/2$, $\kappa_7=282/1001$ and $\kappa_8=381/4004$, but $\kappa_N<0$ for $9\leq N\leq 13$.

Since

$$\frac{\zeta(4n)}{\zeta(2n)}=\prod_{p\geq 2}\frac{1}{1+\frac{1}{p^{2n}}}\leq L(2n,\psi)=\prod_{p\geq 2}\frac{1}{1-\frac{\psi(p)}{p^{2n}}}\leq\prod_{p\geq 2}\frac{1}{1-\frac{1}{p^{2n}}}=\zeta(2n), \tag{13}$$

recalling (12) we have

$$\begin{aligned}C_N(f,d,\psi)&\geq C_N(f,d)\\ &:=\sum_{1\leq n<N/2}(-1)^{n-1}r_na_{2n-1}(d)\frac{(N-2)!}{(N-2n)!}P_{N,n}(f).\end{aligned} \tag{14}$$

Now, if the A_n's are non-negative and satisfy $A_{n+1}\leq 4A_n$ for $1\leq n<N/2-1$ odd, in using

$$0\leq a_{2(n+1)-1}(d)-1=\sum_{\substack{\delta\mid f\\ \delta>1}}\frac{1}{\delta^{2n+1}}\leq\frac{1}{4}\sum_{\substack{\delta\mid f\\ \delta>1}}\frac{1}{\delta^{2n-1}}=\frac{1}{4}(a_{2n-1}(d)-1)$$

we obtain

$$\begin{aligned}&\sum_{1\leq n<N/2}(-1)^{n-1}(a_{2n-1}(d)-1)A_n\\ \geq&\sum_{\substack{1\leq n<N/2-1\\ n\text{ odd}}}\big((a_{2n-1}(d)-1)A_n-(a_{2(n+1)-1}(d)-1)A_{n+1}\big)\\ \geq&\sum_{\substack{1\leq n<N/2-1\\ n\text{ odd}}}(a_{2n-1}(d)-1)\big(A_n-\frac{1}{4}A_{n+1}\big)\geq 0\end{aligned}$$

and

$$\sum_{1\leq n<N/2}(-1)^{n-1}a_n(d)A_n\geq\sum_{1\leq n<N/2}(-1)^{n-1}A_n,$$

with equality for $d=1$, i.e., for χ primitive.

In our situation we have $A_n=r_n\frac{(N-2)!}{(N-2n)!}P_{N,n}(f)$ and $1\leq P_{N,n+1}(f)\leq P_{N,n}(f)$. It follows that for n odd and $1\leq n<N/2-1$ we have

$$\begin{aligned}\frac{A_{n+1}}{A_n}&\leq\frac{(N-2n)(N-2n-1)r_{n+1}}{r_n}\\ &=(N-2n)(N-2n-1)\frac{\zeta(2n)\zeta(2n+2)}{4\pi^2\zeta(4n)}\in\mathbb{Q}.\end{aligned}$$

Hence, in the range $N\leq 12$, we do have $A_{n+1}\leq 4A_n$ for $1\leq n<N/2-1$ odd (for $N=13$ and $n=1$ we have $A_2/A_1=110\frac{r_2}{r_1}\frac{P_{13,2}(f)}{P_{13,1}(f)}=\frac{55}{12}+O(f^{-1})$). Hence, recalling (14), for $3\leq N\leq 12$ we have

$$C_N(f,d,\psi)\geq C_N(f,d)\geq C_N(f):=\sum_{1\leq n<N/2}(-1)^{n-1}r_n\frac{(N-2)!}{(N-2n)!}P_{N,n}(f).$$

Now, using any software for algebraic computation, we used Maple, the reader can easily check that for $3 \leq N \leq 8$, the $C_N(f)$'s are linear combinations of $1, f^{-1}, \dots, f^{-(N-3)}$ with non-negative rational coefficients. This is not true for $N = 9$. Consequently, for $3 \leq N \leq 8$ we do have $C_N(f) \geq C_N(+\infty) = \kappa_N$. Indeed, we have

$$C_3(f) = P_{3,1}(f) = 1,\ C_4(f) = P_{4,1}(f) = 1 + \frac{4}{f},\ C_5(f) = \frac{3}{4} + \frac{15}{2f} + \frac{35}{2f^2},$$

$$C_6(f) = \left(1 + \frac{6}{f}\right) \times \left(\frac{1}{2} + \frac{6}{f} + \frac{15}{f^2}\right),\ C_7(f) = \frac{282}{1001} + \frac{35}{4f} + \frac{175}{2f^2} + \frac{735}{2f^3} + \frac{1624}{3f^4},$$

and

$$C_8(f) = \left(1 + \frac{8}{f}\right) \times \left(\frac{381}{4004} + \frac{6}{f} + \frac{67}{f^2} + \frac{304}{f^3} + \frac{469}{f^4}\right)$$

(See Corollary 4.2 for an explanation of the factorisations of the $C_N(f)$ for N even).

2. The case $9 \leq N \leq 12$. Now, let $\mathcal{P} = \{p_1, \dots, p_m\}$ be a given finite set of $m \geq 1$ prime integers. For $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_m) \in \{-1, 0, 1\}^m$, set

$$\Pi_n(\mathcal{P}, \vec{\epsilon}, -1) := \prod_{p \in \mathcal{P}} \frac{p^{2n} - 1}{p^{2n} - \epsilon_k} \text{ and } \Pi_n(\mathcal{P}, \vec{\epsilon}, +1) := \prod_{p \in \mathcal{P}} \frac{p^{2n} + 1}{p^{2n} - \epsilon_k}.$$

Therefore, $\Pi_n(\mathcal{P}, \vec{\epsilon}, -1) \leq 1 \leq \Pi_n(\mathcal{P}, \vec{\epsilon}, +1)$. We have

$$\left\{\prod_{p \in \mathcal{P}} \frac{p^{2n} + 1}{p^{2n} - \psi(p)}\right\} \frac{\zeta(4n)}{\zeta(2n)} \leq L(2n, \psi) \leq \left\{\prod_{p \in \mathcal{P}} \frac{p^{2n} - 1}{p^{2n} - \psi(p)}\right\} \zeta(2n),$$

an improvement on (13). Therefore, for the choice $\vec{\epsilon} = (\psi(p_1), \dots, \psi(p_m))$, recalling (12) we have $C_N(f, d, \psi) \geq \lambda_N(f, d, \mathcal{P}, \vec{\epsilon})$, where

$$\begin{aligned} &\lambda_N(f, d, \mathcal{P}, \vec{\epsilon}) \\ := &\sum_{1 \leq n < N/2} (-1)^{n-1} r_n a_{2n-1}(d) \frac{(N-2)!}{(N-2n)!} P_{N,n}(f) \Pi_n(\mathcal{P}, \vec{\epsilon}, (-1)^{n-1}) \end{aligned}$$

is of the form $\sum_{1 \leq d < N/2} (-1)^{n-1} a_{2n-1}(d) A_n$, with

$$A_n = r_n \frac{(N-2)!}{(N-2n)!} P_{N,n}(f) \Pi_n(\mathcal{P}, \vec{\epsilon}, (-1)^{n-1}).$$

It follows that for n odd and $1 \leq n < N/2 - 1$ we have

$$\begin{aligned} \frac{A_{n+1}}{A_n} &\leq \frac{(N-2n)(N-2n-1) r_{n+1}}{r_n} \frac{\Pi_{n+1}(\mathcal{P}, \vec{\epsilon}, -1)}{\Pi_n(\mathcal{P}, \vec{\epsilon}, +1)} \\ &\leq \frac{(N-2n)(N-2n-1) r_{n+1}}{r_n}. \end{aligned}$$

Hence, in the range $N \leq 12$, we do have $A_{n+1} \leq 4A_n$ for $1 \leq n < N/2-1$ odd, and

$$\begin{aligned}\lambda_N(f,d,\mathcal{P},\vec{\epsilon}) &\geq \lambda_N(f,\mathcal{P},\vec{\epsilon})\\ &:= \sum_{1\leq n<N/2}(-1)^{n-1}r_n\frac{(N-2)!}{(N-2n)!}P_{N,n}(f)\Pi_n(\mathcal{P},\vec{\epsilon},(-1)^{n-1}),\end{aligned}$$

a linear combination of $1, f^{-1},\dots,f^{-(N-3)}$ with rational coefficients.

Assume that these coefficients are all non-negative rational numbers for all the 3^m possible choices of $\vec{\epsilon}\in\{-1,0,1\}^m$, which can be checked by using any software for algebraic computation. Then each $f\mapsto\lambda_N(f,\mathcal{P},\vec{\epsilon})$ is a decreasing function of f and we obtain that for any Dirichlet character χ as in the third point of Theorem 2.2 we have

$$C_N(f,d,\psi)\geq\kappa_N(\mathcal{P}):=\min_{\vec{\epsilon}\in\{-1,0,1\}^m}\lambda_N(\infty,\mathcal{P},\vec{\epsilon}),$$

where

$$\kappa_N(\mathcal{P})=\min_{\vec{\epsilon}\in\{-1,0,1\}^m}\sum_{1\leq n<N/2}(-1)^{n-1}r_n\frac{(N-2)!}{(N-2n)!}\Pi_n(\mathcal{P},\vec{\epsilon},(-1)^{n-1}).$$

2A. The case $9\leq N\leq 10$. Take $\mathcal{P}=\{2\}$. There are 3 choices for $\vec{\epsilon}$ to consider. In each case, $\lambda_N(f,\{2\},\vec{\epsilon})$ is indeed a linear combination of $1,f^{-1},\dots,f^{-(N-3)}$ with non-negative rational coefficients. This is not true for $N=11$. Consequently, we have $C_N(f,d,\psi)\geq\kappa_N(\{2\})$, with $\kappa_9(\{2\})=\lambda_9(\infty,\{2\},-1)=\frac{2057479}{14994408}$ and $\kappa_{10}(\{2\})=\lambda_{10}(\infty,\{2\},-1)=\frac{209135}{3748602}$. Indeed, for example we have

$$\lambda_9(f,\{2\},-1)=\frac{2057479}{14994408}+\frac{148383}{19448f}+\frac{58161}{374f^2}+\frac{53865}{34f^3}+\frac{1189797}{136f^4}+\frac{50463}{2f^5}+\frac{29531}{f^6}.$$

2B. The case $11\leq N\leq 12$. Take $\mathcal{P}=\{2,3\}$. There are 9 choices for $\vec{\epsilon}$ to consider. In each case, $\lambda_N(f,\{2,3\},\vec{\epsilon})$ is a linear combination of $1,f^{-1},\dots,f^{-(N-3)}$ with non-negative rational coefficients. This is not true for $N=13$. Consequently, we have $C_N(f,d,\psi)\geq\kappa_N(\{2,3\})$, with $\kappa_{11}(\{2,3\})=\lambda_{11}(\infty,\{2,3\},(-1,-1))=\frac{261847204793}{4696592207450}$ and $\kappa_{12}(\{2,3\})=\lambda_{12}(\infty,\{2,3\},(-1,-1))=\frac{17351027073}{939318441490}$.

2B. The case $N=13$. Take $\mathcal{P}=\{2,3,5\}$. There are 27 choices for $\vec{\epsilon}$ to consider. In each case, $\lambda_N(f,\{2,3,5\},\vec{\epsilon})$ is a linear combination of $1,f^{-1},\dots,f^{-(N-3)}$ with non-negative rational coefficients. This is not true for $N=14$. Consequently, we have $C_{13}(f,d,\psi)\geq\kappa_{13}(\{2,3,5\})=\lambda_{13}(\infty,\{2,3,5\},(-1,-1,-1))=\frac{4180538598139829643562193049}{130939161450974958367263101761\,0}$. □

7.2. Proof of the last assertion of Theorem 2.3

Now, we suppose that χ modulo $f > 2$ is real, even and primitive. From the previous section, we have

$$\chi_N(f) = \left(1 + O\left(f^{-1}\right)\right) \frac{f^{N-1}\sqrt{f}}{30 \cdot (N-2)!} \kappa_N(\chi),$$

where the implied constants in this $O(f^{-1})$ depend on N only and where

$$\kappa_N(\chi) := \sum_{1 \leq n < N/2} (-1)^{n-1} \frac{(N-2)!}{(N-2n)!} \frac{\zeta(2)L(2n,\chi)}{(4\pi^2)^{n-1}\zeta(4)}.$$

Proposition 7.2. *Let χ run over the real, even and non principal Dirichlet character of prime conductors $p \equiv 1 \pmod 4$. Then,* (i) *there are infinitely many prime numbers $p \equiv 5 \pmod 8$ for which $\chi_{14}(p) < 0$ and* (ii) *for any given $N \geq 3$, there are infinitely many prime numbers $p \equiv 1 \pmod 8$ for which $\chi_k(p) > 0$ for $3 \leq k \leq N$.*

Proof. Fix $N \geq 3$. Let $M \geq 1$ be chosen large enough at the end of the proof. Let p range over the infinite set of prime numbers $p \equiv 5 \pmod 8$ for which $\left(\frac{q_k}{p}\right) = -1$ (Legendre's symbols) for $1 \leq k \leq M$, where $3 = q_1 < 5 = q_2 < \cdots < q_M$ are the first M prime numbers. Take $\chi = \left(\frac{\bullet}{p}\right)$. Since $\chi(q) = -1$ for $2 \leq q \leq q_M$, by choosing M large enough we have that $L(2n,\chi)$ is as close as desired to $\zeta(4n)/\zeta(2n)$ for $1 \leq n < N/2$. Therefore, $\kappa_N(\chi)$ is as close as desired to the rational number

$$\kappa_N^- := \sum_{1 \leq n < N/2} (-1)^{n-1} \frac{(N-2)!}{(N-2n)!} \frac{\zeta(2)\zeta(4n)}{(4\pi^2)^{n-1}\zeta(4)\zeta(2n)}.$$

Consequently, if $\kappa_N^- < 0$, then there are infinitely many prime numbers $p \equiv 5 \pmod 8$ for which $\chi_N(p) < 0$. The least such N is $N = 14$, for which $\kappa_{14}^- = -\frac{6902151}{667347070} < 0$. The first assertion follows.

In the same way, let p range over the infinite set of prime numbers $p \equiv 1 \pmod 8$ for which $\left(\frac{q_k}{p}\right) = +1$ (Legendre's symbols) for $1 \leq k \leq M$. By taking M large enough we obtain that $\kappa_N(\chi)$ is as close as desired to the rational number κ_N^+ for p large enough in a suitable infinite set of prime numbers $p \equiv 1 \pmod 8$, where

$$\kappa_N^+ := \sum_{1 \leq n < N/2} (-1)^{n-1} \frac{(N-2)!}{(N-2n)!} \frac{\zeta(2)\zeta(2n)}{(4\pi^2)^{n-1}\zeta(4)}.$$

To prove the second assertion, we now prove that for $N \geq 3$ we have

$$\kappa_N^+ = \frac{615(N-2)}{N(N-1)} > 0.$$

Using the functional equation for $\zeta(s)$ and (6), we obtain

$$\zeta(2n)=(-1)^{n-1}\frac{(4\pi^2)^n}{2\cdot(2n)!}B_{2n}(1)\qquad(n\geq 1),$$

$\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$ and

$$\kappa_N^+=\frac{30}{N(N-1)}\sum_{1\leq n<N/2}\binom{N}{2n}B_{2n}(1),$$

a rational number. Now, on the one hand we have

$$\left(\sum_{n\geq 0}B_{2n}(1)\frac{t^{2n}}{(2n)!}\right)\left(\sum_{k\geq 1}\frac{t^k}{k!}\right)=\sum_{N\geq 0}\left(\sum_{0\leq n<N/2}\binom{N}{2n}B_{2n}(1)\right)\frac{t^N}{N!}.$$

On the other hand, by (5), the left hand side of this identity is equal to

$$\frac{1}{2}\left(\frac{te^t}{e^t-1}+\frac{-te^{-t}}{e^{-t}-1}\right)(e^t-1)=\frac{1}{2}t+\frac{1}{2}te^t=t+\sum_{N\geq 2}\frac{t^N}{2\cdot(N-1)!}.$$

Hence, using $B_0(1)=1$, for $N\geq 3$ we have

$$\sum_{1\leq n<N/2}\binom{N}{2n}B_{2n}(1)=-1+\sum_{0\leq n<N/2}\binom{N}{2n}B_{2n}(1)=-1+\frac{N}{2},$$

and the desired result follows. □

Remark 7.3. Notice that p need not be that large, for example, $\chi_{14}(61613)<0$. According to our computation, the least positive fundamental discriminants $D>0$ for which $\chi_{14}(D)<0$ are $D=24653$, 31037 and 39437, and the least prime fundamental discriminants $p\equiv 5 \pmod 8$ for which $\chi_{14}(p)<0$ are $p=61613$, 66293 and 73757.

8. Numerical experimentations

We deal with non-trivial (but non necessarily primitive) real and even Dirichlet characters χ's. Our numerical computation is based on point 2 of Proposition 1.1 and on Corollary 1.2. We used UBASIC on a PC Optiplex 780 with Intel Core 2 Duo E7500, 2.93 Ghz. UBASIC allows fast computation with large integers up to a little more than 2600 figures in decimal expression. Let $m\geq 2$ be given. Define the property $P_m(\chi)\in\{true,false\}$ for χ modulo f as follows. For $m=2$ the property $P_2(\chi)$ is true if and only if $\chi_2(n)\geq 0$ for $1\leq n\leq (f-2)/2$ (which is equivalent to having $\chi_2(n)\geq 0$ for $1\leq n\leq f$ but is twice as fast to test). For $m\geq 3$ the property $P_m(\chi)$ is true if and only if $\chi_k(f)\geq 0$ for $1\leq k<m$ and $\chi_m(n)\geq 0$ for $1\leq n\leq f$. If $P_m(\chi)$ is true, then $m(\chi)\leq m$ and $L(s,\chi)>0$ for $s>0$. If some $P_m(\chi)$ is true, then all the $P_{m'}(\chi)$ are true for $m'\geq m$.

First, let χ run over $N(B)$ the real, even and primitive Dirichlet characters of conductors $\leq B$. Hence, χ is well determined by its conductor. We test whether

$P_m(\chi)$ is true, in which case $L(s,\chi) > 0$ for $s > 0$. Suppose that $P_m(\chi)$ is false. Let $2 \leq p < q$ be the least prime integers for which $\chi(p) = \chi(q) = -1$. We then test whether $P_m(\chi')$ is true, where χ' is the character modulo pf induced by χ. In which case, $L(s,\chi') > 0$ for $s > 0$, hence $L(s,\chi') > 0$ for $s > 0$, by (3). If it is false, we then test whether $P_m(\chi'')$ is true, where χ'' is the character modulo pqf induced by χ. In which case, $L(s,\chi'') > 0$ for $s > 0$, hence $L(s,\chi') > 0$ for $s > 0$, by (3). The drawback of this procedure it that the moduli get larger and larger and the numerical computation slower and slower. Let $N_m(B)$, $N_m(B)'$ and $N_m(B)'$ be the numbers of real, even and primitive Dirichlet characters χ's of conductors $\leq B$ for which $P_m(\chi)$, $P_m(\chi')$ or $P_m(\chi'')$ is true, respectively. Set $\rho_m(B)'' = 100N_m(B)/N(B)$.

TABLE 1

B	10^2	10^3	10^4	10^5	10^6
$N(B)$	30	302	3043	30394	303957
$N_2(B)$	28	244	2080	17928	159727
$N_2(B)'$	29	260	2324	20663	185631
$N_2(B)''$	30	279	2534	22778	206871
$\rho_2(B)''$	100	$92.384\cdots$	$83.273\cdots$	$74.942\cdots$	$68.059\cdots$
T_2	$0sec$	$0sec$	$3sec$	$5mn34sec$	$8h59mn13sec$
T_2'	$0sec$	$0sec$	$6sec$	$11mn29sec$	$18h31mn59sec$
T_2''	$0sec$	$0sec$	$16sec$	$37mn17sec$	$71h21mn47sec$
$N_3(B)$	29	257	2260	19893	177662
$N_3(B)'$	29	268	2440	21814	197090
$N_3(B)''$	30	283	2592	23457	214603
$\rho_3(B)''$	100	$93.708\cdots$	$85.179\cdots$	$77.176\cdots$	$70.603\cdots$
T_3	$0sec$	$0sec$	$4sec$	$6mn58sec$	$11h18mn09sec$
T_3'	$0sec$	$0sec$	$7sec$	$12mn28sec$	$20h14mn39sec$
T_3''	$0sec$	$0sec$	$24sec$	$54mn7sec$	$111h22mn49sec$
$N_4(B)$	29	259	2327	20623	184743
$N_4(B)'$	30	273	2478	22192	201093
$N_4(B)''$	30	283	2613	23723	217603
$\rho_4(B)''$	100	$93.708\cdots$	$85.869\cdots$	$78.051cdots$	$71.590\cdots$
T_4	$0sec$	$0sec$	$4sec$	$7mn35sec$	$12h20mn39sec$
T_4'	$0sec$	$0sec$	$7sec$	$12mn51sec$	$20h54mn46sec$
T_4''	$0sec$	$0sec$	$25sec$	$59mn52sec$	$128h45mn00sec$
$N_5(B)$	29	264	2360	21006	188597
$N_5(B)'$	30	274	2491	22387	203291
$N_5(B)''$	30	283	2622	23868	219200
$\rho_5(B)''$	100	$93.708\cdots$	$86.164\cdots$	$78.528\cdots$	$72.115\cdots$
T_5	$0sec$	$0sec$	$4sec$	$7mn56sec$	$12h57mn44sec$
T_5'	$0sec$	$0sec$	$7sec$	$13mn5sec$	$21h19mn32sec$
T_5''	$0sec$	$0sec$	$27sec$	$1h04mn17sec$	$138h42mn20sec$
$N_{10}(B)$	30	269	2427	21619	195208
$N_{10}(B)'$	30	279	2532	22763	207322
$N_{10}(B)''$	30	284	2641	24133	221981
$\rho_{10}(B)''$	100	$94.039\cdots$	$86.789\cdots$	$79.400\cdots$	$73.030\cdots$
T_{10}	$0sec$	$0sec$	$5sec$	$8mn48sec$	$14h31mn34sec$
T_{10}'	$0sec$	$0sec$	$8sec$	$13mn41sec$	$22h24mn51sec$
T_{10}''	$0sec$	$0sec$	$30sec$	$1h14mn13sec$	$160h28mn33sec$

In Table 1, we computed $N_m(B)$, $N_m(B)'$, $N_m(B)''$ and $\rho_m(B)''$ for $m \in \{2,3,4,5,10\}$ and $B \in \{10^2,10^3,10^4,10^5\}$. For a given m the time needed to complete these computations are denoted by T_m, T'_m and T''_m, respectively.

Second, let χ run over the 28 real, even non-principal primitive characters of conductors $f_\chi \le 10^3$ for which $m(\chi)=\infty$ (see Section 1.1). We computed the least product $D_{10}(\chi)$ of the consecutive primes $p \in E(\chi)$ for which the $D_{10}(\chi)$-induced character χ' modulo $f' = D_{10}(\chi)f_\chi$ satisfies $\chi'_{10}(n) \ge 0$ for $1 \le n \le f'$. Hence $m(\chi') \le 10 < \infty$ and $L(s,\chi)>0$ for these 28 primitive characters χ. We could also sometimes find some $d < D_{10}(\chi)$ for which the d-induced character χ'' modulo $f'' = df_\chi$ satisfies $m(\chi'')<\infty$.

TABLE 2 (28 cases)

f_χ	$D_{10}(\chi)$	T	d	$m(\chi'')$
173	30		10	82
188	3		3	3
248	15		5	5
293	2310		715	25
332	105		15	32
413	330	$4sec$	22	12
437	510510	$18mn17sec$	102102	22
453	70		35	5
488	105		35	12
552	5		5	4
572	105		105	8
573	2		2	8
629	6		2	19
668	15015	$19mn06sec$	15015	5
677	746130	$1h23sec$	746130	8
717	10010		2002	31
728	165		33	16
773	2730		1365	47
797	6		3	54
813	10		2	40
853	10		5	2
860	231		77	3
888	5	$1h1mn10sec$	5	10
908	111546435	$144h20mn$	8580495	10
920	3		3	7
941	462		42	23
957	15470		910	12
965	42	$144h20mn$	11	11

9. A faster algorithm

We conclude this paper with a new method for proving that $L(s,\chi)>0$ for $s>0$ for primitive, real and even Dirichlet characters. When it applies it is much faster than Chowla's method. However, at the moment we do not know how to generalize our result to non-primitive characters. A suitable generalization applied to d-induced characters would provide an efficient method for testing whether $L(s,\chi)>0$ for $s>0$ for more than 41% of the primitive even Dirichlet characters of prime conductors $p\equiv 1 \pmod 8$ and $p\leq 10^9$.

Theorem 9.1. *Let χ be a primitive even Dirichlet character modulo $f\geq 5$. Let $N\geq 2$ be the least integer satisfying $1-e^{-2\pi(N+1)/f}-e^{-\pi N(N+2)/f}\geq 0$. Hence, N is asymptotic to $\sqrt{\frac{f}{2\pi}\log(\frac{f}{\pi})}$. If $\chi_1(n)\geq 1$ for $1\leq n\leq N$, then $L(s,\chi)>0$ for $s>0$.*

Proof. Recall that we have the following integral representation

$$\Lambda(s,\chi):=(f/\pi)^{s/2}\Gamma(s/2)L(s,\chi)=\int_1^\infty S(\pi x/f,\chi)(x^{s/2}+x^{(1-s)/2})\frac{dx}{x},$$

where $S(x,\chi)=\sum_{n\geq 1}\chi(n)e^{-n^2x}$ (e.g. see [12, Chapter 9]). It suffices to prove that under our hypothesis we have $S(x,\chi)\geq 1$ for $x\geq \pi/f$. We have

$$\begin{aligned}
S(x,\chi)&=\sum_{n=1}^{N-1}(e^{-n^2x}-e^{-(n+1)^2x})\chi_1(n)+e^{-N^2x}\chi_1(N)+\sum_{n\geq N+1}\chi(n)e^{-n^2x}\\
&\geq \sum_{n=1}^{N-1}(e^{-n^2x}-e^{-(n+1)^2x})+e^{-N^2x}-\sum_{n\geq N+1}e^{-n^2x}\\
&=e^{-x}-\sum_{n\geq N+1}e^{-n^2x}\\
&=e^{-x}-\sum_{n\geq 0}e^{-(N+1+n)^2x}\geq e^{-x}-e^{-(N+1)^2x}\sum_{n\geq 0}e^{-2(N+1+n)x}\\
&=\frac{e^{-x}h_N(x)}{1-e^{-2(N+1)x}},
\end{aligned}$$

where $h_N(x)=1-e^{-2(N+1)x}-e^{-N(N+2)x}$ is an increasing function of $x>0$. Hence, $h_N(x)\geq h_N(\pi/f)\geq 0$ for $x\geq\pi/f$ and we do have $S(x,\chi)\geq 1$ for $x\geq \pi/f$. □

To estimate the efficiency of Theorem 9.1, we computed Table 3. For a given B, we give the number $\pi'(B;1,8)$ of primes among the $\pi(B;1,8)$ primes $p\equiv 1 \pmod 8$ with $p\leq B$ for which Theorem 9.1 gives $L(s,\left(\frac{\bullet}{p}\right))>0$ fort $s>0$. Hence, Theorem 9.1 applies to 41% of the primitive even Dirichlet characters of prime conductors $p\equiv 1 \pmod 8$ and $p\leq 10^9$.

TABLE 3

B	$\pi(B;1,8)$	$\pi'(B;1,8)$	$\pi'(B;1,8)/\pi(B;1,8)$	T
10^2	5	5	1	$0sec$
10^3	37	26	$0.7027\cdots$	$0sec$
10^4	295	197	$0.6677\cdots$	$0sec$
10^5	2384	1419	$0.5952\cdots$	$0sec$
10^6	19552	10463	$0.5351\cdots$	$8sec$
10^7	165976	80971	$0.4878\cdots$	$3mn17sec$
10^8	1439970	644451	$0.4475\cdots$	$1h29mn19sec$
10^9	12711220	5256723	$0.4135\cdots$	$42h32mn57sec$

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