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# A RESEARCH ON LINEAR $(p,q)$-DIFFERENCE EQUATIONS OF HIGHER ORDER

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Abstract. In this paper, we investigate solutions of equations which are linear $(p,q)$-difference equation of higher order by using $(p,q)$-derivative and integral. We also derive the solution of equation in the case of second order.

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## 1. Introduction

Quantum calculus($q$-calculus) have been studied by many researchers. They investigated some classical theory and several results and properties for $q$-calculus. Several of them obtained various generalizations of operators based on $q$-calculus (see[1-6]). After that, several authors introduced and researched many expansions of positive linear operators, using $q$-integers. The recent research trend treated the applications of $q$-calculus importantly in the field of number theory, approximation theory, and physics. Some scholars have published papers in the area, developing stage, related to approximation theory(see[3-5]).

Recently, $(p,q)$-calculus is introduced as the post quantum calculus of $q$-calculus(see[7-13]). R. Chakrabarti and R. Jagannathan mentioned $(p,q)$-number as two-parameter quantum in the area of physics. Wachs and White introduced the $(p,q)$-number in the mathematics by certain combinatorial problems that is irrelevant to the quantum group(see[13]). The $(p,q)$-integer was introduced to generalize or integrate several forms of $q$-oscillator algebras in the theoretical physics that are related with the representation of single-parameter quantum algebras(see[9]).

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Katriel and Kibler defined the $(p,q)$-binomial coefficients and derived a $(p,q)$-binomial theorem. Burban and Klimyk studied $(p,q)$-differentiation, $(p,q)$-integration(see[11,12]). In [8], R. Jagannathan and K. S. Rao introduced the $(p,q)$-extensions related to two parameter quantum algebras from $q$-identities. P. N. Sadjang expressed two relevant polynomials of the $(p,q)$-derivative and derived the formula of $(p,q)$-integration by part and two $(p,q)$-Taylor formulas of polynomials(see[10]).

Throughout this paper, we use that $0 < q < p \leq 1$ and $p,q \in \mathbf{C}$ where $\mathbb{C}$ is the set of complex numbers . At first, we introduce some basic notations about $(p,q)$-calculus which is found in [7-13].

**Definition 1.1.** (1) For any $n \in \mathbb{C}$ and $0 < q < p \leq 1$, we define the $(p,q)$-number by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{1.1}$$

Note that the $(p,q)$ number is reduced to $q$-number, $\lim_{p\to 1}[n]_{p,q} = [n]_q$ for $q \neq 1$ .

(2) The $(p,q)$-binomial coefficients are defined by $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{\left[n\right]_{p,q}!}{\left[k\right]_{p,q}!\left[n-k\right]_{p,q}!}$, $0 \leq k \leq n$ where $\left[n\right]_{p,q}! = \left[n\right]_{p,q}\left[n-1\right]_{p,q}\cdots\left[1\right]_{p,q}$ for $n = 1, 2, \cdots$, and $\left[0\right]_{p,q}! = 1$.

**Definition 1.2.** Let $f$ be a function on the set of the complex numbers. We define the $(p,q)$-derivative of the function $f$ as follows

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x},\ (x \neq 0). \tag{1.2}$$

Since $D_{p,q}f(0) = f'(0)$, it is provided that $f$ is differentiable at 0.

From the Definition, we have

$$D_{p,q}f(x) = D_{1,\frac{p}{q}}f(qx),\ D_{p,q}f(x) = D_{1,\frac{q}{p}}f(px),\ D_{p,q}f(x) = D_{p^{-1},q^{-1}}f(pqx) \tag{1.3}$$

Since $D_{p,q}z^n = [n]_{p,q}z^{n-1}$, if $t(x) = \sum_{k=0}^{n} a_k x^k$, then

$$D_{p,q}t(x) = \sum_{k=0}^{n-1} a_{k+1}[k+1]_{p,q}x^k.$$

The operator of $(p,q)$-difference equation, $D_{p,q}$, has the following properties.

**Theorem 1.3.** *The derivative of a product and the derivative of a ratio are given by*

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)$$
$$= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x),$$
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}$$
$$= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In [10], P. N. Sadjang introduced the definition of $(p,q)$-integral as below.

**Definition 1.4.** Let $f$ be an arbitrary function. $(p,q)$-integral is defined

$$\int f(x)d_{p,q}x = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}x\right).$$

**Theorem 1.5.** *The $(p,q)$-integration by parts is defined*

$$\int_a^b f(px)D_{p,q}g(x)d_{p,q}x = [f(x)g(x)]_a^b - \int_a^b g(qx)D_{p,q}f(x)d_{p,q}x.$$

**Definition 1.6.** For $z$ in complex number with $|z| < 1$, the $(p,q)$-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!},$$
$$E_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}.$$

Observe that $E_{p,q}(x) = e_{p^{-1},q^{-1}}(x)$.

A general linear $(p,q)$-difference equations of first order is represented by

$$D_{p,q}y(x) = a(x)y(qx) + b(x), \tag{1.4}$$

and a non homogeneous equation that is concerned with the corresponding homogeneous one has

$$D_{p,q}y(x) = a(x)y(qx). \tag{1.5}$$

In [14], We investigated the general solution of linear $(p,q)$-difference equation of first order and the system of the equation.

The purpose of this paper is to explore $(p,q)$-difference equations of higher order and to find the solution of the equations. In section 2, we investigate solutions about the system of linear $(p,q)$-differential equations of higher order in various case. In section 3, we derive solutions about the systems of linear $(p,q)$-difference equation of higher order with constant coefficients including simple case. Add to that, we consider a simple example that is the second order $(p,q)$-difference equations.

## 2. General theory on linear $(p, q)$-difference equations of higher order

In this section, we define the $k$-order linear non-homogeneous $(p, q)$-difference equation and corresponding homogeneous equation which is derived from a certain equation. We also investigate the solution of the equation.

We consider the equation which is $k$-order nonconstant coefficients linear non-homogeneous $(p, q)$-difference equation of order $k$ and corresponding homogeneous equation as follows :

$$[D^{(k)}_{1,\frac{p}{q}} + a_1(x)D^{(k-1)}_{1,\frac{p}{q}} + \cdots + a_{k-1}(x)D_{1,\frac{p}{q}} + a_k(x)]y(x) = b(x), \tag{2.1}$$

and

$$[D^{(k)}_{1,\frac{p}{q}} + a_1(x)D^{(k-1)}_{1,\frac{p}{q}} + \cdots + a_{k-1}(x)D_{1,\frac{p}{q}} + a_k(x)]y(x) = 0. \tag{2.2}$$

A scalar equation of (2.1) and (2.2) are reduced to a system of the (1.4) and (1.5) in [14]. Add that, It gives general theory for the system.

Assume that

$$z_1(x) = y(x),\ z_2(x) = D_{1,\frac{p}{q}}\, y(x),\ \cdots,\ z_k(x) = D^{(k-1)}_{1,\frac{p}{q}} y(x). \tag{2.3}$$

We obtain the result as follows :

$$\begin{aligned} D_{1,\frac{p}{q}} z(qx) &= \begin{pmatrix} z_2(qx) \\ z_3(qx) \\ \vdots \\ z_k(qx) \\ -a_k(qx)z_1(qx) - a_{k-1}(qx)z_2(qx) - \cdots - a_1(qx)z_k(qx) + b(qx) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_k(qx) & -a_{k-1}(qx) & \cdots & -a_1(qx) \end{pmatrix} \begin{pmatrix} z_1(qx) \\ z_2(qx) \\ \vdots \\ z_{k-1}(qx) \\ z_k(qx) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(qx) \end{pmatrix}. \end{aligned}$$

From (1.3), the above matrix form is represented by

$$D_{p,q}z(x) = A(x)z(qx) + B(x), \tag{2.4}$$

where

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k(qx) & -a_{k-1}(qx) & -a_{k-2}(qx) & \cdots & -a_1(qx) \end{pmatrix},$$

$z(x) = (z_1(x), \cdots, z_k(x))^t$, and $B(x) = (0, 0, \cdots, 0, b(qx))^t$.

Let the initial conditions be

$$y(x_0) = y_0,\ D_{1,\frac{p}{q}}\, y(x_0) = y_1,\ D^{(2)}_{1,\frac{p}{q}} y(x_0) = y_2,\ \cdots,\ D^{(k-1)}_{1,\frac{p}{q}} y(x_0) = y_{k-1}.$$

Then we can consider the existence of a unique solution of (2.1). By (2.3), it is equivalent to the existence of a unique solution of (2.4) under the constraints $(z_1(x_0), \cdots, z_k(x_0))^t = (y_0, y_1, \cdots, y_{k-1})^t$. Consequently, like a fundamental system of solutions $y_1(x), y_2(x), \ldots, y_k(x)$ of (2.2), we can get a fundamental system

$$\left(y_1(x), D_{1,\frac{p}{q}}\, y_1(x), \ldots, D^{(k-1)}_{1,\frac{p}{q}} y_1(x)\right)^t, \cdots, \left(y_k(x), D_{1,\frac{p}{q}}\, y_k(x), \ldots, D^{(k-1)}_{1,\frac{p}{q}} y_k(x)\right)^t$$

of the corresponding homogeneous equation $D_{p,q}z(x) = A(x)z(qx)$, including the fundamental matrix

$$\Phi(x) = \begin{pmatrix} y_1(x) & \cdots & y_k(x) \\ D_{1,\frac{p}{q}}\, y_1(x) & \cdots & D_{1,\frac{p}{q}}\, y_k(x) \\ \vdots & \cdots & \vdots \\ D^{(k-1)}_{1,\frac{p}{q}} y_1(x) & \cdots & D^{(k-1)}_{1,\frac{p}{q}} y_k(x) \end{pmatrix}. \tag{2.5}$$

If $\sum_{i=1}^k \alpha_i y_i(x) = 0$, then $\sum_{i=1}^k \alpha_i\, D_{1,\frac{p}{q}}\, y_i(x) = 0, \cdots, \sum_{i=1}^k \alpha_i D^{(k-1)}_{1,\frac{p}{q}} y_i(x) = 0.$

From above result, we have $\alpha\Phi(x) = 0$ where $\Phi(x)$ is a fundamental matrix and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)^t$
Therefore, the fundamental matrix in (2.5) is non-singular, that is

$$det\,[\Phi(x)] = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ D_{1,\frac{p}{q}}\, y_1(x) & \cdots & D_{1,\frac{p}{q}}\, y_n(x) \\ \vdots & \cdots & \vdots \\ D^{(k-1)}_{1,\frac{p}{q}} y_1(x) & \cdots & D^{(k-1)}_{1,\frac{p}{q}} y_n(x) \end{vmatrix} \neq 0$$

if and only if the fundamental solutions, $y_i (i = 1, 2, \cdots, k)$ , are linear independent.

If $y_i(x)(i = 1, 2, \cdots, n)$ is a fundamental system of solution of the homogeneous (2.2), for the fundamental matrix $\Phi(x)$, then the general solution of the system,

$$\Phi(px)D_{p,q}C(x) = B(x), \tag{2.6}$$

is found as below

$$z(x) = \Phi(x)C(x), \tag{2.7}$$

where $C(x) = (C_1(x), \cdots, C_k(x))^t$.

From (2.4) and (2.7), we have

$$\begin{aligned} B(x) &= C(qx)D_{p,q}\Phi(x) + \Phi(px)D_{p,q}C(x) - A(x)z(qx) \\ &= \Phi(px)D_{p,q}C(x). \end{aligned}$$

Therefore, we get the result of (2.6) and the following equation,

$$D_{p,q}C(x) = \Phi^{-1}(px)B(x). \tag{2.8}$$

By (2.8) and definition of $(p, q)$-derivative, we have

$$C(px) = C(qx) + (p - q)x\,\Phi^{-1}(px)B(x),$$

and by replacing $px$ by $x$, ons has

$$C(x) = C(\frac{q}{p}x) + (1 - \frac{q}{p})x\,\Phi^{-1}(x)B(\frac{1}{p}x). \tag{2.9}$$

**Theorem 2.1.** *Consider the equations $D_{p,q}C(x) = \Phi^{-1}(px)B(x)$. Then we obtain*

$$\begin{aligned} C(x) &= C\left(\frac{q^N}{p^N}x\right) + \left(1 - \frac{q}{p}\right)x\sum_{i=o}^{N-1}\left(\frac{q^i}{p^i}\right)\Phi^{-1}\left(\frac{q^i}{p^i}x\right)B\left(\frac{q^i}{p^{i+1}}x\right) \\ &= C(x_0) + \sum_{t=(\frac{q}{p})^i x_0}^{x}(1 - \frac{q}{p})t\,\Phi^{-1}(t)B\left(\frac{1}{p}t\right). \end{aligned}$$

*Proof.* By the recurrence relation of (2.9), we have

$$C(x) = C\left(\frac{q^N}{p^N}x\right) + \left(1 - \frac{q}{p}\right)x\sum_{i=o}^{N-1}\left(\frac{q^i}{p^i}\right)\Phi^{-1}\left(\frac{q^i}{p^i}x\right)B\left(\frac{q^i}{p^{i+1}}x\right).$$

If we put $(\frac{q}{p})^N x = x_0$, then the above equation is represented by the following result.

$$C(x) = C(x_0) + \sum_{t=(\frac{q}{p})^i x_0}^{x}\left(1 - \frac{q}{p}\right)t\,\Phi^{-1}(t)B\left(\frac{1}{p}t\right).$$

. □

**Corollary 2.2.** *If $N \to \infty$ for $0 < \frac{q}{p} < 1$, then $(\frac{q}{p})^N$ approaches 0.*

*Hence, we have*

$$C(x) = C(0) + (p - q)x\sum_{i=o}^{\infty}\left(\frac{q^i}{p^{i+1}}\right)\Phi^{-1}\left(\frac{q^i}{p^i}x\right)B\left(\frac{q^i}{p^{i+1}}x\right)$$

*and the general solutions of (2.1) reads*

$$y(x) = \sum_{i=1}^{n} C_i(x)y_i(x).$$

Note that for $(p,q)$-difference equation

$$[D^{(k)}_{1,\frac{q}{p}} + a_1(x)D^{(k-1)}_{1,\frac{q}{p}} + \cdots + a_{k-1}(x)D_{1,\frac{q}{p}} + a_k(x)]y(x) = b(x)$$

and

$$[D^{(k)}_{1,\frac{q}{p}} + a_1(x)D^{(k-1)}_{1,\frac{q}{p}} + \cdots + a_{k-1}(x)D_{1,\frac{q}{p}} + a_k(x)]y(x) = 0.$$

Suppose that

$$t_1(x) = y(x),\ t_2(x) = D_{1,\frac{q}{p}}y(x),\ \cdots,\ t_k(x) = D^{(k-1)}_{1,\frac{q}{p}}y(x).$$

Then we also have as follows :

$$D_{p,q}z(x) = P(x)t(px) + Q(x), \tag{2.10}$$

where

$$P(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k(px) & -a_{k-1}(px) & -a_{k-2}(px) & \cdots & -a_1(px) \end{pmatrix},$$

$t(x) = (t_1(x), \cdots, t_k(x))^t$ and $Q(x) = (0, 0, \cdots, 0, b(px))^t$.

Similar to (2.5), we can get a fundamental system

$$\Big(y_1(x), D_{1,\frac{q}{p}}y_1(x), \ldots, D^{(k-1)}_{1,\frac{q}{p}}y_1(x)\Big)^t, \cdots, \Big(y_k(x), D_{1,\frac{q}{p}}y_k(x), \ldots, D^{(k-1)}_{1,\frac{q}{p}}y_k(x)\Big)^t$$

of the homogeneous equation of (2.10),

$$D_{p,q}t(x) = P(x)t(px),$$

where

$$\Phi(x) = \begin{pmatrix} y_1(x) & \cdots & y_k(x) \\ D_{1,\frac{q}{p}}y_1(x) & \cdots & D_{1,\frac{q}{p}}y_k(x) \\ \vdots & \cdots & \vdots \\ D^{(k-1)}_{1,\frac{q}{p}}y_1(x) & \cdots & D^{(k-1)}_{1,\frac{q}{p}}y_k(x) \end{pmatrix}. \tag{2.11}$$

If $y_i(x)(i = 1, 2, \cdots, n)$ is a fundamental system of solution of the homogeneous equation (2.2), with the fundamental matrix $\Phi(x)$, then the general solution of the system

$$\Phi(qx)D_{p,q}R(x) = Q(x) \tag{2.12}$$

is found as below

$$t(x) = \Phi(x)R(x) \tag{2.13}$$

where $R(x) = (R_1(x), \cdots, R_k(x))^t$. By (2.10) and (2.13), we notice the result of (2.12),

$$\begin{aligned} Q(x) &= D_{p,q}t(x) - P(x)t(px) \\ &= \Phi(qx)D_{p,q}R(x). \end{aligned}$$

From that, we get the following equation,

$$D_{p,q}R(x) = \Phi^{-1}(qx)Q(x). \tag{2.14}$$

The (2.14) and the definition of $(p, q)$-derivative are gives

$$R(px) = R(qx) + (p - q)x\,\Phi^{-1}(qx)Q(x).$$

By replacing $px$ by $x$, one has

$$R(x) = R(\frac{q}{p}x) + (1 - \frac{q}{p})x\,\Phi^{-1}(\frac{q}{p}x)B(\frac{1}{p}x).$$

**Theorem 2.3.** *Consider the equations $D_{p,q}R(x) = \Phi^{-1}(qx)Q(x)$. Then we obtain*

$$\begin{aligned} R(x) &= R\left(\frac{q^N}{p^N}x\right) + \left(1 - \frac{q}{p}\right)x\sum_{i=o}^{N-1}\left(\frac{q^i}{p^i}\right)\Phi^{-1}\left(\frac{q^{i+1}}{p^{i+1}}x\right)B\left(\frac{q^i}{p^{i+1}}x\right) \\ &= R(x_0) + \sum_{t=(\frac{q}{p})^i x_0}^{x}(1 - \frac{q}{p})r\,\Phi^{-1}(r)B\left(\frac{1}{p}r\right). \end{aligned}$$

*Proof.* The first part is proved by the recurrence relation of (2.9). If we put $(\frac{q}{p})^N x = x_0$, then the above equation is represented by the second part. $\square$

**Corollary 2.4.** *If $N \to \infty$ for $0 < \frac{q}{p} < 1$, then $(\frac{q}{p})^N$ approaches 0. Hence, we have*

$$R(x) = R(0) + (1 - \frac{q}{p})x\sum_{i=o}^{\infty}\left(\frac{q^i}{p^i}\right)\Phi^{-1}\left(\frac{q^{i+1}}{p^{i+1}}x\right)B\left(\frac{q^i}{p^{i+1}}x\right)$$

*and the general solutions of (2.1) reads*

$$y(x) = \sum_{i=1}^{n} R_i(x)y_i(x).$$

## 3. Linear $(p, q)$-difference equations of second order

In this section, we derive a general solutions of linear $(p, q)$-difference equation of second order. We investigate linear $(p, q)$-difference equation with constant coefficients and some example of the second order. Furthermore, we research the equations which have the solution in series is concerned with analytic functions.

Consider linear non homogeneous $(p,q)$-difference equation of order $n$ with constant coefficients,

$$\left[D_{p,q}^{k}+a_1 D_{p,q}^{k-1}+\cdots+a_{k-1}D_{p,q}+a_k\right]y(x)=b(x) \tag{3.1}$$

and the corresponding homogeneous equation,

$$\left[D_{p,q}^{k}+a_1 D_{p,q}^{k-1}+\cdots+a_{k-1}D_{p,q}+a_k\right]y(x)=0. \tag{3.2}$$

Now, consider the equation, $D_{p,q}y(x)=\lambda y(x)$, then it's solution read $y(x)=e_{p,q}(\lambda x)$. So, we obtain the characteristic equation of (3.2),

$$\lambda^k+a_1\lambda^{k-1}+\cdots+a_{k-1}\lambda+a_k=0. \tag{3.3}$$

**Theorem 3.1.** *If the characteristic equation has k-distinct roots, $\lambda_i (i=1,2,\cdots,k)$, then the solutions of (3.3) is obtained by $y_i(x)=e_{p,q}(\lambda_i x)$ for $i=1,2,\cdots,k$.*

**Theorem 3.2.** *In the case of the characteristic equation has some roots that are not distinct, then (3.3) admits as k-linear independent solutions. So, we can write the solution of the $(p,q)$-defference equation by $y(x)=\sum_{n=0}^{\infty}c_n x^n$ where the coefficients $c_n$, satisfies*

$$\sum_{i=0}^{m}\left[\binom{m}{i}(-\lambda)^i\left(\prod_{k=0}^{i-1}\frac{p^{(n+i)-k}-q^{(n+i)-k}}{p-q}\right)\right]c_{n+i}=0$$

*a homogeneous $(p,q)$-difference equation of order $m$.*

*Proof.* To solve (3.1) and (3.2), it is enough to prove following equation, $(D_{p,q}-\lambda)^m y(x)=0$ that include the equation $y(x)=\sum_{n=0}^{\infty}c_n x^n$.

So, we obtain

$$\left[\binom{m}{0}D_{p,q}^{m}+\binom{m}{1}D_{p,q}^{m-1}(-\lambda)+\cdots+\binom{m}{m-1}D_{p,q}(-\lambda)^{m-1}+\binom{m}{m}(-\lambda)^m\right] \times\sum_{n=0}^{\infty}c_n x^n=0. \tag{3.4}$$

Using the definition of $(p,q)$-derivative operator, we get

$$\binom{m}{m}\sum_{n=0}^{\infty}(-\lambda)^m c_n x^n+\binom{m}{m-1}\sum_{n=0}^{\infty}(-\lambda)^{m-1}c_{n+1}\frac{p^{n+1}-q^{n+1}}{p-q}x^n + \cdots+\binom{m}{0}\sum_{n=0}^{\infty}c_{n+m}\frac{(p^{n+m}-q^{n+m})\cdots(p^{n+1}-q^{n+1})}{(p-q)^m}x^n=0.$$

From the above equation, we have the following result by comparison of coefficients

$$\sum_{i=0}^{m}\left[\binom{m}{i}(-\lambda)^i\left(\prod_{k=0}^{i-1}\frac{p^{(n+i)-k}-q^{(n+i)-k}}{p-q}\right)\right]c_{n+i}=0. \tag{3.5}$$

□

Now, assume that all the coefficients $a_i(x)$ of (3.1) are the form $a_i(x)=x^i d_i$ where $i=0,\cdots,k$ and $d_i$ is constants. Then the equation is represented as below formula

$$\begin{aligned} b(x)=\sum_{n=0}^{\infty}d_k c_n x^{n+k}+\sum_{n=0}^{\infty}d_{k-1}c_{n+1}\frac{p^{n+1}-q^{n+1}}{p-q}x^{n+k-1}+\cdots \\ +\sum_{n=0}^{\infty}d_0 c_{n+k}\frac{(p^{n+k}-q^{n+k})\cdots(p^{n+1}-q^{n+1})}{(p-q)^k}x^n. \end{aligned} \tag{3.6}$$

**Example 3.1** Consider a equation $[D^{(2)}_{1,\frac{q}{p}}-5D_{1,\frac{q}{p}}+6]y(x)=x^2$, that its solution is a combination of exponential function. Find the general solution of the second order linear difference equation.

*Solution.* We assume that

$$z_1(x)=y(x),\ z_2(x)=D_{1,\frac{p}{q}}\,y(x).$$

Then we have

$$D_{1,\frac{p}{q}}z(qx)=\begin{pmatrix}0&1\\-6&5\end{pmatrix}\begin{pmatrix}z_1(qx)\\z_2(qx)\end{pmatrix}+\begin{pmatrix}0\\x^2\end{pmatrix}.$$

From (1.3), the above matrix form is represented by

$$D_{p,q}z(x)=A(x)z(qx)+B(x), \tag{3.7}$$

where $A(x)=\begin{pmatrix}0&1\\-6&5\end{pmatrix}$, $z(x)=(z_1(x),z_2(x))^t$, and $B(x)=(0,x^2)^t$.

To solve the difference equation, we consider the characteristic equation of that,

$$\lambda^2-5\lambda+6=0, \tag{3.8}$$

has solutions $\lambda_1=2$ and $\lambda_2=3$. So, we can write the particular solution of (3.7) by $y_1(x)=e_{1,\frac{q}{p}}(2x)$ and $y_2(x)=e_{1,\frac{q}{p}}(3x)$. Let $y_1(x),y_2(x)$ be a fundamental system of solution of the homogeneous equation, $[D^{(2)}_{1,\frac{q}{p}}-5D_{1,\frac{q}{p}}+6]y(x)=x^2$, corresponding to the fundamental matrix $\Phi(x)$ where

$$\Phi(x)=\begin{pmatrix}e_{1,\frac{q}{p}}(2x)&e_{1,\frac{q}{p}}(3x)\\2e_{1,\frac{q}{p}}(2x)&3e_{1,\frac{q}{p}}(3x)\end{pmatrix}.$$

The general solution of (3.7) is obtained as $z(x)=\Phi(x)C(x)$ where $C(x)=(c_1(x),c_2(x))$. This leads to the general solution is $y(x) = c_1(x)e_{1,\frac{q}{p}}(2x) + c_2(x)e_{1,\frac{q}{p}}(3x)$, where

$$\Phi(qx)\begin{pmatrix} D_{p,q}c_1(x) \\ D_{p,q}c_2(x)\end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.$$

So, we obtain $D_{p,q}c_1(x) = -x^2 e_{1,(\frac{q}{p})^{-1}}(-2qx)$ and $D_{p,q}c_2(x) = x^2 e_{1,(\frac{q}{p})^{-1}}(-3qx)$. Using the $(p,q)$-integrations by parts, we have the general solution as below.

$$\begin{aligned} y(x) = &- c_1\, e_{1,\frac{q}{p}}(2x) + \left(\frac{1}{2p^2}x^2 + \frac{p+q}{4p^3}x + \frac{p+q}{8p^3}\right) - \frac{p+q}{8p^3}e_{1,\frac{q}{p}}(2x) \\ &+ c_2\, e_{1,\frac{q}{p}}(3x) - \left(\frac{1}{3p^2}x^2 + \frac{p+q}{9p^3}x + \frac{p+q}{27p^3}\right) + \frac{p+q}{27p^3}e_{1,\frac{q}{p}}(3x) \end{aligned}$$

where

$$c_1(x) = -c_1 + e_{1,(\frac{q}{p})^{-1}}(-2x)\left(\frac{1}{2p^2}x^2 + \frac{p+q}{4p^3}x + \frac{p+q}{8p^3}\right) - \frac{p+q}{8p^3}$$

and

$$c_2(x) = c_2 - e_{1,(\frac{q}{p})^{-1}}(-3x)\left(\frac{1}{3p^2}x^2 + \frac{p+q}{9p^3}x + \frac{p+q}{27p^3}\right) + \frac{p+q}{27p^3}.$$

**Example 3.2** Consider another equation $[D^{(2)}_{1,\frac{p}{q}} - 3D_{1,\frac{p}{q}} + 2]y(x) = x^2$, with $y(x)$ is expressed exponential function. The characteristic equation has solutions, $\lambda_1 = 1$ and $\lambda_2 = 2$. This leads to the general solution of the equation is $y(x) = c_1(x)e_{1,\frac{p}{q}}(x) + c_2(x)e_{1,\frac{p}{q}}(2x)$, where $\Phi(px)\begin{pmatrix} D_{1,\frac{p}{q}}c_1(x) \\ D_{1,\frac{p}{q}}c_2(x)\end{pmatrix} = \begin{pmatrix} 0 \\ x^2\end{pmatrix}$ and $\Phi(x) = \begin{pmatrix} e_{1,\frac{p}{q}}(x) & e_{1,\frac{p}{q}}(2x) \\ 2e_{1,\frac{p}{q}}(x) & 3e_{1,\frac{p}{q}}(2x)\end{pmatrix}$.

We also can get the general solution by the result of $D_{p,q}c_1(x)$ and $D_{p,q}c_2(x)$ and the $(p,q)$-integrations by parts in the same method of example (3.1).

**Example 3.3** Given a $(p,q)$-difference equation,

$$[D^3_{p,q} - 3D^2_{p,q} + 2D_{p,q}]y(x) = 4x, \tag{3.9}$$

if the equation is differentiated three times, then we obtain the characteristic equation, $m^4(m-1)(m-2) = 0$ that has solutions, $m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 0, m_5 = 1, m_6 = 2$.

Thus, the general solution of (3.9) is expressed by $y(x) = y_p(x) + c_4\, e_{p,q}(x) + c_5\, e_{p,q}(2x)$ where $y_p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. By substituting $y_p(x)$ into equation (3.9), we find the unknown coefficient $c_i$, $i = 0, \cdots, 3$. If initial conditions are given, then we also get the other coefficients $c_5$ and $c_6$.

Now, consider the second order case when the coefficients in (3.2) are analytic functions.

**Theorem 3.3.** *Let $f(x)$, $h(x)$ and $y(x)$ be analytic functions at $x = 0$ ($\in \mathbb{C}$). Linear homogeneous $(p,q)$-difference equation of second order,*

$$[D_{p,q}^2 + f(x)D_{p,q} + h(x)]y(x) = 0,$$

*with $f(x)$ and $h(x)$, allows two linear independent analytic solution at $x = 0$.*

*Proof.* Since $f(x)$ and $h(x)$ are analytic functions at $x = 0$, we can expressed the function in series as below :

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,\ h(x) = \sum_{n=0}^{\infty} h_n x^n \tag{3.10}$$

where $f_n = \frac{f^n(0)}{n!}$ and $h_n = \frac{h^n(0)}{n!}$.
If the function $y(x)$ is analytic at $x = 0$, then we can write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{3.11}$$

By using (3.10) and (3.11), we obtain

$$D_{p,q}^2 \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} f_n x^n D_{p,q} \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} h_n x^n \sum_{n=0}^{\infty} a_n x^n = 0. \tag{3.12}$$

From the calculation of (3.12), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} a_{n+2} \frac{(p^{n+2} - q^{n+2})(p^{n+1} - q^{n+1})}{(p-q)^2} x^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{n-k} a_{k+1} \frac{(p^{k+1} - q^{k+1})}{(p-q)} x^n \\ &+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} h_{n-k} a_k x^n = 0\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&a_{n+2} \frac{(p^{n+2} - q^{n+2})(p^{n+1} - q^{n+1})}{(p-q)^2} \\ &= -\sum_{k=0}^{n} (f_{n-k} a_{k+1} \frac{(p^{k+1} - q^{k+1})}{(p-q)} + \sum_{k=0}^{n} h_{n-k} a_k, n = 0, 1, 2, \ldots.\end{aligned}$$

So, the coefficient $a_n, n = 2, 3 \ldots$ is determined by (3.13). Since the coefficient $a_0$ and $a_1$ are arbitrary complex numbers, we obtain two linear independent analytic solutions. □

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