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A RELATIVE RÉNYI OPERATOR ENTROPY†

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Abstract. We define an operator version of the relative Rényi entropy as the generalization of relative von Neumann entropy, and provide its fundamental properties and the bounds for its trace value. Moreover, we see an effect of the relative Rényi entropy under tensor product, and show the sub-additivity for density matrices.

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1. Introduction

Claude Shannon was the originator of information theory which has had a lot of influence on physics, medical science, and economics and he systematized information theory from a mathematical perspective rather than a physical one. In [12], Shannon defined the expected value when m characters are expected with the probability p_i as the (Shannon) entropy given by

$$H(p) := -\sum_{i=1}^{m} p_i \log_2 p_i.$$

As the definition suggests, the entropy signifies the average of the amount of information, while the relative entropy means the closeness of two probability distributions and is defined as

$$D(p|q) := H(p,q) - H(p) = -\sum_{i=1}^{m} p_i \log_2 q_i + \sum_{i=1}^{m} p_i \log_2 p_i = -\sum_{i=1}^{m} p_i \log_2 \frac{q_i}{p_i},$$

where $H(p,q)$ is the joint entropy of p and q. This is called the Kullback-Leibler divergence [9].

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In quantum mechanics the role of a probability vector $(p_1, p_2, \ldots, p_m)$ is played by a *density matrix*, which is a positive semi-definite Hermitian matrix of trace 1. Density matrices play a crucial role in quantum mechanics, quantum decoherence, and quantum information and computation. The quantum entropy of a density matrix ρ is defined as

$$S(\rho) := -\mathrm{tr}\rho \log \rho.$$

One can see easily that $S(\rho) = H(\lambda)$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is an m-tuple of non-negative eigenvalues of ρ.

The quantum relative entropy, also called the von Neumann entropy, is defined by extending the Kullback-Leibler divergence from classical mechanics to quantum mechanics. The relative von Neumann entropy is defined as

$$D(\rho|\sigma) := \mathrm{tr}(\rho(\log \rho - \log \sigma))$$

for density matrices ρ and σ. Such quantum relative entropy plays an important role in the entanglement of a quantum state and the Markov chain.

Let $\rho, \sigma, \tau, \omega$ be positive semi-definite operators, where ρ, τ be non-zero. Suppose $\ker \sigma \subset \ker \rho$ and $\ker \omega \subset \ker \tau$, where $\ker \rho$ is a kernel of ρ. Rényi introduced six axioms of the relative entropy (or divergence) in the quantum information theory and proved that the Rényi divergence satisfies the six axioms (see [10]).

(1) $D(\rho|\sigma)$ is continuous in ρ, σ;
(2) $D(U\rho U^*|U\sigma U^*) = D(\rho|\sigma)$ for any unitary matrix U;
(3) $D\left(I|\frac{1}{2}I\right) = 1$;
(4) $D(\rho|\sigma) \geq 0$ if $\rho \geq \sigma$ and $D(\rho|\sigma) \leq 0$ otherwise;
(5) $D(\rho \otimes \tau|\sigma \otimes \omega) = D(\rho|\sigma) + D(\tau|\omega)$;
(6) there exists a strictly monotone and continuous function g such that if $\mathrm{tr}(\rho + \tau) \leq 1$ and $\mathrm{tr}(\sigma + \omega) \leq 1$ then

$$D(\rho \otimes \tau|\sigma \otimes \omega) = g^{-1}\left(\frac{\mathrm{tr}\rho}{\mathrm{tr}(\rho+\tau)} \cdot g(D(\rho|\sigma)) + \frac{\mathrm{tr}\tau}{\mathrm{tr}(\rho+\tau)} \cdot g(D(\tau|\omega))\right).$$

In this paper, we consider new type of quantum relative Rényi entropy as the parameterized version of the relative von Neumann entropy $D(\rho|\sigma)$:

$$R_\alpha(\rho|\sigma) = \frac{1}{\alpha - 1}\left[\mathrm{tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha - 1\right]$$

for $0 < \alpha < 1$, and will examine whether R_α satisfies several properties of Rényi's axioms. More precisely, we prove that R_α is invariant under a unitary congruence transformation corresponding to (2), is non-negative corresponding to (4), and has a subadditivity corresponding to (5). Before we show this properties, we define the operator version T_α of the relative Rényi entropy and introduce the relationship with the von Neumann entropy and the boundedness of T_α for any two positive definite matrices.

2. Preliminaries on relative Rényi entropy

Let $\mathbb{H}_m$ be a real vector space of all $m \times m$ Hermitian matrices, and let $\mathbb{P}_m \subset \mathbb{H}_m$ be the open convex cone of all positive definite matrices. Note that the closure $\overline{\mathbb{P}_m}$ coincides with the closed convex cone of all positive semi-definite matrices. For $A, B \in \mathbb{H}_m$ we define as $A \leq B$ if and only if $B - A$ is positive semi-definite, as $A < B$ if and only if $B - A$ is positive definite. This is known as the Loewner partial order.

Let $\alpha > 0$ and $\alpha \neq 1$. The *standard Rényi entropy* (or *standard Rényi divergence*) of A and B in $\mathbb{P}_m$ is given [11] by

$$D_\alpha(A|B) = \frac{1}{\alpha - 1} \log \frac{\mathrm{tr} B^{\frac{1-\alpha}{2}} A^\alpha B^{\frac{1-\alpha}{2}}}{\mathrm{tr} A},$$

meanwhile, the *sandwiched Rényi entropy* (or *sandwiched Rényi divergence*) of A and B is given [10, 13] by

$$\tilde{D}_\alpha(A|B) = \frac{1}{\alpha - 1} \log \frac{\mathrm{tr}\left(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}}\right)^\alpha}{\mathrm{tr} A}.$$

In addition, we have from [8, Corollary 8.2]

$$\tilde{D}_\alpha(A|B) \leq D_\alpha(A|B) \leq \hat{D}_\alpha(A|B)$$

for $0 < \alpha \leq 2$ and $\alpha \neq 1$, where

$$\hat{D}_\alpha(A|B) = \frac{1}{\alpha - 1} \log \frac{\mathrm{tr} B \#_\alpha A}{\mathrm{tr} A},$$

and $B \#_\alpha A = B^{1/2}(B^{-1/2} A B^{-1/2})^\alpha B^{1/2}$ is known as the weighted geometric mean of A and B. In this sense, $\hat{D}_\alpha(A|B)$ is called the *maximal Rényi entropy* of A and B.

For density matrices ρ and σ, the sandwiched Rényi entropy (or quantum Rényi divergence) is defined by

$$\tilde{D}_\alpha(\rho|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{tr}(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha$$

for any $\alpha \in (0,1) \cup (1, \infty)$. In [6] the factor $(\mathrm{tr}\rho)^{-1}$ was appeared in the definition of sandwiched Rényi entropy $\tilde{D}_\alpha(\rho|\sigma)$, but it could be dropped for any density matrix ρ.

Remark 2.1. Note that $\tilde{D}_\alpha(\rho|\sigma)$ is the relative von Neumann entropy for $\alpha = 1$, that is,

$$\tilde{D}_1(\rho|\sigma) := \lim_{\alpha \to 1} \tilde{D}_\alpha(\rho|\sigma) = \mathrm{tr}\rho(\log \rho - \log \sigma).$$

For $\alpha = 1/2$, $D_\alpha(\rho|\sigma)$ is closely related to the fidelity $F(\rho, \sigma) = \mathrm{tr}(\sigma^{1/2} \rho \sigma^{1/2})^{1/2}$, that is,

$$D_{1/2}(\rho|\sigma) = -2 \log F(\rho, \sigma).$$

3. Relative Rényi operator entropy

We consider in this article the operator version of relative Rényi entropy. Let $A, B \in \mathbb{P}_m$ and $\alpha > 0$, $\alpha \neq 1$. The *relative Rényi operator entropy* of A and B is defined by

$$T_\alpha(A|B) = \frac{1}{\alpha - 1}\left[\left(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}}\right)^\alpha - A\right].$$

Theorem 3.1. *For any $A, B \in \mathbb{P}_m$*

$$\lim_{\alpha \to 1} T_\alpha(A|B) = A \log A - \frac{1}{2}[(\log B)A + A(\log B)].$$

Proof. Let $\beta = \dfrac{1-\alpha}{2\alpha}$. Then

$$\frac{d}{d\alpha}(B^\beta A B^\beta)^\alpha = \frac{d}{d\alpha} e^{\alpha \log(B^\beta A B^\beta)}$$
$$= e^{\alpha \log(B^\beta A B^\beta)}\left[\log(B^\beta A B^\beta) - \frac{1}{2\alpha}(B^\beta A B^\beta)^{-1}\{(\log B)B^\beta A B^\beta + B^\beta A B^\beta (\log B)\}\right].$$

So

$$\lim_{\alpha \to 1} T_\alpha(A|B) = \lim_{\alpha \to 1} \frac{d}{d\alpha} e^{\alpha \log(B^\beta A B^\beta)} = A \log A - \frac{1}{2}[(\log B)A + A(\log B)].$$

□

Since the trace map is continuous and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we obtain the following.

Corollary 3.2. *For any $A, B \in \mathbb{P}_m$*

$$\lim_{\alpha \to 1} trT_\alpha(A|B) = trA(\log A - \log B).$$

We see some fundamental properties of the relative Rényi operator entropy.

Theorem 3.3. *The relative Rényi operator entropy T_α satisfies*

(1) $T_\alpha(A|B) = 0$ *if and only if* $A = B$,
(2) $T_\alpha(UAU^*|UBU^*) = UT_\alpha(A|B)U^*$ *for any unitary matrix* U,
(3) $T_\alpha(\cdot|B) : \mathbb{P}_m \to \mathbb{P}_m$ *for given* $\alpha \in (0, 1)$ *and* $B \in \mathbb{P}_m$ *is convex: for any* $A_1, A_2, B \in \mathbb{P}_m$ *and* $\lambda \in [0, 1]$

$$T_\alpha((1-\lambda)A_1 + \lambda A_2|B) \leq (1-\lambda)T_\alpha(A_1|B) + \lambda T_\alpha(A_2|B).$$

Proof. (1) Assume that $T_\alpha(A|B) = 0$ for $\alpha > 0$, $\alpha \neq 1$. Then $B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}} = A^{\frac{1}{\alpha}}$, and taking congruence transformation by $A^{1/2}$ we have

$$\left(A^{\frac{1}{2}} B^{\frac{1-\alpha}{2\alpha}} A^{\frac{1}{2}}\right)^2 = A^{\frac{1+\alpha}{\alpha}}.$$

Taking square root on both sides and simplifying yield $A^{\frac{1-\alpha}{2\alpha}} = B^{\frac{1-\alpha}{2\alpha}}$, and hence, $A = B$. The converse is trivial.

(2) The invariance under unitary congruence transformation (2) is obvious, since $UA^tU^* = (UAU^*)^t$ for any $A \in \mathbb{P}_m$ and $t \in \mathbb{R}$.

(3) Let $0 < \alpha < 1$. One can write the relative Rényi operator entropy as

$$T_\alpha(A|B) = \frac{1}{\alpha-1}\left[B^{\frac{1-\alpha}{2\alpha}}(B^{\frac{\alpha-1}{\alpha}}\#_\alpha A)B^{\frac{1-\alpha}{2\alpha}} - A\right].$$

By joint concavity of two-variable weighted geometric mean, we have

$$B^{\frac{\alpha-1}{\alpha}}\#_\alpha((1-\lambda)A_1+\lambda A_2) \geq (1-\lambda)B^{\frac{\alpha-1}{\alpha}}\#_\alpha A_1 + \lambda B^{\frac{\alpha-1}{\alpha}}\#_\alpha A_2.$$

Since the congruence transformation is operator monotone,

$$\begin{aligned}&B^{\frac{1-\alpha}{2\alpha}}\left(B^{\frac{\alpha-1}{\alpha}}\#_\alpha((1-\lambda)A_1+\lambda A_2)\right)B^{\frac{1-\alpha}{2\alpha}}\\ &\geq (1-\lambda)(B^{\frac{1-\alpha}{2\alpha}}A_1B^{\frac{1-\alpha}{2\alpha}})^\alpha + \lambda(B^{\frac{1-\alpha}{2\alpha}}A_2B^{\frac{1-\alpha}{2\alpha}})^\alpha.\end{aligned}$$

Since the translation by $-((1-\lambda)A_1+\lambda A_2)$ is order preserving and a multiplication by a negative scalar $\alpha - 1$ is order reversing, we obtain (3) after a simple calculation.

□

The following provides the boundedness for trace value of relative Rényi operator entropy.

Theorem 3.4. *For any $A, B \in \mathbb{P}_m$ and $\alpha \in (0,1)$*

$$tr(A-B) \leq trT_\alpha(A|B) \leq tr(AB^{-1}A - A).$$

Proof. Note from [3, Theorem 10] that for any $A, B \in \mathbb{P}_m$ and $\alpha \in (0,1)$

$$\mathrm{tr}B\#_\alpha A \leq \mathrm{tr}B^{1-\alpha}A^\alpha \leq \mathrm{tr}(B^{\frac{1-\alpha}{2\alpha}}AB^{\frac{1-\alpha}{2\alpha}})^\alpha \leq \mathrm{tr}[(1-\alpha)B+\alpha A].$$

So we obtain

$$\frac{1}{\alpha-1}\mathrm{tr}[B\#_\alpha A - A] \geq \frac{1}{\alpha-1}\mathrm{tr}[B^{1-\alpha}A^\alpha - A] \geq \mathrm{tr}T_\alpha(A|B) \geq \mathrm{tr}(A-B). \quad (1)$$

Since

$$A - AB^{-1}A \leq \frac{A\#_tB - A}{t} \leq B - A$$

for any $t \in (0,1)$ from [4, 5], replacing α be $1-t$ in (1) and applying the symmetry of weighted geometric mean: $A\#_tB = B\#_{1-t}A$ we obtain

$$\begin{aligned}\mathrm{tr}(A-B) \leq \mathrm{tr}T_\alpha(A|B) &\leq \frac{1}{\alpha-1}\mathrm{tr}[B\#_\alpha A - A]\\ &= -\mathrm{tr}\frac{A\#_tB - A}{t} \leq \mathrm{tr}(AB^{-1}A - A).\end{aligned}$$

□

Remark 3.1. Taking the limit as $\alpha \to 1$ to the first inequality in Theorem 3.4 we obtain

$$\mathrm{tr}A(\log A - \log B) \geq \mathrm{tr}(A-B).$$

This is known as the Klein's inequality in [2, Exercise 4.3.4].

The quantity $F_\alpha(A,B) = \mathrm{tr}(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}})^\alpha$ appeared in [13] is a parameterized version of fidelity, since the usual fidelity is the special case for $\alpha = 1/2$. Several extremal representations for the quantity $F_\alpha(A,B)$ have been shown in [3].

Theorem 3.5. [3, Theorem 8] *Let $A, B \in \mathbb{P}_m$ and $\alpha \in (0,1)$. Then*

(i) $F_\alpha(A,B) = \min_{X\in\mathbb{P}_m} tr\left[(1-\alpha)(B^{\frac{\alpha-1}{2\alpha}} X B^{\frac{\alpha-1}{2\alpha}})^{\frac{\alpha}{\alpha-1}} + \alpha XA\right]$,

(ii) $F_\alpha(A,B) = \min_{X\in\mathbb{P}_m} \left[tr(B^{\frac{\alpha-1}{2\alpha}} X B^{\frac{\alpha-1}{2\alpha}})^{\frac{\alpha}{\alpha-1}}\right]^{1-\alpha} [tr(XA)]^\alpha$,

(iii) $F_\alpha(A,B) = \min_{X\in\mathbb{P}_m} tr\left[\alpha B^{\frac{1-\alpha}{\alpha}} X + (1-\alpha)\left(A^{-1/2} X A^{-1/2}\right)^{\frac{\alpha}{\alpha-1}}\right]$,

(iv) $F_\alpha(A,B) = \min_{X\in\mathbb{P}_m} \left[tr\left(B^{\frac{1-\alpha}{\alpha}} X\right)\right]^\alpha \left[tr\left(A^{-1/2} X A^{-1/2}\right)^{\frac{\alpha}{\alpha-1}}\right]^{1-\alpha}$.

Corollary 3.6. *Let $A, B \in \mathbb{P}_m$ and $\alpha \in (0,1)$. Then*

$$trT_\alpha(A^{1-\alpha}|B^\alpha) \geq \frac{1}{1-\alpha}\left[trA^{1-\alpha} - K\right],$$

where

$$K = \min\left\{\left[trA^{1-\alpha}\right]^\alpha \left[trB^\alpha\right]^{1-\alpha}, \left[trA^\alpha\right]^{1-\alpha}\left[trB^{1-\alpha}\right]^\alpha\right\}.$$

Proof. Note that for any $A, B \in \mathbb{P}_m$ and $\alpha \in (0,1)$

$$\mathrm{tr}T_\alpha(A|B) = \frac{1}{1-\alpha}\left[\mathrm{tr}A - F_\alpha(A,B)\right]. \tag{2}$$

Taking $X = I$ in Theorem 3.5 (ii) yields

$$F_\alpha(A,B) \leq [\mathrm{tr}A]^\alpha [\mathrm{tr}B]^{1-\alpha}.$$

Similarly, from Theorem 3.5 (iv)

$$F_\alpha(A,B) \leq \left[\mathrm{tr}\left(A^{\frac{\alpha}{1-\alpha}}\right)\right]^{1-\alpha}\left[\mathrm{tr}\left(B^{\frac{1-\alpha}{\alpha}}\right)\right]^\alpha.$$

So it holds after replacing A and B by $A^{1-\alpha}$ and B^α in (2). ☐

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be arbitrary matrices with certain sizes. The *tensor product* (or *Kronecker product*) $A \otimes B$ of A and B is the matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1nB} \\ a_{21}B & a_{22}B & \cdots & a_{2nB} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mnB} \end{bmatrix}.$$

One can see easily that the tensor product is bilinear and associative, but not commutative. Moreover, it preserves the positivity: the tensor product of two positive semi-definite (positive definite) matrices is positive semi-definite (positive definite, respectively).

Lemma 3.7. *The following holds:*

(1) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ *provided the matrix multiplications are valid,*

(2) $(A \otimes B)^s = A^s \otimes B^s$ *for* $A, B \in \mathbb{P}_m$ *and* $s \in \mathbb{R}$.

Theorem 3.8. *Let* $A_1, A_2, B_1, B_2 \in \mathbb{P}_m$ *and let* $\alpha > 0, \alpha \neq 1$. *Then*

$$\begin{aligned} &T_\alpha(A_1 \otimes A_2 | B_1 \otimes B_2) \\ &= (\alpha - 1) T_\alpha(A_1|B_1) \otimes T_\alpha(A_2|B_2) + A_1 \otimes T_\alpha(A_2|B_2) + T_\alpha(A_1|B_1) \otimes A_2. \end{aligned}$$

Proof. Let $\beta = \dfrac{1-\alpha}{2\alpha}$. Then we have from Lemma 3.7

$$\begin{aligned} &[(B_1 \otimes B_2)^\beta (A_1 \otimes A_2)(B_1 \otimes B_2)^\beta]^\alpha - A_1 \otimes A_2 \\ &= (B_1^\beta A_1 B_1^\beta)^\alpha \otimes (B_2^\beta A_2 B_2^\beta)^\alpha - A_1 \otimes A_2 \\ &= [(B_1^\beta A_1 B_1^\beta)^\alpha - A_1] \otimes [(B_2^\beta A_2 B_2^\beta)^\alpha - A_2] + A_1 \otimes [(B_2^\beta A_2 B_2^\beta)^\alpha - A_2] \\ &\quad + [(B_1^\beta A_1 B_1^\beta)^\alpha - A_1] \otimes A_2. \end{aligned}$$

Thus, we obtain the desired identity. □

4. Quantum relative Rényi entropy

For any density matrices ρ and σ the *quantum relative Rényi entropy* of ρ and σ is given by

$$R_\alpha(\rho|\sigma) = \frac{1}{\alpha - 1}\left[\mathrm{tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha - 1\right]$$

for $\alpha > 0$ and $\alpha \neq 1$. Note that $R_\alpha(\rho|\sigma) = \mathrm{tr} T_\alpha(\rho|\sigma)$. For density matrices ρ and τ, it can not be possible that $\mathrm{tr}(\rho + \tau) \leq 1$. Thus, we do not want to consider that whether $R_\alpha(\rho|\sigma)$ satisfies the axiom (6). Moreover, the trace of the identity matrix is the dimension of the matrix so this is not the case with the axiom (3) in this paper. But, from the definition of the quantum relative Rényi entropy, we obtain that $R_\alpha(\rho|\sigma)$ is nonnegative and unitarily invariant.

Remark 4.1. The notations $\langle x|$ and $|x\rangle$, known as a *bra vector* and a *ket vector*, represent row and column vectors in $\mathbb{C}^m$ respectively. An important linear functional is a bra vector obtained by a ket vector. Given $|x\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{C}^m$ we define a bra vector $\langle x|$ associated to $|x\rangle$ by

$$|x\rangle \mapsto \langle x| = (\overline{x_1}, \ldots, \overline{x_m}) \in (\mathbb{C}^m)^*,$$

where $(\mathbb{C}^m)^*$ is a dual space of $\mathbb{C}^m$. Note that for given $|x\rangle, |y\rangle \in \mathbb{C}^m$ their inner product is given by

$$\langle x|y\rangle = (|x\rangle)^* |y\rangle = \sum_{i=1}^m \overline{x_i} y_i.$$

Note that a density matrix whose eigenvalues are only 1 and 0 is called a pure state, and the other is called a mixed state. Assume that σ is a density matrix representing pure state. So there exist a unit vector $|a\rangle$ in $\mathbb{C}^m$ such that $\sigma = |a\rangle\langle a|$. Then

$$\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}} = |a\rangle\langle a|\rho|a\rangle\langle a|,$$

so $\mathrm{tr}(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha = \langle a|\rho|a\rangle^\alpha$. Thus,

$$R_\alpha(\rho|\sigma) = \frac{\langle a|\rho|a\rangle^\alpha - 1}{\alpha - 1}. \tag{3}$$

In addition, if ρ is also a density matrix representing pure state, say $\rho = |b\rangle\langle b|$ for a unit vector $|b\rangle$ in $\mathbb{C}^m$, then (3) reduces to

$$R_\alpha(\rho|\sigma) = \frac{|\langle a|b\rangle|^{2\alpha} - 1}{\alpha - 1} = \frac{\cos^{2\alpha}\theta - 1}{\alpha - 1},$$

where θ is the angle between two unit vectors $|a\rangle$ and $|b\rangle$.

Lemma 4.1. *For any density matrices ρ, σ, and $\alpha \in (0,1)$, we have $R_\alpha(\rho|\sigma) \geq 0$.*

Proof. One can see that $\rho + \frac{1}{n}I, \sigma + \frac{1}{n}I \in \mathbb{P}_m$ for all $n \in \mathbb{N}$. By Theorem 3.4

$$R_\alpha\left(\rho + \frac{1}{n}I \,\middle|\, \sigma + \frac{1}{n}I\right) = \mathrm{tr}T_\alpha\left(\rho + \frac{1}{n}I \,\middle|\, \sigma + \frac{1}{n}I\right) \geq 0.$$

By continuity we obtain that $R_\alpha(\rho|\sigma) \geq 0$. $\square$

Remark 4.2. From Theorem 3.4 we have

$$0 \leq R_\alpha(\rho|\sigma) \leq \mathrm{tr}(\rho\sigma^{-1}\rho) - 1$$

for any invertible density matrices ρ and σ.

The following hold from Theorem 3.3 together with continuity such that $\rho + \frac{1}{n}I \to \rho$ as $n \to \infty$.

Theorem 4.2. *The quantum relative Rényi entropy R_α satisfies*

(i) $R_\alpha(U\rho U^*|U\sigma U^*) = UR_\alpha(\rho|\sigma)U^*$ *for any unitary matrix U,*
(ii) $R_\alpha(\rho|\sigma)$ *for $\alpha \in (0,1)$ is convex on ρ: for any density matrices ρ_1, ρ_2, σ and $\lambda \in [0,1]$*

$$R_\alpha((1-\lambda)\rho_1 + \lambda\rho_2|\sigma) \leq (1-\lambda)R_\alpha(\rho_1|\sigma) + \lambda R_\alpha(\rho_2|\sigma).$$

One can easily see that the tensor product of density matrices is again a density matrix. We now show the subadditivity of the quantum relative Rényi entropy under the tensor product.

Theorem 4.3. *For any density operators $\rho_1, \rho_2, \sigma_1, \sigma_2$, and $\alpha \in (0,1)$,*

$$R_\alpha(\rho_1 \otimes \rho_2|\sigma_1 \otimes \sigma_2) \leq R_\alpha(\rho_1|\sigma_1) + R_\alpha(\rho_2|\sigma_2).$$

Proof. Note that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$ for any square matrices A and B. Since $R_\alpha(\rho|\sigma) = \mathrm{tr}T_\alpha(\rho|\sigma)$, Theorem 3.8 and Lemma 4.1 yield

$$\begin{aligned} R_\alpha(\rho_1 \otimes \rho_2|\sigma_1 \otimes \sigma_2) &= \mathrm{tr}T_\alpha(\rho_1 \otimes \rho_2|\sigma_1 \otimes \sigma_2) \\ &= (\alpha - 1)R_\alpha(\rho_1|\sigma_1)R_\alpha(\rho_2|\sigma_2) + R_\alpha(\rho_2|\sigma_2) + R_\alpha(\rho_1|\sigma_1) \\ &\le R_\alpha(\rho_2|\sigma_2) + R_\alpha(\rho_1|\sigma_1). \end{aligned}$$

□

5. Final remark and open questions

In Section 3 we introduced the relative Rényi operator entropy $T_\alpha(A|B)$ of positive definite Hermitian matrices A and B for $\alpha > 0$ and $\alpha \neq 1$. On the other hand, many properties of convexity in Theorem 3.3 and boundedness in Theorem 3.4 and Corollary 3.6 have been shown for $\alpha \in (0, 1)$. Also in Section 4 we studied the subadditivity of the quantum relative Rényi entropy $R_\alpha(\rho|\sigma)$ of density matrices ρ and σ for $\alpha \in (0, 1)$. It is an interesting problem what happens on consequences in this paper for the case of $\alpha \in (1, \infty)$.

We have seen in Theorem 3.8 the effect of the relative Rényi operator entropy T_α under tensor product. Note from Schur's product theorem in [1, Lemma 4] that the Hadamard product (or entry-wise product) of positive (semi-)definite Hermitian matrices is again positive (semi-)definite. Indeed, there exists a strictly positive and unital linear map Φ such that for any square matrices A, B

$$\Phi(A \otimes B) = A \circ B.$$

Analogous to Theorem 3.8 and Theorem 4.3, it is an interesting problem to find the relationship between $T_\alpha(A_1 \circ A_2|B_1 \circ B_2)$ and $T_\alpha(A_1|B_1) \circ T_\alpha(A_2|B_2)$ for positive definite Hermitian matrices A_1, A_2, B_1, B_2, also for density matrices.

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