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ACCESS TO LAPLACE TRANSFORM OF $fg^{\dagger}$

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Abstract. We would like to consider Laplace transform of the form of fg, the form of product, and applies it to Burger's equation in general case. This topic has not yet been addressed, and the methodology of this article is done by considerations with respect to several approaches about the transform of the form of fg and the mean value theorem for integrals. This paper has meaning in that the integral transform method is applied to solving nonlinear equations.

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1. Introduction

Theories on integral transforms provide a reasonable tool for solving differential equations[3]. Among these transforms, Laplace transform is recognized as the most reasonable and easy to use due to its simplicity. The advantage of the integral transform method is that it gives a simple tool which is represented by an algebraic representation. Of course, these integral transforms are meaningful not only in the given space, but also in the transformed space in itself. The utility of integral transforms can be easily seen in computed tomography scan or magnetic resonance imaging[14]. Normally, we obtain the projection data by integral transform, and produce the image with the inverse transform[11].

To begin with, let us see the intrinsic structure of integral transforms. The structure of it is of the form

$$\int_0^\infty k(s,t)f(t)dt,$$

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and the Laplace transform has the kernel $k(s,t) = e^{-st}$ as we know already[15]. It is interesting that the kernel has the form of exponential function. It is considered that the reason is to use the property that the function decreases gently and e^{-st} converges to 0 when t approaches ∞. Since the Laplace transform can be rewritten as

$$\int_0^\infty e^{-\frac{t}{u}} f(t)dt$$

by $s = 1/u$, we can propose the general form of Laplace-typed transform by

$$u^\alpha \int_0^\infty e^{-\frac{t}{u}} f(t)dt$$

as a natural extension[11]. In this comprehensive form, the integer value can be suitably selected in various problems. The value $\alpha = 0, -1$,and 1 corresponds to Laplace[15-16, 18], Sumudu[6, 19], and Elzaki transform[9, 12-13], respectively in the above form. Of course, the integer value α can be extended to real value[17].

On the other hand, $\pounds(f)\pounds(g)$ can be solved by $\pounds(f)\pounds(g) = \pounds(f * g)$ for $f * g$ is the convolution of f and g. However, $\pounds(fg)$ is pretty hard to deal with. Therefore, we would like to cover this topic here. The main objective of this paper is to approach with respect to Laplace transform of $\pounds(fg)$, and this approach has an important value because it is a key issue in the transform theory and the first attempt in this topic. According to the research results so far, the integral transform method was not used to find the solutions of nonlinear equations. This paper can be interpreted as a new attempt to break away from this point of view.

The obtained results are as follows; There exists a point $c \in (0, \infty)$ such that

$$\pounds(u \cdot u_x) = \frac{1}{s}\ u(x,c)u_x(x,c) = \frac{1}{s}\ [(\frac{u^2}{2})_x]_{t=c}.$$

Moreover, if $u \geq 0$, then there exists a point c such that

$$\pounds(u \cdot u_x) = u_x(x,c)\pounds(u).$$

The latter can be generalized as $\pounds(fg) = g(c)\pounds(f)$ on the condition that f and g are integrable and f is non-negative. Additionally, $\pounds(u \cdot u_x)$ can be represented as

$$\frac{1}{s}\ (\frac{u^2}{2})_x(x,0) + \frac{1}{s}\ \pounds[\ d(\frac{u^2}{2})_x],$$

$$-(\frac{u^2}{2})(0,t) + s\pounds(\frac{u^2}{2}),$$

$$\frac{1}{s}\ (\frac{u^2}{2})_x\ (0,t) + \frac{1}{s}\ \pounds[d(\frac{u^2}{2})_x],$$

$$-\int_0^\infty [\{u(x,0)\}^2/2]_x\ dt + \pounds[\{u(x,\infty)\}^2/2]_x,$$

or

$$\frac{1}{s}\ u(x,0)u_x(x,0) + \frac{1}{s}\pounds[d\ (u \cdot u_x)]$$

for f_x is the partial derivative with respect to x.

2. Access to Laplace transform of fg

We would like to approach Laplace transform of fg, $\pounds(fg)$, mainly $\pounds(u \cdot u_x)$ for application.

Lemma 2.1. *(Lagrange's method[15]) Lagrange's method states that a particular solution y_p of $y'' + p(x)y' + q(x)y = r(x)$ on open interval I is*

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$ on I, and W is the Wronskian of y_1, y_2.

Lemma 2.2. *(Monotone convergence theorem(MCT)[2, 5]). Let M^+ be the collection of all non-negative measurable function and let μ be a measure. If (f_n) is a monotone increasing sequence of functions in M^+ which converges to f, then*

$$\int f\ d\mu = \lim \int f_n\ d\mu.$$

Let us put $f_n = g_1 + \cdots + g_n$ and apply the MCT. Then we obtain the following lemma.

Lemma 2.3. *(Beppo Livi's theorem[2]). Let (g_n) be a sequence in M^+, then*

$$\int \sum_{n=1}^{\infty} g_n\ d\mu = \sum_{n=1}^{\infty} \int g_n\ d\mu.$$

Lemma 2.4. *(The Mean Value Theorem for Integrals[1]). Let f be continuous on an interval $I = [a,b]$ and let p be integrable on I and such that $p(x) \geq 0$ for all $x \in I$. Then there exists a point $c \in I$ such that*

$$\int_a^b\ f(x)p(x)\ dx = f(c)\ \int_a^b\ p(x)dx.$$

Note that a continuous function on a closed bounded interval is integrable on there.

Lemma 2.5. *(The Mean Value Theorem for Double Integrals[1]) Let $z = f(x,y)$ be a continuous function on the closed, bounded, and connected subset $D \subseteq D(f)$, and let A be the positive area of D. Then there exists a point $(x_0, y_0) \in D$ such that*

$$\int\int_D f(x,y)\ dA = Af(x_0, y_0).$$

Theorem 2.6.

$$\frac{\partial}{\partial x}\pounds(\frac{u^n}{n}) = \pounds(u^{n-1}\cdot u_x)$$

for an arbitrary integer n.

Proof.

$$\begin{aligned}\frac{\partial}{\partial x}\pounds(\frac{u^n}{n}) &= \int_0^\infty e^{-st}\ nu^{n-1}\ u_x/n\ dt\\ &= \pounds(u^{n-1}\cdot u_x).\end{aligned}$$

□

For example, if $u = x^2t^2$, then

$$\frac{\partial}{\partial x}\pounds(\frac{u^2}{2}) = \frac{1}{2}\pounds(4x^3t^4) = \pounds(2x^3t^4)$$

and $\pounds(u\cdot u_x) = \pounds(x^2t^2\cdot 2xt^2) = \pounds(2x^3t^4)$, and so

$$\frac{\partial}{\partial x}\pounds(\frac{u^2}{2}) = \pounds(u\cdot u_x).$$

Theorem 2.6 can be naturally rewritten as

$$\frac{\partial}{\partial x}\pounds(\sum_{k=1}^{n}\frac{u^k}{k}) = \pounds(\sum_{k=1}^{n}u^{k-1}\cdot u_x)$$

for an arbitrary integer n.

Theorem 2.7. *Let f_x be the partial derivative with respect to x. Then*

$$(A)\quad \pounds(u\cdot u_x) = \frac{1}{s}\ (\frac{u^2}{2})_x(x,0) + \frac{1}{s}\ \pounds[\ d(\frac{u^2}{2})_x].$$

$$(B)\quad \pounds(u\cdot u_x) = -(\frac{u^2}{2})(0,t) + s\pounds(\frac{u^2}{2}).$$

$$(C)\quad \pounds(u\cdot u_x) = \frac{1}{s}\ (\frac{u^2}{2})_x\ (0,t) + \frac{1}{s}\ \pounds[d(\frac{u^2}{2})_x].$$

$$(D)\quad \pounds(u\cdot u_x) = -\int_0^\infty [\{u(x,0)\}^2/2]_x\ dt + \pounds[\{u(x,\infty)\}^2/2]_x.$$

$$(E)\quad \pounds(u\cdot u_x) = \frac{1}{s}\ u(x,0)u_x(x,0) + \frac{1}{s}\pounds[d\ (u\cdot u_x)]$$

Proof. (A) Let us consider two cases. First, let us only differentiate for t. Since

$$\pounds(u\cdot u_x) = \pounds\{(\frac{u^2}{2})_x\} = \int_0^\infty e^{-st}\ (\frac{u^2}{2})_x\ dt,$$

integration of parts gives

$$\begin{aligned}\pounds(u\cdot u_x) &= -\frac{1}{s}(\frac{u^2}{2})_x\ e^{-st}]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\cdot(\frac{u^2}{2})_{xt}\ dt\\ &= \frac{1}{s}\ (\frac{u^2}{2})_x(x,0) + \frac{1}{s}\ \pounds(\frac{u^2}{2})_{xt}.\end{aligned}$$

Next, if we differentiate for x and t, in a similar way, we have

$$\begin{aligned}\pounds(u\cdot u_x) &= \frac{1}{s}\ (\frac{u^2}{2})_x(x,0) + \frac{1}{s}\ \pounds[(\frac{u^2}{2})_{xt} + (\frac{u^2}{2})_{xx}]\\ &= \frac{1}{s}\ (\frac{u^2}{2})_x(x,0) + \frac{1}{s}\ \pounds[\ d(\frac{u^2}{2})_x].\end{aligned}$$

(B) Since $\int_A f(x)dx = \int_A f(t)dt$,

$$\pounds(u\cdot u_x) = \pounds\{(\frac{u^2}{2})_x\} = \int_0^\infty e^{-sx}\ (\frac{u^2}{2})_x\ dx.$$

If we integrate with respect to x, we have

$$\begin{aligned}\pounds\{(\frac{u^2}{2})_x\} &= e^{-sx}\cdot(\frac{u^2}{2})]_0^\infty + s\int_0^\infty e^{-sx}\ (\frac{u^2}{2})\ dx\\ &= -(\frac{u^2}{2})(0,t) + s\pounds(\frac{u^2}{2}).\end{aligned}$$

(C) Doing integration by parts, if we change the form of the derivative, we have

$$\begin{aligned}&= -\frac{1}{s}(\frac{u^2}{2})_x\ e^{-st}]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\cdot d[(\frac{u^2}{2})_x]\ dx\\ &= \frac{1}{s}\ (\frac{u^2}{2})_x\ (0,t) + \frac{1}{s}\ \pounds[d(\frac{u^2}{2})_x],\end{aligned}$$

where

$$\pounds(u\cdot u_x) = \int_0^\infty e^{-sx}\ (\frac{u^2}{2})_x\ dx.$$

(D) If we take the form of a derivative as $(u^2/2)_x$ in the process of integration by parts, we have

$$\begin{aligned}\pounds(u\cdot u_x) &= [e^{-st}\int(\frac{u^2}{2})_x\ dt]_0^\infty + s\int_0^\infty e^{-st}\ (\int_0^\infty(\frac{u^2}{2})_x\ dt)\ dt\\ &= -[\int(\frac{u^2}{2})_x\ dt]_{t=0} + s\pounds[\int_0^\infty(\frac{u(x,t)^2}{2})_x\ dt]\\ &= -[\int(\frac{u^2}{2})_x\ dt]_{t=0} + \lim s\pounds[\int_0^t(\frac{u(x,\tau)^2}{2})_x\ d\tau]\end{aligned}$$

as $t\to\infty$. Thus,

$$\begin{aligned}\pounds(u\cdot u_x) &= -[\int(\frac{u^2}{2})_x\ dt]_{t=0} + \lim\pounds[(\frac{u(x,t)^2}{2})_x]\\ &= -[\int(\frac{u^2}{2})_x\ dt]_{t=0} + \pounds[\{u(x,\infty)\}^2/2]_x.\end{aligned}$$

In the proof of (D), we would like to check on two matters. One is the commutativeness of lim and $\int$, the other is

$$\pounds[\int_0^t(\frac{u(x,\tau)^2}{2})_x\ d\tau] = \frac{1}{s}\pounds[(\frac{u(x,t)^2}{2})_x].$$

First, $(u(x,t)^2\ /2)$ is non-negative measurable, and so, by Beppo Levi's theorem(lemma 2.3), the commutativeness of lim and $\int$ is valid[4, 10]. Note that Beppo Levi's theorem can be restated as the following;

$$\int \lim_{n\to\infty}\sum_{k=1}^{n} g_n\ d\mu = \lim_{n\to\infty}\sum_{k=1}^{n}\int g_n\ d\mu = \lim_{n\to\infty}\int\sum_{k=1}^{n} g_n\ d\mu.$$

Second, let us put

$$g(t) = \int_0^t (\frac{u(x,\tau)^2}{2})_x\ d\tau.$$

Then $g'(t) = (u(x,t)^2\ /2)_x$, and so

$$£[(\frac{u(x,t)^2}{2})_x] = £(g'(t)) = s£(g(t)) - g(0) = s£(g(t))$$

$$= s£[\int_0^t (\frac{u(x,\tau)^2}{2})_x\ d\tau].$$

(E) By the simple calculation, we have

$$£(fg) = \int_0^\infty e^{-st}\ fg\ dt$$

$$= -\frac{1}{s}\ [e^{-st}\ fg]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}(f'g + fg')\ dt$$

$$= \frac{1}{s}\ f(x,0)g(x,0) + \frac{1}{s}£[d\ (fg)].$$

By changing f to u and g to u_x we get the result

$$\frac{1}{s}\ u(x,0)u_x(x,0) + \frac{1}{s}£[d\ (u\cdot u_x)].$$

□

In the above theorem, (C) shows that $(x,0)$ can be changed to $(0,t)$ in the first term of the right side of (A).

Theorem 2.8. *(Laplace transform of $u\cdot u_x$ by means of mean value theorem for integrals) There exists a point $c\in(0,\infty)$ such that*

$$£(u\cdot u_x) = \frac{1}{s}\ u(x,c)u_x(x,c) = \frac{1}{s}\ [(\frac{u^2}{2})_x]_{t=c}.$$

Moreover, if $u\geq 0$, then there exists a point c such that

$$£(u\cdot u_x) = u_x(x,c)£(u).$$

Proof. For mean value theorem, the continuity of f is not important, but the integrability of f. Since $uu_x = (u^2/2)_x$ is integrable, and e^{-st} is integrable and non-negative, the following equality holds according to lemma 4. Hence, there exists a point $c \in (0, \infty)$ such that

$$\pounds(u \cdot u_x) = \int_0^{\infty} e^{-st}\ (u \cdot u_x)\ dt$$

$$= u(x, c)\ u_x(x, c) \lim_{h \to \infty} \int_0^{h} e^{-st}\ dt$$

$$= u(x, c)\ u_x(x, c)\pounds(1)$$

$$= \frac{1}{s}\ u(x, c)\ u_x(x, c),$$

equivalently,

$$\pounds(u \cdot u_x) = \frac{1}{s}\ [(\frac{u^2}{2})_x]_{t=c}.$$

In a similar fashion, there exists a point c in the given interval such that

$$\pounds(u \cdot u_x) = \int_0^{\infty} (e^{-st} u_x) u\ dt$$

$$u_x(x, c) \int_0^{\infty} e^{-st}\ u\ dt$$

$$= u_x(x, c)\pounds(u)$$

for $u \geq 0$. □

For example, let us check an example of the latter. The latter can be generalized as $\pounds(fg) = g(c)\pounds(f)$ on the condition that f and g are integrable and f is non-negative. Taking $f(t) = t$ and $g(t) = 1$, we have

$$\pounds(fg) = \pounds(t) = \frac{1}{s^2} = 1 \cdot \frac{1}{s^2} = 1(c)\pounds(f) = g(c)\pounds(f).$$

Similarly, if $f(t) = t$ and $g(t) = t^2$, then

$$\pounds(fg) = \pounds(t^3) = \frac{6}{s^4} = \frac{6}{s^2} \cdot \frac{1}{s^2} = g(c)\pounds(f) = c^2\pounds(f)$$

for $c = \sqrt{6}/s$. Note that the scope of this transform is $[0, \infty)$. These examples show that there is an appropriate value c as a non-negative value.

On one hand, the solution of Burger's equation[7] was obtained using the Laplace transform in [8]. However, since it was done only if $u(x, t)$ be considered as a function of x, we would like to improve the proof and extend it to some more general case.

Theorem 2.9. *(Burger's equation) The solution $u(x,t) = w^{-1}(x,s)$ of the Burger's equation*

$$u_t + u \cdot u_x = \nu \cdot u_{xx} \tag{1}$$

can be obtained by

$$w(x,s) = A(s)e^{\sqrt{s/v}\ x} + B(s)e^{-\sqrt{s/v}\ x} + \frac{1}{2}\sqrt{v/s}\ e^{\sqrt{s/v}\ x}\int e^{-\sqrt{s/v}\ x}\ r\ dx$$
$$-\frac{1}{2}\sqrt{v/s}\ e^{-\sqrt{s/v}\ x}\int e^{\sqrt{s/v}\ x}\ r\ dx, \tag{2}$$

where u_0 is a constant, u is a given velocity, ν is viscosity coefficient, and

$$r = s\pounds(\frac{u^2}{2}) - (\frac{u^2}{2})(0,t) - u(x,0).$$

Proof. Taking the Laplace transform with respect to t on both sides and writing $w(x,s) = \pounds[u(x,t)]$, we have

$$sw - u(x,0) - (\frac{u^2}{2})(0,t) + s\pounds(\frac{u^2}{2}) = \nu\frac{\partial^2 w}{\partial x^2}$$

by (B) of theorem 7. Organizing this equality, we have

$$\nu\frac{\partial^2 w}{\partial x^2} - sw = s\pounds(\frac{u^2}{2}) - (\frac{u^2}{2})(0,t) - u(x,0). \tag{3}$$

Let us put

$$r = s\pounds(\frac{u^2}{2}) - (\frac{u^2}{2})(0,t) - u(x,0).$$

Then by Lagrange's method(lemma 1), a solution of equation of (3) is

$$w(x,s) = A(s)e^{\sqrt{s/v}\ x} + B(s)e^{-\sqrt{s/v}\ x} + w_p$$

for w_p is a particular solution of the equation (3). Calculating the Wronskian W of $e^{\sqrt{s/v}\ x}$ and $e^{-\sqrt{s/v}\ x}$, we have

$$W = -2\sqrt{s/v}.$$

By Lagrange's method, we have

$$w_p = -e^{\sqrt{s/v}\ x}\int \frac{e^{-\sqrt{s/v}\ x}\ r}{-2\sqrt{s/v}}\ dx + e^{-\sqrt{s/v}\ x}\int \frac{e^{\sqrt{s/v}\ x}\ r}{-2\sqrt{s/v}}\ dx$$
$$= \frac{1}{2}\sqrt{v/s}\ e^{\sqrt{s/v}\ x}\int e^{-\sqrt{s/v}\ x}\ r\ dx - \frac{1}{2}\sqrt{v/s}\ e^{-\sqrt{s/v}\ x}\int e^{\sqrt{s/v}\ x}\ r\ dx.$$

Thus a general solution of (3) is

$$w(x,s) = A(s)e^{\sqrt{s/v}\ x} + B(s)e^{-\sqrt{s/v}\ x} + \frac{1}{2}\sqrt{v/s}\ e^{\sqrt{s/v}\ x}\int e^{-\sqrt{s/v}\ x}\ r\ dx$$
$$-\frac{1}{2}\sqrt{v/s}\ e^{-\sqrt{s/v}\ x}\int e^{\sqrt{s/v}\ x}\ r\ dx,$$

where

$$r = s\pounds(\frac{u^2}{2}) - (\frac{u^2}{2})(0,t) - u(x,0).$$

It is clear that the Laplace transform of the inviscid Burger's equation($\nu = 0$) has a form of

$$w(x,s) = -\pounds(\frac{u^2}{2}) + \frac{1}{s}(\frac{u^2}{2})(0,t) + \frac{1}{s}u(x,0)$$

from (3). On the other hand, from (1), this equation can be represented by

$$\frac{du}{dt} + uu_x = 0.$$

Organizing this equation, we have

$$\frac{du}{u} = -u_x dt.$$

Thus

$$\ln|u| = -\int u_x \; dt + c*,$$

and so

$$u = ce^{-\int u_x \; dt} \; (c = \pm e^{c*}).$$

Of course, if $u > 0$, then $c = e^{c*}$, and if $u < 0$, then $c = -e^{c*}$. □

Corollary 2.10. *(Burger's equation) From the perspective of the mean value theorem for integrals, Laplace transform of the Burger's equation*

$$u_t + u \cdot u_x = \nu \cdot u_{xx} \tag{2}$$

can be represented by

$$w(x,s) = A(s)e^{\sqrt{s/v}\; x} + B(s)e^{-\sqrt{s/v}\; x} + \frac{1}{2}\sqrt{v/s}\; e^{\sqrt{s/v}\; x}\int e^{-\sqrt{s/v}\; x}\; r\; dx$$

$$-\frac{1}{2}\sqrt{v/s}\; e^{-\sqrt{s/v}\; x}\int e^{\sqrt{s/v}\; x}\; r\; dx,$$

where

$$r = \frac{1}{s}u(x,c)u_x(x,c) - u(x,0)$$

for some point c. In case of $u > 0$, the above r can be changed to

$$u_x(x,c)\pounds(u) - u(x,0).$$

Proof. Taking Laplace transform with respect to t on both sides, writing $w(x,s) = \pounds[u(x,t)]$, and applying the mean value theorem for integrals to the equation, we obtain the result in a manner similar to theorem 2.9. □

Of course, the obtained research results can be naturally extended to the generalized Laplace-type equation.

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