

DIFFUSIVE AND STOCHASTIC ANALYSIS OF LOKTA-VOLTERRA MODEL WITH BIFURCATION[†]

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ABSTRACT. The paper presents a critical analysis of selected topics related to the modeling of interacting species in which prey has nonlinear reproduction, which is in competition with predator. The mathematical model's stochastic stability is investigated. The method of designing appropriate Lyapunov functions is used to identify permanence conditions among the parameters of the model and conditions for the structure to no longer be extinct. The system's two-dimensional diffusive stability is regarded and studied. The system experiences the process of saddle-node bifurcation by varying the death rate of predator parameter. Further effects of parameters that undergo inherent oscillations are numerically investigated, revealing that as the intensity of predation parameter b is increased, the device encounters non-periodic and damped oscillations.

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1. Introduction

The most fundamental challenge of interacting population evolution is determining the conditions for long-term species coexistence. The predator-prey relationship is a well-known basic organized phenomenon in population dynamics, and numerous predator-prey models have been researched for their importance in the theory and applications of dynamical structures in population biology and ecology. Furthermore, the predator-prey paradigm is a crucial component of the reaction-diffusion mechanism. Turing (1952) was one of the first scientists to

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recognize the significance of biomorphogenesis trends in non-equilibrium diffusion reactions [1, 2]. In this regard, since the beginning of ecological studies for ecosystem balance, stability and volatility of predator-prey models with different growth rates or different functional responses [3] have been the focus of attention. On a mathematical level, the complex interaction between predators and their prey is one of the most prominent topics in mathematical ecology. Investigations into predator-prey models have been established during the last thirty years, and more practical models have been derived to explain chaotic coexistence and bifurcation study of predator-prey environments. In prey-predator simulations, turbulent oscillations behind diffusive fronts were also examined [4, 5, 6]. For prey-predator models with intra-specific rivalry, a requirement for permanent coexistence was created. For more information on some latest research on complex predator-prey models and their interacting activities in environments, we refer to readers ([7, 8, 9, 10, 11, 12]). The key goal of this paper is to look at the issue of coexistence between two interacting organisms dispersed through spatially heterogeneous regions to see how spatial dispersal and environmental heterogeneity influence coexistence.

1.1. Fundamentals. Predation is an omnipresent population phenomenon that has arisen in the metazoan several times. The process of plundering will influence the ecosystem's distribution, prosperity and dynamics. In recent decades a considerable number of researchers have been working to understand the emergence of spatiotemporal patterns and their potential mechanisms for developing a pattern or structure due to the unstable homogeneous equilibrium caused by the random disturbances caused by Turin's seminal work.

The distribution of the west tussock, believed to be infected by a parasite wiping, is a clear illustration of predation [13]. The relationships between predators have an integrated propensity to change and to show oscillating behaviour. If the number of predators is initially small or very small it would certainly raise the size of the prey community in the actual environment. With the population growth of prey, the food supplies are abundant for predators as the populations of predators also start growing, which in turn has an adverse impact on the population of prey which leads to a reduction of the population of prey. With the population of prey being scarce, the population of predator's decreases and then the cycle begins to repeat. One of the simplest mathematical models that Lotka-Volterra has proposed [14] will clarify the intuitive dynamics of prey-predator interactions. In the area of mathematical ecology, this classic two-style predator-prone model proposed by Volterra and Lotka is a landmark. With its variations, the underlying model captured and analysed the non-linear relationships of predatory and proxy species and their densities. An interaction between a predator (N_2) and its prey (N_1) is a continuous time model which

can be represented in the following as

$$\begin{aligned}\frac{dN_1}{dt} &= rN_1 - \alpha_1 N_1 N_2 \\ \frac{dN_2}{dt} &= \alpha_1 \alpha_2 N_1 N_2 - dN_2\end{aligned}\tag{1}$$

where r is the natural growth rate of prey in the absence of predators, α_1 is the predator attack rate to get the prey, α_2 is the impact of prey on predators and d is the death rate of predators in the absence of prey resource. Bailey and Nicholson's basic mathematical model in 1935[15] is a groundwork in which predator interactions can be analyzed [16]. These interactions are now formulated to investigate insect parasitoid and its hosts' dynamics as given by

$$\begin{aligned}\frac{dN_1}{dt} &= \alpha_1 N_1 f(N_1, N_2) \\ \frac{dN_2}{dt} &= \alpha_2 N_1 (1 - f(N_1, N_2))\end{aligned}\tag{2}$$

where α_1 is the per capita rate for increase of hosts in the absence of parasitoids, α_2 is the conversion efficiency of hosts for new parasitoids and $f(N_1, N_2)$ is the functional response acceptable and true in the literature of mathematical ecology.

If the nullclines of the prey and predator are linear, the relations between the prey and predator are very susceptible to the inclines of the respective balance lines. But the assumption that certain paths are still constant is unreasonable. Some predators are very effective in searching for and capturing their bearings but are prevented by their density when their species compete with each other's hunting activities. The balance of such predators may initially be very steep, but their slope may eventually decline as they are directly related to their density. While these predators are highly effective, their contact with the prey can be very stable if their density of saturation does not exceed the extinction point of the prey. In actual fact, we see a significant significance in the biologic regulation of the pests as the effective predators with self-inhibiting interactions may manage their prey at very low densities.

Any predators have extremely powerful search mechanisms [17, 18, 19, 20, 21]. This applies especially to pathogenic microorganisms which enter their hosts through passive atmospheric transmission or direct contact between infected and uninfected persons. While these depredators can live at low densities by entering a sleeping state, i.e. their bodily processes are natural and have been interrupted or slowed down for a certain time. Their population won't grow until their density is very large on a given host. If they achieve this crucial mass, they will replicate large amounts of descendants very rapidly, such that the pitch of their balance is very steep.

1.2. Mathematical Background. The environmental parameters are well established positive constants for population models in deterministic modelling

settings. Both species organisms have time-independent values in the group balance, i.e. all the net rate of development is negative. Such a balance may be stable if ecosystems are disturbed and those affected populations are returned in due course to their equilibrium. Either by damp oscillation or monotonously, the returning phase may be accomplished. Conversely, the device can be referred to as chaotic if such disruption appears to intensify itself. Again this instability could occur in disturbance as oscillating or as a monotonous rise.

Suppose the multispecies population dynamics are given by a set of m equations as

$$\frac{dN_i(t)}{dt} = F_i(N_1^*, N_2^*, N_3^* \dots N_m^*) \quad (3)$$

The growth rate of the i^{th} species at time t is given by some nonlinear function F_i of interacting species. The equilibrium populations N_i^* follow from m algebraic equations, obtained by making all the growth rates equal to zero such as

$$F_i(N_1^*, N_2^*, N_3^* \dots N_m^*) = 0$$

Expanding about this equilibrium for each species population we are substituting $N_i(t) = N_i^* + x_i(t)$, where $x_i(t)$ is small perturbation to the i^{th} population. Writing the Taylors series expansion to the respective equations around this equilibrium and neglecting all terms which are of second or higher order in the population perturbations $x_i(t)$, a linearized approximation is given by the formula

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^m a_{ij} x_j(t) \quad (4)$$

The set of m equations describes the population dynamics in the neighborhood of equilibrium point. Expressing these m equations in matrix notation, $\frac{dX(t)}{dt} = AX(t)$, where $X(t)$ is the $m \times 1$ matrix of x_i and A is the $m \times m$, community matrix [22]. The elements of community matrix a_{ij} explain the effect of species j , upon species i near equilibrium, where $a_{ij} = \left(\frac{\partial F_i}{\partial N_j}\right)^*$ are the partial derivatives of F_i keeping all populations except N_j constant. These a_{ij} are to be evaluated with all populations having their equilibrium values. For the set of linear equations in (4), the solutions are given by

$$x_i(t) = \sum_{j=1}^m c_{ij} e^{\lambda_j(t)} \quad (5)$$

where C_{ij} are constants which depends on initial values of perturbations and time dependence is completely contained in the m exponentials (λ_j) factors ($\lambda_1, \lambda_2, \lambda_3 \dots \lambda_m$), these factors characterize the temporal behavior of the system. $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_m$ which are the so called eigen values of the matrix A . Then they are found by $(A - \lambda I)X(t) = 0$. This set of equations has a nontrivial

solution if and only if

$$|A - \lambda I| = 0 \quad (6)$$

The equation given in (6) is an m^{th} order polynomial equation in λ and it determines the eigen values λ . The general form of λ is in the form of complex numbers, so the real part h produces exponent growth or decay and the imaginary part k produces sinusoidal oscillation. It is clear that the perturbations to the equilibrium populations will die away in time if and only if all eigen values λ have negative real parts. If anyone eigen value has positive real part the exponential factor will grow ever larger as time goes on and consequently the equilibrium is unstable. The special case of neutral stability is attained if one or more eigen values are purely imaginary, numbers and the rest have negative real parts. We observe that an equilibrium configuration in the multispecies system will have neighborhood stability if and only if all eigen values of the community matrix lie in the left hand half of the plane of complex numbers. It is convenient to define Λ as minus the largest real part of all the eigen values of the community matrix $-\Lambda = [real(\lambda)]_{max}$, when the stability criterion then becomes $\Lambda > 0$. If one or more eigen values have positive real parts ($\Lambda < 0$), all we can say with certainty is that there is not a stable equilibrium point. Perturbations will initially grow, but the neighborhood analysis leaves their ultimate chance uncertain. Terms of order x^2 and higher become important and nonlinearities decide whether the perturbations will grow until extinctions are produced, or otherwise the system may settle into some limit cycle [23].

1.3. Effect of Diffusion. The basic objective of ecology is to consider the interaction between human species and the climate. Empirical evidence indicates that the spatial seal and environmental structure will affect population and community relations [24, 25]. In the past few decades, considerable attention has been paid to the importance of spatial effects in preserving biodiversity [26, 27]. The models of Reaction Diffusion[28, 29, 30, 31, 32, 33] provide an approach for the conversion into global conclusions of organisms, survival or disappearance and coexistences in the climate, local assumptions concerning the migration, death and reproduction of individuals.

1.4. Diffusion Process. The specifics of the dispersal process remain ignored by most population models. One basic approximation leading to curious findings is that people spread from a source group. This may explain the growth of invading animals into appropriate habitats or the migration of people through an uncolonizable habitat amongst the local community. Diffusion is a spontaneous and ongoing phenomenon in an atmosphere that takes each particle or person from its source location on a random footing. Diffusion models are dynamic since all spontaneous gestures need a mechanism to synchronise each time. In 1995 Morris et al [34] dissertation on the dispersal of pollens and the movement of insects and marine ecologists, often researching algae transport in water bodies,

involved applications of spatial diffusion ecology models. In 1973, Segel and Jackson [35] introduced bacterial action diffusion models. Partial differential equations may also explain diffusion. These are necessary since the distribution or excess of people depends on two factors in space location and time. These are often necessary. Smith identified the spread of an ingredient down a tube in 1968 [36]. The change in concentration with a time t is related to the change in concentration with distance S following the partial differential equation

$$\frac{\partial x}{\partial t} = \mu \frac{\partial^2 x}{\partial s^2}$$

where μ is a constant. This is well known in the mathematical literature as the one dimensional heat equation. An ecological use of this equation is the dispersal of individuals along a linear route such as plants dispersing along a road side. The general mathematical system which describes the interaction between two interacting species is given by

$$\begin{aligned} \frac{dx}{dt} &= P(x) - Q(x, y) \\ \frac{dN_2}{dt} &= -S(y) + D(x, y) \end{aligned}$$

In which the functions of the associated population densities are substituted by both names. We may make several changes to the general setup by means of this overall design of the mathematical model for the dynamics of two competing populations. Changes are possible for the community of prey, prey mortality, predator saturation, predator rivalry for prey, etc. The purpose of this current topic is to include nonlinear prey reproduction and prey mortality and the use of a negative exponential is commonly allowed to predator rivalry. Second degree term into the equation for the rate of change density of predator, where the functions $S(y)$ takes the form as $-cy - ey^2$ and $P(x)$ takes the form as $\frac{ax^2}{\alpha+x}$ and with this modifications the above model can take the following form as given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{ax^2}{\alpha+x} - bxy - hx \\ \frac{dy}{dt} &= dxy - cy - ey^2 \end{aligned} \tag{7}$$

It is worth mentioning that all the parameters in this model (7) are positive constants. Moreover, the model (7) is also subject to the initial non-negative conditions and under the above consideration; the model (7) has the following domain as

$$D = [(x, y) \in R_+^2 : x \geq 0, y \geq 0] \tag{8}$$

Now the solution of the system (7) can be proved to be uniformly bounded in vein of the following theorem.

2. Boundedness of the Solution

It is important to ensure that the model (7) is biologically well behaved and well posed one, so we need to demonstrate the positivity and boundedness of solutions for the model (7). In this regard, we need to prove the following theorem:

Theorem 2.1 ((Bendixson's - Dulac's negative criterion)). *Let, $H(x, y)$ be a smooth function on $D \subset R^2$. If $\nabla \cdot (Hf_1, Hf_2)$ is of one sign in D , then no closed orbit is contained within D under the condition $e > \min\{1, \frac{h\alpha}{xy}\}$*

Proof. Let us consider the smooth function,

$$H(x, y) = \frac{\alpha + x}{x^2 y} \quad (9)$$

From the system (7), we have,

$$\begin{aligned} f_1(x, y) &= \frac{ax^2}{\alpha + x} - bxy - hx \\ f_2(x, y) &= dxy - cy - ey^2 \\ \frac{\partial H f_1}{\partial x} &= \frac{\alpha}{x^2} + \frac{h\alpha}{x^2 y} \text{ and } \frac{\partial H f_2}{\partial x} = \frac{-e\alpha}{x^2} + \frac{-e}{x} \\ \frac{\partial H f_1}{\partial x} + \frac{\partial H f_2}{\partial x} &= \frac{\alpha}{x^2}(1 - e) + \frac{1}{x^2 y}(h\alpha - e\alpha x) < 0 \end{aligned}$$

The last expression is strictly negative if $e > \min\{1, \frac{h\alpha}{xy}\}$ in the interior of the first quadrant. So there cannot be a closed orbit which satisfies (7) and it entirely lies within the interior of the first quadrant. \square

3. Equilibrium Point and its Stability Analysis

The positive equilibrium point of (7) is given by $\frac{dx}{dt} = \frac{dy}{dt} = 0$ and it is denoted by $E^*(x^*, y^*)$ where $x^* = \frac{c+ey^*}{d}$ and y^* is the positive root of the quadratic equation $A_1 y^2 + A_2 y + A_3 = 0$ i.e, $y^* = \frac{-A_2 + \sqrt{-A_2^2 - 4A_1 A_3}}{2A_1}$.

3.1. Stability at Positive Equilibrium point E^* . The Jacobian matrix of the system (7) is given by

$$\mathbf{J} = \begin{bmatrix} \frac{ax^*}{\alpha+x^*} - \frac{ax^{*2}}{(\alpha+x^*)^2} & -bx^* \\ dy^* & -ey^* \end{bmatrix}$$

Also

$$\begin{aligned} tr(J) &= \frac{ax^*}{\alpha+x^*} - \frac{ax^{*2}}{(\alpha+x^*)^2} - ey^* = \frac{ax^*}{\alpha+x^*} - \frac{(\alpha+x^*)[(a-ey^*)x^* - e\alpha y^*]}{(\alpha+x^*)^2}, tr(J) < 0 \\ \text{if, } y^* &> \frac{a}{e} \end{aligned}$$

$$\det(J) = \frac{-eax^*y^*}{\alpha+x^*} + \frac{eay^*x^{*2}}{(\alpha+x^*)^2} + bdx^*y^* = \frac{x^*y^*}{\alpha+x^*} [bd\alpha + bc + bey^* - ea] + \frac{eay^*x^{*2}}{(\alpha+x^*)^2} \text{ and}$$

$$\det(J) > 0 \text{ if } (\alpha + x^*)(bd\alpha + bc + bey^* - ea) + eax^* > 0$$

Hence the system (7) is asymptotically stable
if $y^* > \frac{a}{e}$ and $(\alpha + x^*)(bd\alpha + bc + bey^* - ea) + eax^* > 0$

4. Diffusion Instability of the System

In the study of ecology, diffusion means the movement of species in any direction. Here, we go with the assumption that if the movement of species is only in the vertical direction, then the population density variables are $x = x(s, t)$ and $y = y(s, t)$, where s is the space variable and t is the time variable. To analyze the role of diffusion, consider the following diffusive model,

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{ax^2}{\alpha + x} - bxy - hx + D_1 \frac{\partial^2 x}{\partial s^2} \\ \frac{\partial y}{\partial t} &= dxy - cy - ey^2 + D_2 \frac{\partial^2 y}{\partial s^2} \end{aligned} \quad (10)$$

The non-zero initial conditions are given by

$$x(s, 0) > 0, \text{ and, } y(s, 0) > 0 \text{ for } s \in [0, R] \quad (11)$$

The zero-flux boundary conditions are given by

$$[(x)_s]_{s=0, R} = [(y)_s]_{s=0, R} = 0 \quad (12)$$

To linearize the model (10) with the following substitutions and with respect to the conditions (11) and (12), we obtain the following equations,

$$x(s, t) = x^* + \eta_1(s, t), y(s, t) = y^* + \eta_2(s, t)$$

$$\frac{\partial \eta_1}{\partial t} = b_{11}\eta_1 + b_{12}\eta_2 + D_1 \frac{\partial^2 \eta_1}{\partial s^2} \quad (13)$$

$$\frac{\partial \eta_2}{\partial t} = b_{21}\eta_1 + b_{22}\eta_2 + D_2 \frac{\partial^2 \eta_2}{\partial s^2} \quad (14)$$

where, $b_{11} = \frac{ax^*}{\alpha+x^*} - \frac{ax^{*2}}{(\alpha+x^*)^2}$; $b_{12} = -bx^*$; $b_{21} = dy^*$; $b_{22} = -ey^*$

Assume the solutions of the equations (13)-(14) are in the form of $\eta_1 = ae^{\lambda t} \cos ks$, $\eta_2 = be^{\lambda t} \cos ks$ where λ and k are frequency and wave numbers respectively. In this case, the characteristic equation of the model (13)-(14) is given by

$$\mu^2 + \rho_1\mu + \rho_2 = 0 \quad (15)$$

where, $\rho_1 = (D_1 + D_2)k^2 - b_{11} - b_{22}$, $\rho_2 = -(b_{11}D_1 + b_{22}D_2)k^2 + b_{11}b_{22} - b_{12}b_{21} + D_1D_2k^4 = F(k^2)$

Here the idea is to get the condition of instability due to diffusion. If one of the roots of the characteristic equation (15) is positive, then the system will be unstable. If both the roots of the quadratic equation (15) are negative, then the system can be represented by the model

$$k^2 > \frac{b_{11} + b_{22}}{D_1 + D_2} \quad (16)$$

The sufficient condition for a root of the quadratic equation (15) to be positive is

$$F(k^2) = -(b_{11}D_1 + b_{22}D_2)k^2 + b_{11}b_{22} - b_{12}b_{21} + D_1D_2k^4 < 0$$

Since $F(k^2)$ is a quadratic expression in k^2 and $k \geq 0$, the minimum value of $F(k^2)$ can be obtained for the value of k^2 given by

$$[k^2]_{min} = -\frac{(D_2b_{11} + D_1b_{22})^2}{2D_1D_2}$$

Hence the respective minimum value of $F(k^2)$ is given by

$$F(k_{min}^2) = -\frac{D_2b_{11} + D_1b_{22}}{4D_1D_2} + b_{11}b_{22} - b_{12}b_{21}$$

Thus the condition for the existence of a positive root of (7) $F(k_{min}^2) < 0$ i.e.,

$$4D_1D_2(b_{11}b_{22} - b_{12}b_{21}) - (D_2b_{11} + D_1b_{22})^2 < 0 \quad (17)$$

Biologically, this can be explained as follows: the species x is an activator because it stimulates population expansion, whereas the species y is an inhibitor because it reduces its own rate of growth.

5. Two Dimensional Stability

In two dimensional spaces the model system (15) reduces to

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{ax^2}{\alpha + x} - bxy - hx + \nabla^2 x \\ \frac{\partial y}{\partial t} &= dxy - cy - ey^2 + D\nabla^2 y \end{aligned} \quad (18)$$

Here, $\nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ represents the Laplacian operator.

We can investigate the above model with the following initial and boundary conditions:

$$\begin{aligned} x(u, v, 0) &> 0, y(u, v, 0) > 0, (x, y) \in \Psi \\ \frac{\partial x}{\partial n} &= \frac{\partial y}{\partial n} = 0, (x, y) \in \partial\Psi, t > 0 \end{aligned} \quad (19)$$

On these conditions, n is the outward normal to $\partial\Psi$. Next, we show that in presence of diffusion the global stability behavior of the positive equilibrium.

For this, we will consider

$$w_1(t) = \iint_{\Psi} w(x, y) dA \quad (20)$$

where $w = [x - x^* - x^* \ln(\frac{x}{x^*})] + [y - y^* - y^* \ln(\frac{y}{y^*})]$ Differentiating w_1 with respect to time t , we have

$$\frac{dw_1}{dt} = \iint_{\Psi} \frac{dw}{dt} dA + \iint_{\Psi} (\frac{\partial w}{\partial x} \nabla^2 x + D \frac{\partial w}{\partial y} \nabla^2 y) dA = P_1 + P_2$$

By using Green's first identity in the plane we have,

$$\iint_{\Psi} F \nabla^2 G dA = \int_{\partial \Psi} F \frac{\partial G}{\partial n} ds - \iint_{\Psi} F \frac{\partial G}{\partial n} ds - \iint_{\Psi} (\nabla F \cdot \nabla G) dA$$

And under the analysis similar to Dubey and Hussain [40], one can show that

$$\begin{aligned} \iint_{\Psi} (\frac{\partial w}{\partial x} \nabla^2 x) dA &= - \iint_{\Psi} \frac{\partial^2 w}{\partial x^2} [(\frac{\partial x}{\partial u})^2 + (\frac{\partial x}{\partial v})^2] dA \leq 0 \\ \iint_{\Psi} D (\frac{\partial w}{\partial y} \nabla^2 y) dA &= - \iint_{\Psi} \frac{\partial^2 w}{\partial y^2} [(\frac{\partial y}{\partial u})^2 + (\frac{\partial y}{\partial v})^2] dA \leq 0 \end{aligned} \quad (21)$$

This shows that $P_2 \leq 0$ and also we observe that $P_1 \leq 0$, so then $\frac{dw_1}{dt} \leq 0$

This implies that if in the absence of diffusion E^* globally asymptotically stable, then in the presence of diffusion it will remain globally asymptotically stable.

6. Stochastic Analysis

A significant aspect of an ecosystem is the environmental fluctuation. The majority of natural processes are not purely deterministic laws; instead, certain average meaning oscillates spontaneously, such that the deterministic equilibrium is not a fully stable condition [39, 42]. May [43] pointed out that the environmental changes have generated greater or lesser spontaneous fluctuations in birth rates, carrying power, competitive coefficients and other parameters involved in the model structure. As a result, the distribution of the balance population spontaneously fluctuates over certain mean values. In deterministic environments, we search for the community of constant equilibrium and then examine their stability as a result of interaction dynamics within and within organisms. For the structures motivated by environmental stochasticity, the solution of the controlling stochastic differing equations cannot be found by a time-independent equilibrium stage.

By allowing stochastic fluctuations to the variables and around their values at the positive equilibrium E^* , as an outcome the obligatory mutualism system becomes a stochastic differential equation (SDE) as given by

$$\begin{aligned} dx &= \left(\frac{ax^2}{a+x} - bxy - hx \right) dt + \sigma_1 (x - x^*) d\xi_t^1 \\ dy &= (dxy - cy - ey^2) dt + \sigma_2 (y - y^*) d\xi_t^2 \end{aligned} \quad (22)$$

where $\sigma_i, i = 1, 2$ are real constants, $\xi_t^i, i = 1, 2$ are independent standard Wiener processes. To analyze the stochastic stability of E^* , we consider the linear system of (22) around E^* as follows [41, 37, 38].

where

$$\begin{aligned} u(t) &= (u_1(t), u_2(t))^T \text{ and } f(u(t)) = Ju(t)g(u) \begin{bmatrix} \sigma_1 p_1 & 0 \\ 0 & \sigma_2 p_2 \end{bmatrix} \\ d\xi(t) &= \text{col}(\xi_1(t), \xi_2(t)) \quad u_1 = x - x^*; u_2 = y - y^* \end{aligned} \quad (23)$$

Let $U = (t \geq t_0) \times \mathfrak{R}^2, t_0 \in \mathfrak{R}^+$ Hence $V \in C_2^0(U)$ is a continuous function with respect to t and a twice continuously differentiable function with respect to u , so we have.

$$\begin{aligned} LV(t, u) &= \frac{\partial V(t, u)}{\partial t} + f^T(u) \frac{\partial V(t, u)}{\partial u} + \frac{1}{2} \text{Tr} \left[g^T(u) \frac{\partial^2 V(t, u)}{\partial u^2} g(u) \right] \quad (24) \\ \frac{\partial V}{\partial u} &= \text{col} \left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2} \right) \text{ and } \frac{\partial^2 V(t, u)}{\partial u^2} = \left(\frac{\partial^2 V}{\partial u_j \partial u_i} \right) \quad i, j = 1, 2, T \text{ denotes transpose.} \end{aligned}$$

Theorem 6.1. *If there exists a function $V(u, t) \in C_2^0(U)$ satisfying the following inequalities as given by*

$$K_1 |u|^p \leq V(t, u) \leq K_2 |u|^p; LV(t, u) \leq -K_3 |u|^p, K_i > 0, p > 0 \quad (25)$$

Then the trivial solution of (23) is exponentially p -stable for . Note that, if in (25), $p = 2$, then the trivial solution of (23) is also called asymptotically mean square stable and it is globally asymptotically stable in probability.

Theorem 6.2. *Suppose that $\left(\left(\frac{ax^{*2}}{(a+x^*)^2} - \frac{ax^*}{a+x^*} \right) - \frac{1}{2} \sigma_1^2 \right) > 0, (ey^* - \frac{1}{2} \sigma_2^2) > 0$ then the zero solution of (23) are true when $p = 2$ is asymptotically mean square stable.*

Proof: Let us consider the Lyapunov function

$$V(u) = \frac{1}{2} [v_1 u_1^2 + v_2 u_2^2] \quad (26)$$

here v_1, v_2 are nonnegative constants are to be taken as in the given below.

$$\begin{aligned} LV(u) &= v_1 \left(\left(\frac{ax^*}{a+x^*} - \frac{ax^{*2}}{(a+x^*)^2} \right) - bx^* u_2 \right) u_1 \\ &+ v_2 (dy^* u_1 - ey^* u_2) u_2 + \frac{1}{2} \text{Tr} \left[g^T(u) \frac{\partial^2 V(t, u)}{\partial u^2} g(u) \right] \end{aligned} \quad (27)$$

we can easily observe that

$$\frac{\partial^2 V(t, u)}{\partial u^2} = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$$

and hence

$$g^T(u) \frac{\partial^2 V(t, u)}{\partial u^2} = \begin{bmatrix} v_1 \sigma_1^2 u_1^2 & 0 \\ 0 & v_2 \sigma_2^2 u_2^2 \end{bmatrix}$$

with

$$\frac{1}{2} Tr \left[g^T(u) \frac{\partial^2 V(t, u)}{\partial u^2} g(u) \right] = \frac{1}{2} [v_1 \sigma_1^2 u_1^2 + v_2 \sigma_2^2 u_2^2] \quad (28)$$

If in (5.5) we choose $bx^*v_1 = dy^*v_2$, then from (28), we have

$LV(u) = - \left(\left(\frac{ax^{*2}}{(a+x^*)^2} - \frac{ax^*}{a+x^*} \right) - \frac{1}{2} \sigma_1^2 \right) v_1 u_1^2 - \left(ey^* - \frac{1}{2} \sigma_2^2 \right) v_2 u_2^2 < 0$ So according to Theorem(2), the proof is completed. Main results are here

7. Numerical and Computer Simulations

In this section we addressed the numerical results for the stochastic and diffusion of the system (7), which are corresponding to the analytical results. Case 1: Taking the following parameter values as given by $a = 0.941; \alpha = 0.06; b = 0.09; h = 0.02; d = 0.309; c = 0.69; e = 0.015$

Figure 1 is the trajectories and Figure 2 phase portrait of the system (7) for the above parameter values, which gives the result that the system is asymptotically stable at positive equilibrium point.

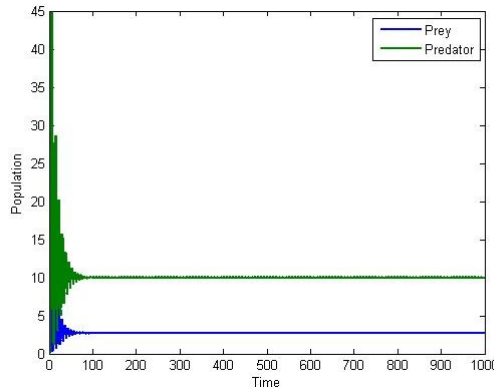


FIGURE 1. Illustrates the stable equilibrium points of the system (7) for $a = 0.941; \alpha = 0.06; b = 0.09; h = 0.02; d = 0.309; c = 0.69; e = 0.015$

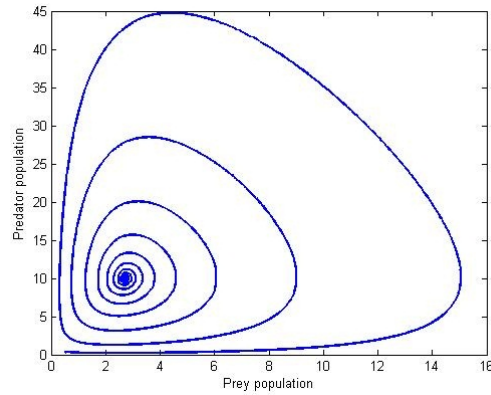


FIGURE 2. Phase portrait of the system (7), which represents stable equilibrium point for $a = 0.941$; $\alpha = 0.06$; $b = 0.09$; $h = 0.02$; $d = 0.309$; $c = 0.69$; $e = 0.015$

Case 2: For the above parameter values along with the $D_1 = 0.05$, $D_2 = 0.03$ and the following graphs are representing steadiness of prey and predator populations with respect to the diffusion in the system

Figure 3 shows the steadiness of prey population and Figure 4 shows steadiness of predator population against time and space

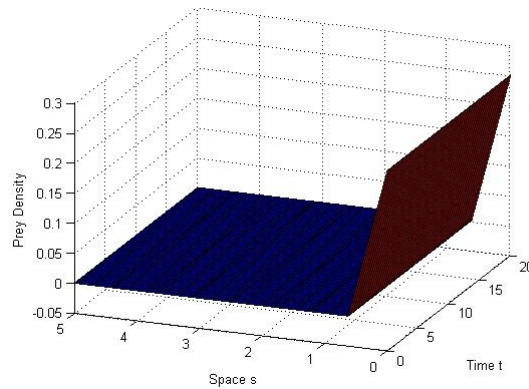


FIGURE 3. Simulated solution of the system which shows the steadiness of prey population for $a = 0.941$; $\alpha = 0.06$; $b = 0.09$; $h = 0.02$; $d = 0.309$; $c = 0.69$; $e = 0.015$

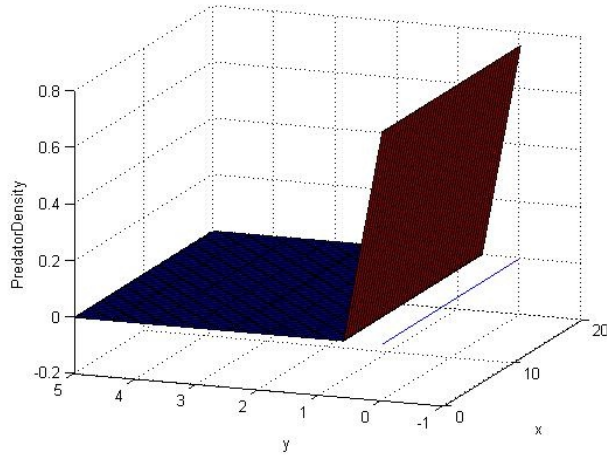


FIGURE 4. Simulated solution of the system which shows the steadiness of predator population for $a = 0.941; \alpha = 0.06; b = 0.09; h = 0.02; d = 0.309; c = 0.69; e = 0.015$ against time and space

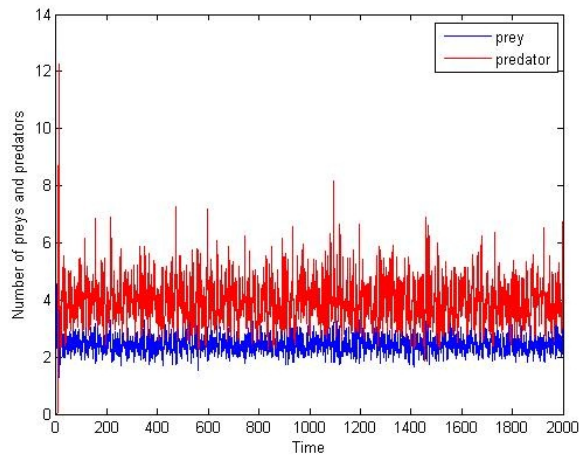


FIGURE 5. The figure represents the trajectories of the model (22) for $a = 0.941; \alpha = 0.06; b = 0.09; h = 0.02; d = 0.309; c = 0.69; e = 0.015$ and $\sigma_1 = 0.08; \sigma_2 = 0.1$

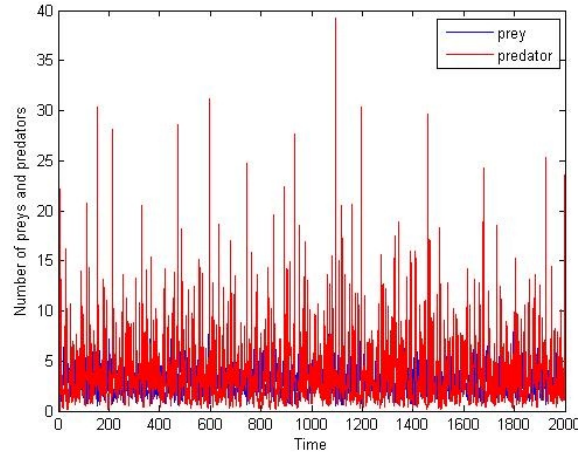


FIGURE 6. The figure represents the trajectories of the model (22) for $a = 0.941$; $\alpha = 0.06$; $b = 0.09$; $h = 0.02$; $d = 0.309$; $c = 0.69$; $e = 0.015$ and $\sigma_1 = 0.08$; $\sigma_2 = 0.1$

7.1. Bifurcation Analysis. In a two dimensional system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ that depends on a parameter λ , suppose that for some value of $\lambda = \lambda_0$ the nullclines intersects as shown in following figures. Each point of intersection corresponds to a fixed point as $\lambda = \lambda_0$ changes the equilibrium points move apart or coincides with each other. At some point $\lambda = \lambda_k$ the two null clines becoming tangent to each other. Then the fixed points coincide with each other when $\lambda = \lambda_k$ after that the null clines pull apart. The equilibrium points disappear. This situation in the dynamical systems observed as saddle node bifurcation. In this present model the null clines of species given by $\dot{x} = 0$ which implies $x = 0$ and $\frac{ax}{a+x} - by - h = 0$, $\dot{y} = 0$, which implies $y = 0$ and $dx - c - ey = 0$. The nullcline $\frac{ax}{a+x} - by - h = 0$ represents a hyperbola for all positive values of a, α, b, h and the nullcline $y = \frac{dx}{e} - \frac{c}{e}$ is straight line with slope $\frac{d}{e}$ and with negative y -intercept $(\frac{-c}{e})$. The y -intercept $\frac{-c}{e}$ becomes closer to zero as e tends to infinity, but biologically which is absurd in reality. So the parameter d chosen as a bifurcation parameter which influences the slope of straight line.

We notice that each intersection corresponds to a fixed point. The fixed points move as d changes from 1.4 to 1.925. The null clines pull away from each other as d varies, becoming tangent at $d = d_c$ and then the fixed points approach each other and collide when $d = d_c$ and after that the null clines pull apart. The saddle node bifurcation observed for this model at $d_c = 1.925$.

TABLE 1. *Nature of bifurcation for different values of bifurcation parameter d for fixing the values of $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$ and $c = 1.3$*

Values of bifurcation parameter (d)	Number of equilibrium points-location -nature	Nature of bifurcation
$d=1.4$	Two(8.64,5.40)- Nodal Sink(2.08,081)- Saddle (Figure 4)	Saddle Node bifurcation
$d=1.6$	Two(6.80,4.79)- Nodal Sink(2.32,1.21)- Saddle (Figure 5)	
$d=1.8$	Two(5.16,3.99)- Nodal Sink(2.72,1.80)- Saddle (Figure 6)	
$d=1.92$	Two(3.71,2.91)- Nodal Sink(3.55,2.75)- Saddle (Figure 7)	
$d=1.925$	Equilibrium points are disappeared(Figure 8)	

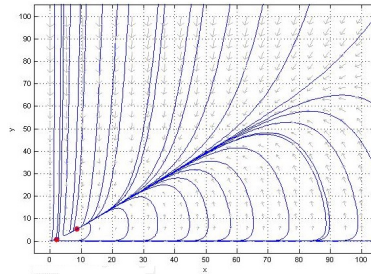


FIGURE 7. The figure represents trajectories of system (7) for $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$, $c = 1.3$ and $d = 1.4$. Two equilibrium points exists which are Nodal Sink (8.64, 5.40) and Saddle (2.08,081).

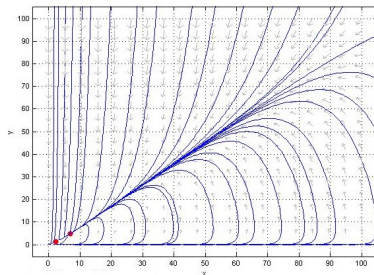


FIGURE 8. The figure represents trajectories of system (7) for $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$, $c = 1.3$ and $d = 1.6$. Two equilibrium points exists which are Nodal Sink (6.80,4.79) and Saddle (2.32,1.21).

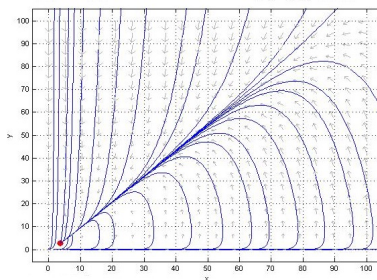


FIGURE 9. The figure represents trajectories of system (7) for $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$, $c = 1.3$ and $d = 1.8$. Two equilibrium points exists which are Nodal Sink (5.16,3.99) and Saddle (2.72,1.80).

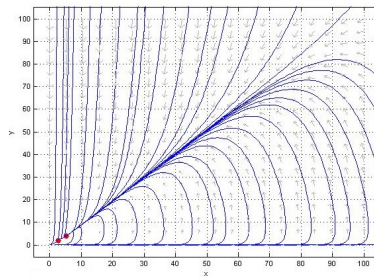


FIGURE 10. The figure represents trajectories of system (7) for $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$, $c = 1.3$ and $d = 1.92$. Two equilibrium points exists which are Nodal Sink (3.71,2.91) and Saddle (3.55,2.75).

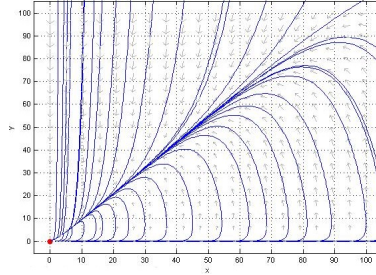


FIGURE 11. The figure represents trajectories of system (7) for $a = 4.5$, $\alpha = 2.1$, $b = 0.3$, $h = 2$, $e = 2$, $c = 1.3$ and $d = 1.925$. Two equilibrium points are disappeared.

8. Conclusions and Ecological Implications

The above analysis leads to the following conclusions. The modification of the Lotka-Volterra system in (1) with nonlinear reproduction of the prey populations leads to the system (7), where it denotes the prey density at which the reproduction rate is half of its possible maximum value. The phase portrait of the system (7). The asymptotic stability of coexisting state exists only if $y^* > \frac{a}{e}$ and $(a + x^*)(bd\alpha + bc + bey^* - ea) + eax^*$. The movement of the prey and predator can be modeled by the presence of spatial diffusion on the system (7). Section 4 examines the system's diffusion instability, which involves prey and predator populations diffusing in a one-dimensional space, possibly with differing diffusivities. The equilibrium point (x^*, y^*) of the system of ordinary differential equations in space, which is the spatially uniform steady state of the system (10). The equilibrium is asymptotically stable for the system of ordinary differential equations (7), but unstable for the system with diffusion (10). Diffusive instability's conditions have been established in (17). Also we have shown the stochastic system (22) is globally asymptotically stable when the intensity of white noise is less than some threshold values stated in Theorem 3. From the analytical and numerical results, it is conclude that the main factor that affects the stability of the stochastic model is the intensity of white noise. Finally we have shown the system has saddle node bifurcation with bifurcation parameter d .

Conflicts of interest : The authors declare that there is no conflict of interests regarding the publication of this article.

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