East Asian Math. J.
Vol. 39 (2023), No. 1, pp. 087–092
http://dx.doi.org/10.7858/eamj.2023.010

EXPONENTIAL BEHAVIOR FOR SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEM WITH EPSILON PERTURBATION

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Abstract. In this paper, we study the nonlinear hyperbolic system with epsilon perturbation

$$s_{tt} - div[c(x)\nabla s] + (\varepsilon - 1)s_t = |s|^\gamma s.$$

Under the conditions of c, γ and other assumptions, we prove the exponential behavior rates of the modified energy perturbation.

1. Introduction

In the present work, we are concerned with the following problem:

$$s_{tt} - div[c(x)\nabla s] + (\varepsilon - 1)s_t = |s|^\gamma s \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1}$$
$$s(x,t) = 0 \qquad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{2}$$
$$s(x,0) = s_0(x), \quad s_t(x,0) = s_1(x) \qquad \text{in} \quad \Omega, \tag{3}$$

where Ω be a bounded open set of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary Γ, γ and other conditions such as c be in next section. In fact, s_0, s_1 are initially given functions and $s(x,t)$ is the transversal displacement of the strip at spatial coordinate x and time t in the real world application. (K is kinetic energy. P is potential energy. δWD_{nc} is the variation of non-conservative work done. δWD_{rb} is the variation of work done at the right boundary. More detailed

Received December 30, 2023; Accepted January 17, 2023.
2010 *Mathematics Subject Classification.* Primary 35L70.
Key words and phrases. exponential behavior, epsilon perturbation, modified energy functional, epsilon perturbation.

(pISSN 1226-6973, eISSN 2287-2833)

information can be found in [1].)

$$K = \frac{1}{2}\int_0^{l(t)} \varpi(x)C(x)[\overline{v}^2 + (\overline{v}s_x + s_t)^2]dx, \tag{4}$$

$$P = \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4} \int_0^1 \left(\frac{\partial s}{\partial x} \right)^2 dx \right) + \frac{1}{2}C(x)\sigma(x,t) \right] \zeta(x,t)dx, \tag{5}$$

$$\delta WD_{nc} = \int_0^{l(t)} \left[\psi(x,t) - \frac{\varpi(x)\overline{v}}{2} s_t(x, t-\tau) \right] \delta s(x,t)dx \\ +\psi_c(t)\delta s(l(t),t), \tag{6}$$

$$\delta WD_{rb} = \varpi(l(t))C(l(t))\overline{v}(\overline{v}s_x(l(t),t) + s_t(l(t),t))\delta s(l(t),t). \tag{7}$$

From (4)-(7), some generalized hyperbolic systems can be derived (See [1]).

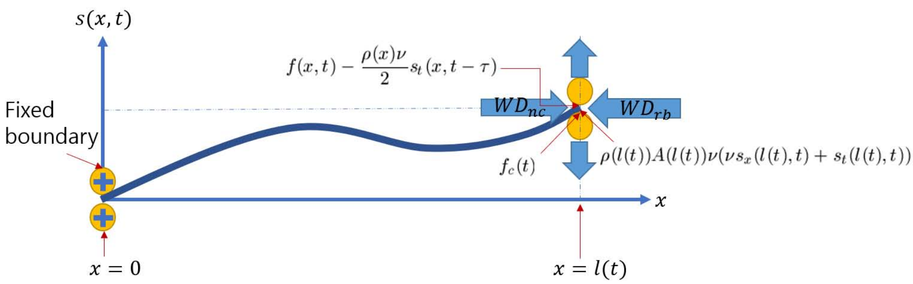

FIGURE 1. the energy system considering WD form on the sense of physical situations.

As opposed to the above FIGURE1, the boundaries in this study are fixed, so the work done can be ignored. Therefore, the roles of the damping term and the forcing term, which affects both position energy and kinetic energy are very important. Unlike previous research results (See [2, 3, 4, 6, 9, 7, 8, 9, 10, 11]), the coefficient of the damping term in the main equation of this study is negative due to the epsilon perturbation ε. In particular, if the signs of the coefficients of the damping term and the forcing term are the same, as in the main system of this paper, it can disrupt energy equilibrium, leading to the blow-up of the solution. Therefore, it is significant to consider systems where the coefficient of the damping term is negative.

In this paper, we will mainly concern on an aspect of decay behavior of the perturbed modified energy of the main system. We get its proof by using the energy perturbation method.

This paper organized as follows. In Section 2, we will present some notations, material needed (assumptions, lemmas and so on) for our work and state a global existence and energy behavior theorem (main result). Section 3 contains the proof of our main result. Finally, in Section 4, we discuss the main conclusion and future research.

2. Preliminaries and main results

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^N$

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}, \|u\|_{L^1(0,+\infty)}, \ \|v\|_{L^\infty(0,+\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively. The assumption for the important variation ε is $0 < \varepsilon << 1$. Finally, the assumption for $c : \Omega \to \mathbb{R}^+$ is a non-negative and uniformly lower bounded (that is $C(x) \geq c_0 \geq 0$) function.

Now we state our result as follows.

Theorem 2.1. *Set either $\gamma > 2$ in case of $N < 2$ or $2 < \gamma \leq \frac{2N-2}{N-2}$ in case of $N \geq 3$ where N in $\Omega \subset \mathbb{R}^N$. Let the above assumptions hold. Then there exists a unique solution u of the problem* (1)-(3) *satisfying*

$$s \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \ s' \in L^\infty(0, T; H_0^1(\Omega)), \ s'' \in L^\infty(0, T; L^2(\Omega)),$$

and

$$s(x,t) \to s_0(x) \ \textit{in} \ H_0^1(\Omega) \cap H^2(\Omega); \qquad s'(x,t) \to s_1(x) \ \textit{in} \ H_0^1(\Omega),$$

as $t \to 0$.

Proof. By using Galerkin's approximation and a routine procedure similar to that of cite [8, 2], we can the global existence result for the solution subject to (1)-(3) under the above assumptions. □

Theorem 2.2. *Let s be the global solution of the problem* (1)-(3) *under either $\gamma > 2$ in case of $N < 2$ or $2 < \gamma \leq \frac{2N-2}{N-2}$ in case of $N \geq 3$ where N in $\Omega \subset \mathbb{R}^N$, and the above all conditions. We define the energy functional $E(t)$ as*

$$E(t) = \frac{1}{2} \left[\|s'(t)\|^2 + \int_\Omega c(x) |\nabla s(x,t)|^2 dx - \frac{2}{\gamma + 2} \|s'(t)\|_{\gamma+2}^{\gamma+2} \right].$$

and

$$\widetilde{E}(t) = E(t) + \varepsilon \chi(t)$$

where $\chi(t) = \langle s_t, s \rangle$ Then the modified energy functional moves exponentially, like as

$$\widetilde{E}(t) \leq e^{t-\tau} \widetilde{E}(\tau), \tag{8}$$

and also

$$\widetilde{E}(t) \geq \frac{\partial}{\partial t}((1 - e^{\tau - t}) \widetilde{E}(\tau)), \tag{9}$$

for all $0 \le \tau \le t < \infty$.

3. Proof of Theorem 2.2 (Energy Behavior)

Proof. Multiplying u' on both sides of (1), integrating the resulting equations over Ω, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|s_t\|^2 + \frac{1}{2}\int_\Omega c(x)|\nabla s|^2 dx - \frac{1}{\gamma+2}\|s\|_{\gamma+2}^{\gamma+2}\right] = (1-\varepsilon)\|s_t\|^2. \tag{10}$$

For positive constant ϵ, let us define the perturbed modified energy by

$$\widetilde{E}(t) = E(t) + \epsilon\chi(t) \tag{11}$$

where $\chi(t) = \langle s_t, s\rangle$. Differentiating with respect to t on both sides of $\chi(t)$, using (1) and $\epsilon \approx \varepsilon$ in (1), we get

$$\begin{aligned}\chi'(t) =& \langle s_{tt}, s\rangle + \|s_t\|^2 \\ \le& \|s_t\|^2 - \int_\Omega c(x)|\nabla s|^2 dx + \|s\|_{\gamma+2}^{\gamma+2}.\end{aligned} \tag{12}$$

By the assumption of c, c is a uniformly lower bounded function, so (12) implies that

$$\chi'(t) \le \|s_t\|^2 - c_0\|\nabla s\|^2 dx + \|s\|_{\gamma+2}^{\gamma+2}. \tag{13}$$

From (10) and (13), the following can be derived.

$$\widetilde{E}'(t) \le \|s_t\|^2 - c_0\varepsilon\|\nabla s\|^2 dx + \varepsilon\|s\|_{\gamma+2}^{\gamma+2}.$$

By the assumption of ε, that is $0 < \varepsilon << 1$, we obtain

$$\widetilde{E}'(t) \le \widetilde{E}(t), \tag{14}$$

for all $0 \le \tau \le t < \infty$. Taking the integral from τ to t in (14), we get

$$\ln\frac{\widetilde{E}(t)}{\widetilde{E}(\tau)} \le t - \tau.$$

So we have (8). And also multiplying both sides of (14) by $e^{\tau-t}$, we get

$$e^{\tau-t}\widetilde{E}'(t) - e^{\tau-t}\widetilde{E}(t) \le 0.$$

Adding both sides of the above inequality by the non-negative value $\widetilde{E}'(t)$ and $-2e^{\tau-t}\widetilde{E}'(t)$ to the left side, we have

$$(1-e^{\tau-t})\widetilde{E}'(t) - e^{\tau-t}\widetilde{E}(t) \le \widetilde{E}'(t).$$

So we deduce

$$\frac{\partial}{\partial t}((1-e^{\tau-t})\widetilde{E}(\tau)) \le \widetilde{E}'(t), \tag{15}$$

for all $0 \le \tau \le t < \infty$. Applying (14) to (15), we have (9). □

4. Conclusion and Further work

We get the energy exponentially behaviors((8) and (9)) by using the energy perturbation method. $-\frac{1}{\gamma+2}\|s'(t)\|_{\gamma+2}^{\gamma+2}$ may cause the energy $E(t)$ to become negative, so range setting of γ is important. The modified energy in (11) is physically associated with states that primarily dampen the total energy. The potential energy or states that increases the internal energy has a negligible effect on the physically modified energy. So, ϵ of the modified energy is approximately equal to the ε of (1). Therefore, in this paper, we set ϵ and ε to be approximately the same. As future research, we will first check if $E(t)$ or $\widetilde{E}(t)$ considering the difference between ϵ and ε decreases exponentially. Then, we will see what happens if the range of γ is broken. It is likely that we will need to study the non-existence of solutions to the main system.

5. Acknowledgment

The author would like to thank the anonymous referee who provided useful and detailed comments on a previous version of the manuscript.

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