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SEMI-RIEMANNIAN MANIFOLDS WITH HOMOGENEOUS GEODESICS

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Abstract. The purpose of this paper is to establish the notion of semi-Riemannian manifolds M with the homogeneous geodesics, and investigate properties about certain homogeneity.

1. Introduction

We adopt the notations in [8]. At first, let's establish the notion of homogeneous geodesics.

Definition 1. A semi-Riemannian manifold M is *homogeneous* provided that, given any points p, q in M, there is an isometry ϕ of M such that $\phi(p) = q$.

Definition 2. Let p be a point in a complete semi-Riemannian manifold M and let A be an linear isometry of T_pM. Consider the map $\tilde{A} : M \longrightarrow M$ defined by $\tilde{A} \circ \exp = \exp \circ A$ where $\exp : T_pM \longrightarrow M$ provided it is well-defined. If, at the point p, M admits the maps defined above for all linear isometries of T_pM, and furthermore they are isometries of M, then M is said to *have homogeneous geodesics at p*.

Examples. (1) The pseudohyperbolic space $H_0^2(r)$ of radius $r > 0$ in R_1^3 is a 2-dimensional nonhomogeneous complete spacelike surface which has homogeneous geodesics at $(1, 0, 0)$. (2) In $\mathbb{R}^3$, the graph of $z = x^2 + y^2$ is a 2-dimensional nonhomogeneous complete Riemannian manifold which has homogeneous geodesics at $(0, 0, 0)$. (3) In $\mathbb{R}^3$, the solution of the equation $(\frac{x}{a})^2 + (\frac{y}{a})^2 + (\frac{z}{b})^2 = 1$ is a 2-dimensional nonhomogeneous complete Riemannian manifold which has homogeneous geodesics at $(0, 0, \pm b)$, where a, b are distinct positive reals.

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2. Semi-Riemannian Manifolds With Homogeneous Geodesics

Proposition 2.1. *Let M be a connected complete semi-Riemannian manifold. If M is a homogeneous space with homogeneous geodesics at a point, then it has homogeneous geodesics at all points of M.*

Proof. Suppose M has homogeneous geodesics at $p \in M$. Let q be an arbitrary point of M. Then there is an isometry ϕ such that $\phi(p) = q$. Since an isometry preserves geodesics, we know

$$\psi \exp(tv) = \exp(t\psi_* v), -\infty < t < \infty, v \in TM$$

for every isometry ψ of M.

Let A be a linear isometry of T_qM. Then

$$\begin{aligned}
\widetilde{A}\exp(tw) &= \exp tAw \\
&= \exp\left(t\left(\phi_*\phi_*^{-1}A\phi_*\phi_*^{-1}\right)(w)\right) \\
&= \exp\left(t\phi_*\left(\phi_*^{-1}A\phi_*\phi_*^{-1}\right)(w)\right) \\
&= \phi\left(\exp t\left(\phi_*^{-1}A\phi_*\phi_*^{-1}\right)(w)\right) \\
&= \phi\left(\exp\left(\left(\phi_*^{-1}A\phi_*\right)\left(\phi_*^{-1}(tw)\right)\right)\right) \\
&= \phi\left(\widetilde{\phi_*^{-1}A\phi_*}\exp\phi_*^{-1}(tw)\right) \\
&= \phi\widetilde{\phi_*^{-1}A\phi_*}\phi^{-1}\exp(tw),
\end{aligned}$$

where $-\infty < t < \infty$, $w \in TM_q$ and $\phi_*^{-1}A\phi_*$ is an isometry of T_pM. Since $\widetilde{\phi_*^{-1}A\phi_*}$ is well-defined and ϕ, $\widetilde{\phi_*^{-1}A\phi_*}$, ϕ^{-1} are all isometries, $\widetilde{A}$ is an isometry of M.

□

Proposition 2.2. *If a connected complete semi-Riemannian manifold M with index 0 has homogeneous geodesics at $p \in M$, then the sectional curvatures of the orthonormal pairs in T_pM are all identical.*

Proof. Let $\{u, v\}$ and $\{z, w\}$ be orthonormal pairs in T_pM. Then extend $\{u, v\}$ and $\{z, w\}$ to $\{u_1 = u, u_2 = v, u_3, \cdots, u_n\}$, $\{z_1 = z, z_2 = w, z_3, \cdots, z_n\}$, which are orthonormal bases of T_pM. Let A be a linear isometry such that $A(u_i) = z_i$ for $i = 1, \ldots, n$. Since $\widetilde{A}\exp tv = \exp tAv$ for $v \in T_pM$, w$-\infty < t < \infty$, we know that $\widetilde{A}_*v = Av$. Then the sectional curvature

$$K(u, v) = K(\widetilde{A_*}u, \widetilde{A_*}v) = K(Au, Av) = K(z, w)$$

because A is an isometry. This completes the proof.

□

The following theorem is due to Schur.

Theorem 2.3. *Let M be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature, $K(p)$, where p is a plane in $T_x(M)$, depends only on x, then M is a space of constant curvature.*

Proof. See Theorem 2.2 in [5]. □

Proposition 2.4. *If a connected complete semi-Riemannian manifold M with index* 0 *has homogeneous geodesics at every point of M and $n = dimM \geq 3$, then M is a space of constant curvature.*

Proof. Since the sectional curvature depends only on the points of M not on any special orthonormal pairs in the tangent space, by Proposition 2.2 and Theorem 2.3, M is actually a space of constant curvature.

□

For the following, we will mean by a geodesic γ from p to q is a geodesic defined on the unit interval $[0, 1]$ such that $\gamma(0) = p$, $\gamma(1) = q$. And if $v \in T_pM$, let γ_v be the geodesic defined by $\gamma_v(t) = \exp(tv)$.The restriction of γ_v to the unit interval, that is, the geodesic from p to $\gamma_v(1)$ in the direction of v will be also denoted by γ_v.

Lemma 2.5. *Let M be a connected complete semi-Riemannian manifold with index* 0*. If $q \in M$ is not a conjugate point of p along any geodesic, and L is a positive number, then there are finitely many geodesics from p to q whose arc lengths are $\leq L$.*

Proof. Suppose there are infinitely many such geodesics. Hence there are infinitely many γ_v's such that $v \in T_pM$, $\gamma_v(1) = q$, $\|v\| \leq L$. Since the set $\{v \in T_pM : ||v|| \leq L\}$ is compact, there is a limit point $z \in T_pM$ with $||z|| \leq L$. Since exp is a continuous map from TM to M, γ_z is also a geodesic from p to q. exp is not critical on z because $q = \exp z$ is not a conjugate point of p along γ_z, when exp is considered as a map from T_pM. Therefore, exp is a local diffeomorphism around z. But, for every neighborhood of $z \in T_pM$, there is a vector v such that $\exp v = \exp z = q$, which is a contradiction.

□

The following theorem is due to Hopf-Rinow.

Theorem 2.6. *For a connected Riemannian manifold M the following conditions are equivalent:*

(1) *As a metric space under Riemannian distance d, M is complete.*
(2) *There exists a point $p \in M$ from which M is geodesically complete.*
(3) *M is geodesically complete.*
(4) *Every closed bounded subset of M is compact.*

Proof. See Theorem 21 in Ch.5 [8]. □

Theorem 2.7. *Let M be a connected complete semi-Riemannian manifold with index* 0*. If M is noncompact and has homogeneous geodesics at $p \in M$, then for all q distinct from p, there are at most finite geodesics from p and q.*

Proof. Suppose there is a minimum point along a certain geodesic γ_w. We let this minimum point $\gamma_v(s) = \exp(sv)$. Let w be another vector in T_pM with $\|w\| = \|v\|$. Then there is a linear isometry A of T_pM such that $A(v) = w$. Since $\gamma_w(t) = \exp tw = \exp tAv = \widetilde{A}\exp tv$, and $\tilde{A}$ is an isometry, $\exp sw$ is also a minimum point of p along the geodesic γ_w. Since every point p' in M is connected to p by a minimal geodesic and every minimal geodesic has length less than or equal to $s\|v\|$, we have $\rho(p,p') \leq s\|v\|$. Hence M is a bounded set and a compact set by the above Hopf-Rinow theorem. Therefore there are no minimum points of p, and hence no conjugate points of p.

If $\exp tw = q$, then $t\|w\| = \rho(p,q)$ because there is no minimum point of p. Therefore, if γ_z is a geodesic from p to q, then arc length must be $\rho(p,q)$.

Since q is not a conjugate point of p along any geodesic, we have the required result by Lemma 2.5.

□

Theorem 2.8. *Let M be a connected complete semi-Riemannian manifold with index 0. If M is compact and homogeneous geodesics at a point p, then for all vector v's in T_pM there is a positive real number t_v such that $\gamma_v(nt_v) = p$, where n is an integer. And if t_v^* is the minimum of such t_v's, then all t_v^*'s are identical.*

Proof. Suppose there is no minimum points of p. Then, for every w in T_pM, $\rho(\exp tw, p) = t\|w\|$. This implies M is unbounded, hence M is noncompact. Therefore, there is a minimum point along some geodesic. Let this minimum point be $\exp \bar{t}v$ along γ_v, where v is the unit vector in T_pM. Since M has homogeneous geodesics at p, for every unit vector w, $\exp \bar{t}w$ is a minimum point along γ_w. And it follows that every minimum point has the same distance $\bar{t}$ from p.

Next, we consider the following two cases.

Case 1. A minimum point of p, $\exp \bar{t}v$ is not a conjugate point of p along γ_v, hence every minimum point $\exp \bar{t}w$ of p is not a conjugate point of p along γ_w :

There is a unique closed geodesic which ends at p and passes through $\exp \bar{t}v$ by Theorem 12 in Ch.11 [4]. Say this geodesic γ_z, where $\|z\| = 1$. Then for every unit vector w in T_pM, γ_w restricted to $[0, t^*z]$ is also a closed geodesic since M has homogeneous geodesics at p. And $t^*z = t^*w$ is obtained by symmetry.

Case 2. A minimum point of p, $\exp \bar{t}v$ is a conjugate point of p along γ_v, hence every minimum point $\exp \bar{t}w$ of p is a conjugate point of p along γ_w :

exp is critical at $\bar{t}v$ when exp is considered as a map defined on T_pM. But we know $T_{\bar{t}v}(T_pM)$ is decomposed as $T_{\bar{t}v}S^{n-1}$ and its 1-dimensional normal space K which is generated by the velocity vector of the curve $\gamma(t) = t(\bar{t}v)$ at $\bar{t}v$, where $S^{n-1} = \{x \in T_pM \| \|x\| = \bar{t}\}$, $n = dim\ M$. Since the curve $\gamma_{\bar{t}v} = \exp\gamma$ has a nonzero velocity vector everywhere, $\exp_*$ restricted to K is an isomorphism. Hence $\exp_*$ restricted to $T_{\bar{t}v}S^{n-1}$ must have a nonzero $\alpha \in T_{\bar{t}v}S^{n-1}$ such that $\exp_*(\alpha) = O$. From this we know that exp restricted to S^{n-1} must have rank less than $n-1$ at $\bar{t}v$.

If A is an isometry of T_pM which maps $\bar{t}v$ to $\bar{t}w$, and if α is a curve in T_pM such that $\alpha(0) = \bar{t}v$ and $\frac{d}{dt}|_0 \exp\alpha(t) = O$, then $\frac{d}{dt}|_0 \tilde{A}\exp\alpha(t) = O$ since $\widetilde{A}$ is a smooth map. But $\widetilde{A}\exp\alpha(t) = \exp A\alpha(t)$ and A_* is an isomorphism of $T_{\bar{t}v}(T_pM)$ and $T_{\bar{t}w}(T_pM)$. The previous argument says that if $\eta \in T_{tv}(T_pM)$ and $\exp_* \eta = O$, then $\exp_* A_*\eta = O$. It follows that the rank of exp at $\bar{t}v$ is either equal to or smaller the the rank of exp at $\bar{t}w$ and that by symmetry they are the same.

By the first part of the proof we know at $\bar{t}w$, $\|w\| = 1$, the rank of exp restricted to S^{n-1} is the rank of exp without restriction minus one. Hence the rank of exp restricted to S^{n-1} is identical at all points of S^{n-1}.

Therefore, there are coordinate systems x of S^{n-1} around $\bar{t}v$ and y around $\exp\bar{t}v$ such that $y\exp x^{-1}(a_1,\cdots,a_{n-1}) = (a_1,\cdots,a_k,0,\cdots,0)$, where k is the constant rank of exp restricted to S^{n-1}.

If $x(\bar{t}v) = (b_1,\cdots,b_k,b_{k+1},\cdots,b_{n-1})$, we set

$$\beta(t) = x^{-1}(b_1,\cdots,b_k,b_{k+1},\cdots,b_{n-1}+t).$$

Then $y\exp\beta(t) = (b_1,\cdots,b_k,0,\cdots,0) = y\exp\bar{t}v$, hence $\exp\beta(t) = \exp\bar{t}v$.

From this we know for some $\epsilon > 0$, for every $0 < \delta < \epsilon$, there is a vector $\bar{t}w$ in S^{n-1} such that $\exp\bar{t}w = \exp\bar{t}v$ and $<\bar{t}v,\bar{t}w> = (\bar{t})^2 - \delta$.

Let $\{u,v\}$ and $\{z,w\}$ be pairs of unit vectors such that $<u,v> = <z,w>$. If we set $v' = \frac{v-<u,v>u}{\|v-<u,v>u\|}$ and $w' = \frac{w-<z,w>z}{\|w-<z,w>w\|}$, then $\{u,v'\}$ and $\{z,w'\}$ are orthonormal pairs, hence there is an isometry A of TM_p such that $A(u) = z$, and $A(v') = w'$. But $v = v' + lu$ and $w = w' + lz$ implies $A(v) = w$, where $l = \frac{<u,v>}{\|v-<u,v>u\|} = \frac{<z,w>}{\|w-<z,w>w\|}$.

Suppose z is such that $<\bar{t}v,\bar{t}z> = (\bar{t})^2 - \delta$ for $0 < \delta < \epsilon$. Then there is an isometry A and a unit vector w such that

$$A(\bar{t}v) = \bar{t}v, A(\bar{t}w) = \bar{t}z, \exp\bar{t}w = \exp\bar{t}v, <\bar{t}v,\bar{t}w> = (\bar{t})^2 - \delta.$$

Thus, $\exp\bar{t}z = \exp A\bar{t}w = \widetilde{A}\exp\bar{t}w = \widetilde{A}\exp\bar{t}v = \exp A\bar{t}v = \exp\bar{t}v$. Hence, the set $F = \{w \in T_pM : \exp\bar{t}w = \exp\bar{t}v\}$ is an open subset of S^{n-1}. But the same argument as above one shows that $S^{n-1} - F$ is also an open set of S^{n-1}. Since S^{n-1} is connected and $v \in F$, we have $S^{n-1} = F$.

Parallel translation preserves inner products. If $v_1,\ldots,v_n$ form an orthonormal basis of T_pM, then the velocity vectors v_i's of γ_{v_i}'s at $q = \exp\bar{t}v$ form an orthonormal basis of TM_q. If $v' = \sum_i c_iv_i$, i.e. $c_i = <v',v_i>$, then v' is the velocity vector of $\gamma_{\sum c_iv_i}$ at q. Hence the geodesics from q are actually the geodesics from p and M has homogeneous geodesics at q.

Since p is a conjugate point and a minimum point of q along all directions, we have $\exp\bar{t}u = p$ for all unit vectors $u \in T_qM$ by the same argument as above. Thus $\exp 2\bar{t}v = p$ for all unit vectors $v \in T_pM$. Every geodesic γ_v from p, $\|v\| = 1$, must have $t_v^* = 2\bar{t}$, where *bart* is the diameter of M with respect to the metric ρ.

□

The following theorem is known as Morse index theorem.

Theorem 2.9. *The index λ of E_{**} is equal to the number of points $\gamma(t)$, with $9 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ; each such conjugate point being counted with its multiplicity. This λ is always finite.*

Proof. See Theorem 15.1 [7]. □

Theorem 2.10. *Let M be a connected complete semi-Riemannian manifold with index 0. If M is compact and has homogeneous geodesics at p in M, and q is a point in M which is not a conjugate point of p along any geodesic, then for every nonnegative integer n there are only finitely many geodesics from p to q with indices less than or equal to n.*

Proof. Since M is compact and has homogeneous geodesics at p in M, all geodesics through p must return to p with the same period, say $\bar{s}$.

Let v be a unit vector in T_pM. Consider a variation α of $\gamma_{m\bar{s}v}$ defined by $\alpha(s,t) = \gamma_{m\bar{s}c(s)}$, where m is a positive integer and c is a smooth curve in the unit sphere of T_pM. Then $\alpha(s,0) = \alpha(s,1) = p$ and α is a variation through geodesics, hence the variation vector field of α is a Jacobi field which vanishes at the end points of $\gamma_{m\bar{s}v}$ by lemma 14.3 [7]. Hence $p = \exp m\bar{s}v$ is a conjugate point of itself along γ_v.

Suppose t is lager than $(n+1)\bar{s}$. Then, for every unit vector w, γ_{tw} has at least $n+1$ conjugate points from p to $\gamma_{tw}(1)$, hence index larger than n by theorem 2.9. Thus the required geodesic must be of length less than or equal to $(n+1)\bar{s}$. Lemma 2.5 can be applied to this. There are only finitely many geodesics from p to q with index $\leq n$. □

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