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# EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO NONLOCAL BOUNDARY VALUE PROBLEMS WITH STRONG SINGULARITY

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Abstract. In this paper, we consider $\varphi$-Laplacian nonlocal boundary value problems with singular weight function which may not be in $L^1(0,1)$. The existence and nonexistence of positive solutions to the given problem for parameter $\lambda$ belonging to some open intervals are shown. Our approach is based on the fixed point index theory.

## 1. Introduction

In this paper, we study the existence of positive solutions to the following boundary value problem

$$\begin{cases} (w(t)\varphi(u'(t)))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 u(x) d\alpha_1(x), u(1) = \int_0^1 u(x) d\alpha_2(x), \end{cases} \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $w \in C([0,1],(0,\infty))$, $\lambda \in [0,\infty) := \mathbb{R}_+$ is a parameter, $f \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $f(s) > 0$ for $s > 0$, $h \in C((0,1),\mathbb{R}_+)$, and the integrator functions $\alpha_i$ $(i = 1,2)$ are nondecreasing on $[0,1]$.

By a solution $u$ to the problem (1), we mean $u \in C^1(0,1) \cap C[0,1]$ with $w\varphi(u') \in C^1(0,1)$ satisfies (1). Throughout this paper, the following hypotheses are assumed, unless otherwise stated.

$(H_1)$ There exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(yx) \le \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{2}$$

$(H_2)$ For $i = 1,2$, $\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0,1)$.

Let us introduce notations $f_0 := \lim_{s\to 0^+} \dfrac{f(s)}{\varphi(s)}$, $f_\infty := \lim_{s\to\infty} \dfrac{f(s)}{\varphi(s)}$ and, for an increasing homeomorphism $\Theta$ on $\mathbb{R}_+$,

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$$\mathcal{D}_\Theta := \left\{ g \in C((0,1),\mathbb{R}_+) : \int_0^1 \left| \Theta^{-1}\left( \int_s^{\frac{1}{2}} g(\tau)d\tau \right) \right| ds < \infty \right\}.$$

It is well known that if an odd increasing homeomorphism $\varphi$ satisfies the assumption $(H_1)$, then

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{3}$$

and $L^1(0,1) \cap C(0,1) \subseteq \mathcal{D}_{\psi_1} \subseteq \mathcal{D}_\varphi \subseteq \mathcal{D}_{\psi_2}$ (see, e.g., ([8], Remark 1)). In the main result (see Theorem 3.1 below), we assume that the weight function $h$ in the problem (1) is in $\mathcal{D}_{\psi_1}$, which implies that $h$ may not be in $L^1(0,1)$. For example, let $\varphi(x) = x + x^2$ for $x \in \mathbb{R}_+$ and define $h : (0,1) \to \mathbb{R}_+$ by $h(t) = t^{-\alpha}$ for $t \in (0,1)$. Then it is easy to check that $(H_1)$ is satisfied with $\psi_1(y) = \min\{y, y^2\}$ and $\psi_2(y) = \max\{y, y^2\}$. From the fact that $\psi_1^{-1}(s) = s$ for $s \ge 1$, it follows that $h \in \mathcal{D}_{\psi_1} \setminus L^1(0,1)$ for any $\alpha \in [1,2)$.

The nonlocal boundary value problems play an important role in physics and applied mathematics (see, e.g., [2, 6, 7]). The existence of solutions for nonlocal boundary value problems have been studied widely. For example, Liu [15] studied the following four-point boundary value problem

$$\begin{cases} u''(t) + h(t)g_1(u(t)) = 0, \ t \in (0,1), \\ u(0) = \mu_0 u(\xi_0), \ u(1) = \mu_1 u(\xi_1), \end{cases} \tag{4}$$

which is a special case of the problem (1). Under various assumptions on the nonlinearity $g_1$, the existence of one or two positive solutions to the problem (4) were shown. Webb and Infante [18], when $\varphi(s) = s$ and $w \equiv 1$, presented some sufficient conditions on the nonlinear term $f = f(t,s)$ for the existence and multiplicity of positive solutions to the problem (1) subject to several nonlocal boundary conditions. Feng, Ge and Jiang [5] studied sufficient conditions on the nonlinear term $f = f(t,s)$ for the existence of multiple positive solutions to the problem (1) subject to multi-point boundary conditions. Kim [10] improved on the results in [5] under the assumption that the weight function $h = h(t)$ may not be in $L^1(0,1)$. Cui [4], under the resonance conditions $\tau_1\tau_2\tau_3\tau_4 \ne 0$ and $\tau_1\tau_4 - \tau_2\tau_3 = 0$, gave some sufficient conditions for the existence of solutions to the problem (1) with $w \equiv 1$, $\varphi(s) = s$ and $\lambda = 1$. Here $\tau_1 = 1 - \int_0^1 (1-x)d\alpha_1(x)$, $\tau_2 = \int_0^1 x d\alpha_1(x)$, $\tau_3 = \int_0^1 (1-x)d\alpha_2(x)$ and $\tau_4 = 1 - \int_0^1 x d\alpha_2(x)$. Note that if $\alpha_1(x) = \alpha_2(x) = x$, then the assumption $(H_2)$ is not satisfied, but the above resonance conditions are satisfied because $\tau_i = \frac{1}{2}$ for $i = 1,2,3,4$. Bougoffa and Khanfer [1] considered the following semilinear problem

$$\begin{cases} u''(t) + g_2(t, u(t)) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 m_1(x)u(x)dx, u(1) = \int_0^1 m_2(x)u(x)dx, \end{cases} \tag{5}$$

where $g_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function and $m_i$ is an integrable function on $[0,1]$ for $i = 1,2$. The authors investigated the sufficient condition on $g_2$ for the uniqueness of solution to the problem (5). Son and Wang [16] studied the existence and multiplicity of positive solutions to $p$-Laplacian systems subject nonlinear boundary conditions. Recently, Kim ([13, 14]) proved the existence of one or two positive solutions to the problem (1) when $f_0 = \beta_1$ and $f_\infty = \beta_2$ for $\beta_1, \beta_2 \in \{0, \infty\}$. For historical development of the theory of the problems with nonlocal boundary conditions, we refer the reader to the survey papers [3, 17, 19]. Motivated by the previous results mentioned above, we study the existence and nonexistence of positive solutions to the problem (1) when either $f_\infty \in (0, \infty)$ and $f_0 \in \{0, \infty\}$ or $f_0 \in (0, \infty)$ and $f_\infty \in \{0, \infty\}$ (see Theorem 3.1 below).

The rest of this paper is organized as follows. In Section 2, preliminary results which are needed for proving the main result (Theorem 3.1) are provided. In Section 3, the main result (Theorem 3.1) is stated and the proof of it is given.

## 2. Preliminaries

Throughout this section, we assume that $(H_1), (H_2)$ and $h \in \mathcal{D}_\varphi \setminus \{0\}$ hold. The usual maximum norm in a Banach space $C[0, 1]$ is denoted by $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$ for $u \in C[0, 1]$, and let $a_h := \inf\{x \in (0, 1) : h(x) > 0\}$, $b_h := \sup\{x \in (0, 1) : h(x) > 0\}$, $\bar{a}_h := \sup\{x \in (0, 1) : h(y) > 0$ for all $y \in (a_h, x)\}$, $\bar{b}_h := \inf\{x \in (0, 1) : h(y) > 0$ for all $y \in (x, b_h)\}$, $c_h^1 := \frac{1}{4}(3a_h+\bar{a}_h)$ and $c_h^2 := \frac{1}{4}(\bar{b}_h + 3b_h)$. Then, since $h \in C((0, 1), \mathbb{R}_+) \setminus \{0\}$, we have two cases: either $0 \le a_h < \bar{a}_h \le \bar{b}_h < b_h \le 1$ or $0 \le a_h = \bar{b}_h < b_h \le 1$ and $0 \le a_h < \bar{a}_h = b_h \le 1$. Consequently,

$$h(t) > 0 \text{ for } t \in (a_h, \bar{a}_h) \cup (\bar{b}_h, b_h), \text{ and } 0 \le a_h < c_h^1 < c_h^2 < b_h \le 1. \tag{6}$$

Let $r_h := r_1 \min\{c_h^1, 1 - c_h^2\} \in (0, 1)$, where

$$w_0 := \min_{t\in[0,1]} w(t) > 0 \text{ and } r_1 := \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{w_0}\right)\right]^{-1} \in (0, 1].$$

Then $\mathcal{P} := \{u \in C([0, 1], \mathbb{R}_+) : u(t) \ge r_h\|u\|_\infty$ for $t \in [c_h^1, c_h^2]\}$ is a cone in $C[0, 1]$. For $m > 0$, let $\mathcal{P}_m := \{u \in \mathcal{P} : \|u\|_\infty < m\}$, $\partial\mathcal{P}_m := \{u \in \mathcal{P} : \|u\|_\infty = m\}$ and $\overline{\mathcal{P}}_m := \mathcal{P}_m \cup \partial\mathcal{P}_m$.

Let $\mathcal{C}_1 := \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)\min\left\{\int_{c_h^1}^{c_h}\psi_2^{-1}\left(\int_s^{c_h} h(\tau)d\tau\right)ds, \int_{c_h}^{c_h^2}\psi_2^{-1}\left(\int_{c_h}^s h(\tau)d\tau\right)ds\right\}$ and $\mathcal{C}_2 := \psi_1^{-1}\left(\frac{1}{w_0}\right)\max\left\{\mathcal{A}_1\int_0^{c_h}\psi_1^{-1}\left(\int_s^{c_h} h(\tau)d\tau\right)ds, \mathcal{A}_2\int_{c_h}^1\psi_1^{-1}\left(\int_{c_h}^s h(\tau)d\tau\right)ds\right\}$.

Here, $c_h := \frac{c_h^1 + c_h^2}{2}$ and $\mathcal{A}_i := (1 - \hat{\alpha}_i)^{-1} > 0$ for $i = 1, 2$. Clearly, by (6), $\mathcal{C}_1 > 0$ and $\mathcal{C}_2 > 0$. Define continuous functions $f_*, f^* : (0, \infty) \to (0, \infty)$ by $f_*(m) := \min\{f(z) : r_h m \le z \le m\}$ and $f^*(m) := \max\{f(z) : 0 \le z \le m\}$ for $m \in (0, \infty)$.

Define $R_1, R_2 : (0, \infty) \to (0, \infty)$ by

$$R_1(s) := \frac{1}{f_*(s)}\varphi\left(\frac{s}{\mathcal{C}_1}\right) \text{ and } R_2(s) := \frac{1}{f^*(s)}\varphi\left(\frac{s}{\mathcal{C}_2}\right) \text{ for } s \in (0, \infty).$$

*Remark* 1. (i) By (3) and $(H_2)$, $\psi_2^{-1}(z) \le \psi_1^{-1}(z)$ for all $z \in \mathbb{R}_+$ and $\mathcal{A}_i = (1-\hat{\alpha}_i)^{-1} \ge 1$ for $i = 1, 2$. Thus, $0 < \mathcal{C}_1 < \mathcal{C}_2$ and

$$0 < R_2(s) < R_1(s) \text{ for all } s \in (0, \infty). \tag{7}$$

(ii) For any $L \in C((0,\infty),(0,\infty))$, let $L_c := \lim_{s\to c} \dfrac{L(s)}{\varphi(s)}$ for $c \in \{0, \infty\}$. Then it is well known that $(f_*)_c = (f^*)_c = 0$ if $f_c = 0$ , and $(f_*)_c = (f^*)_c = \infty$ if $f_c = \infty$ (see, e.g., [11, Remark 2]). For $i \in \{1, 2\}$, it follows from (3) that

$$\lim_{s\to 0} R_i(s) = \infty \text{ if } f_0 = 0, \text{ and } \lim_{s\to\infty} R_i(s) = \infty \text{ if } f_\infty = 0; \tag{8}$$

$$\lim_{s\to 0} R_i(s) = 0 \text{ if } f_0 = \infty, \text{ and } \lim_{s\to\infty} R_i(s) = 0 \text{ if } f_\infty = \infty. \tag{9}$$

For $k \in \mathcal{D}_\varphi$, consider the following problem

$$\begin{cases} (w(t)\varphi(u'(t)))' + k(t) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 u(x)d\alpha_1(x), u(1) = \int_0^1 u(x)d\alpha_2(x). \end{cases} \tag{10}$$

Define $T : \mathcal{D}_\varphi \to C[0,1]$ by $T(0) = 0$ and for $k \in \mathcal{D}_\varphi \setminus \{0\}$,

$$T(k)(t) = \begin{cases} \mathcal{A}_1 \int_0^1 \int_0^x I_k(s,\sigma)dsd\alpha_1(x) + \int_0^t I_k(s,\sigma)ds, & \text{if } 0 \le t \le \sigma, \\ -\mathcal{A}_2 \int_0^1 \int_x^1 I_k(s,\sigma)dsd\alpha_2(x) - \int_t^1 I_k(s,\sigma)ds, & \text{if } \sigma \le t \le 1, \end{cases}$$

where $I_k(s,y) := \varphi^{-1}\left(\dfrac{1}{w(s)}\displaystyle\int_s^x k(\tau)d\tau\right)$ for $s, y \in (0,1)$ and $\sigma = \sigma(k)$ is a constant satisfying

$$\begin{aligned} &\mathcal{A}_1 \int_0^1 \int_0^x I_k(s,\sigma)dsd\alpha_1(x) + \int_0^\sigma I_k(s,\sigma)ds \\ = \ & -\mathcal{A}_2 \int_0^1 \int_x^1 I_k(s,\sigma)dsd\alpha_2(x) - \int_\sigma^1 I_k(s,\sigma)ds. \end{aligned} \tag{11}$$

For any $k \in \mathcal{H}_\varphi$ and any $\sigma$ satisfying (11), $T(k)$ is monotone increasing on $[0,\sigma)$ and monotone decreasing on $(\sigma, 1]$. We notice that $\sigma = \sigma(k)$ is not necessarily unique, but $T(k)$ is independent of the choice of $\sigma$ satisfying (11) (see [9, Remark 2]).

**Lemma 2.1.** ([9, Lemma 2]) *Assume that* $(H_1), (H_2)$ *and* $k \in \mathcal{D}_\varphi$ *hold. Then* $T(k)$ *is a unique solution to the problem* (10) *satisfying the following properties:*

(*i*) $T(k)(t) \ge \min\{T(k)(0), T(k)(1)\} \ge 0$ *for* $t \in [0,1]$*;*
(*ii*) *for any* $k \not\equiv 0$, $\max\{T(k)(0), T(k)(1)\} < \|T(k)\|_\infty$*;*
(*iii*) $\sigma$ *is a constant satisfying* (11) *if and only if* $T(k)(\sigma) = \|T(k)\|_\infty$*;*
(*iv*) $T(k)(t) \ge r_1 \min\{t, 1-t\}\|T(k)\|_\infty$ *for* $t \in [0,1]$ *and* $T(k) \in \mathcal{P}$.

Define a function $G : \mathbb{R}_+ \times \mathcal{P} \to C(0,1)$ by $G(\lambda, u)(t) := \lambda h(t) f(u(t))$ for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$ and $t \in (0,1)$. Clearly, $G(\lambda, u) \in \mathcal{D}_\varphi$ for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$, since $h \in \mathcal{D}_\varphi$. Let us define an operator $H : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ by

$$H(\lambda, u) := T(G(\lambda, u)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}.$$

By Lemma 2.1 (*iv*), $H(\mathbb{R}_+ \times \mathcal{P}) \subseteq \mathcal{P}$, and consequently $H$ is well defined. Moreover, $u$ is a solution to the problem (1) if and only if $H(\lambda, u) = u$ for some $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$.

**Lemma 2.2.** ([12, Lemma 4]) *Assume that $(H_1), (H_2)$ and $h \in \mathcal{D}_\varphi \setminus \{0\}$ hold. Then the operator $H : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ is completely continuous.*

**Theorem 2.3.** ([14, Theorem 3.3]) *Assume that $(H_1), (H_2)$ and $h \in \mathcal{D}_{\psi_1} \setminus \{0\}$ hold. Assume, in addition, that there exist $m_1$ and $m_2$ such that $0 < m_2 < m_1$ (resp., $0 < m_1 < m_2$) and $R_1(m_1) < R_2(m_2)$. Then the problem (1) has a positive solution $u = u(\lambda)$ satisfying $m_2 < \|u\|_\infty < m_1$ (resp., $m_1 < \|u\|_\infty < m_2$) for any $\lambda \in (R_1(m_1), R_2(m_2))$.*

## 3. Main result

In this section, we state and prove the main result of this paper.

**Theorem 3.1.** *Assume that $(H_1), (H_2)$ and $h \in \mathcal{D}_{\psi_1} \setminus \{0\}$ hold.*

(*i*) *If $f_0 = 0$ and $f_\infty \in (0, \infty)$, then there exist positive constants $\lambda^*$ and $\underline{\lambda}$ such that the problem (1) has a positive solution $u(\lambda)$ for any $\lambda \in (\lambda^*, \infty)$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to \infty$, and it has no positive solutions for $\lambda \in (0, \underline{\lambda})$.*

(*ii*) *If $f_0 = \infty$ and $f_\infty \in (0, \infty)$, then there exist positive constants $\lambda_*$ and $\overline{\lambda}$ such that the problem (1) has a positive solution $u = u(\lambda)$ for any $\lambda \in (0, \lambda_*)$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0$, and it has no positive solutions for $\lambda \in (\overline{\lambda}, \infty)$.*

(*iii*) *If $f_0 \in (0, \infty)$ and $f_\infty = 0$, then there exist positive constants $\lambda_1^*$ and $\underline{\lambda}^1$ such that the problem (1) has a positive solution $u(\lambda)$ for any $\lambda \in (\lambda_1^*, \infty)$ satisfying $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$, and it has no positive solutions for $\lambda \in (0, \underline{\lambda}^1)$.*

(*iv*) *If $f_0 \in (0, \infty)$ and $f_\infty = \infty$, then there exist positive constants $\lambda_*^1$ and $\overline{\lambda}_1$ such that the problem (1) has a positive solution $u = u(\lambda)$ for any $\lambda \in (0, \lambda_*^1)$ satisfying $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0$, and it has no positive solutions for $\lambda \in (\overline{\lambda}_1, \infty)$.*

*Proof.* We only give the proofs of (*i*) and (*ii*), since (*iii*) and (*iv*) can be proved in a similar manner.

(*i*) Since $f_0 = 0$, by (8),

$$R_i(m) \to \infty \text{ as } m \to 0 \text{ for } i = 1, 2. \tag{12}$$

From the definition of $R_1$ and (2), it follows that

$$\lim_{m\to\infty} R_1(m) \geq \lim_{m\to\infty} \frac{\varphi(\frac{m}{\mathcal{C}_1})}{f(m)} \geq \lim_{m\to\infty} \frac{\varphi(m)}{f(m)} \psi_1(\frac{1}{\mathcal{C}_1}) = \frac{1}{f_\infty} \psi_1(\frac{1}{\mathcal{C}_1}) > 0.$$

Thus, there exists $\lambda^* := \inf\{R_1(m) : m \in (0,\infty)\} \in (0,\infty)$. For any $\lambda \in (\lambda^*,\infty)$, by (7) and (12), there exist $m_1^\lambda$ and $m_2^\lambda$ such that $0 < m_2^\lambda < m_1^\lambda$ and $R_1(m_1^\lambda) < \lambda < R_2(m_2^\lambda)$. By Theorem 2.3, there exists a positive solution $u_\lambda$ to the problem (1) satisfying $m_2^\lambda < \|u_\lambda\|_\infty < m_1^\lambda$. Moreover, since $R_i(m) \to \infty$ as $m \to 0$ for $i = 1, 2$, we may choose $m_1^\lambda$ and $m_2^\lambda$ satisfying $0 < m_2^\lambda < m_1^\lambda$ and $m_1^\lambda \to 0$ as $\lambda \to \infty$. Consequently, we can choose positive solutions $u_\lambda$ to the problem (1) for all large $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to \infty$.

Now we prove the nonexistence of positive solutions to the problem (1). Let $\lambda > 0$ be a constant for which the problem (1) has a positive solution $u$. Since $f_0 = 0$ and $f_\infty \in (0,\infty)$, there exists $M_1 > 0$ such that $f(s) \le M_1\varphi(s)$ for $s \in \mathbb{R}_+$. Thus,

$$f(u(t)) \le M_1\varphi(u(t)) \le M_1\varphi(u(\sigma)) \text{ for all } t \in [0,1]. \tag{13}$$

Here $\sigma$ is a constant satisfying $u(\sigma) = \|u\|_\infty$. We only consider the case $\sigma \le c_h$, since the proof for the case $\sigma > c_h$ is similar. First we prove

$$u(\sigma) \le \mathcal{A}_1 \int_0^\sigma I_{G(\lambda,u)}(s,\sigma)ds, \tag{14}$$

where $I_{G(\lambda,u)}(s,\sigma) = \varphi^{-1}\left(\frac{1}{w(s)}\int_s^\sigma \lambda h(\tau)f(u_\lambda(\tau))d\tau\right)$. Indeed, from the facts that $I_{G(\lambda,u)}(s,\sigma) \ge 0$ for $s \le \sigma$ and $I_{G(\lambda,u)}(s,\sigma) \le 0$ for $s \ge \sigma$, it follows that

$$\begin{aligned}
&\int_0^1\int_\sigma^r I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r)\\
= \;& -\int_0^\sigma\int_r^\sigma I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_\sigma^1\int_\sigma^r I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r) \le 0.
\end{aligned}$$

Then

$$\begin{aligned}
u(\sigma) &= \mathcal{A}_1\int_0^1\int_0^r I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{G(\lambda,u)}(s,\sigma)ds\\
&= \mathcal{A}_1\left[\int_0^1\int_0^r I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \left(1-\int_0^1 d\alpha_1(r)\right)\int_0^\sigma I_{G(\lambda,u)}(s,\sigma)ds\right]\\
&= \mathcal{A}_1\left[\int_0^1\int_\sigma^r I_{G(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{G(\lambda,u)}(s,\sigma)ds\right]\\
&\le \mathcal{A}_1\int_0^\sigma I_{G(\lambda,u)}(s,\sigma)ds,
\end{aligned}$$

and thus (14) is proved. From (14) and (13), it follows that

$$\begin{aligned}
u(\sigma) &\le \mathcal{A}_1\int_0^\sigma \varphi^{-1}\left(\frac{1}{w(s)}\int_s^\sigma \lambda h(\tau)f(u(\tau))d\tau\right)ds\\
&\le \mathcal{A}_1\int_0^{c_h}\varphi^{-1}\left(\int_s^{c_h} h(\tau)d\tau w_0^{-1}\lambda M_1\varphi(u(\sigma))\right)ds\\
&\le C_0\varphi^{-1}(w_0^{-1}\lambda M_1\varphi(u(\sigma))) \le C_0\psi_1^{-1}(w_0^{-1}\lambda M_1)u(\sigma).
\end{aligned}$$

Here $C_0 := \max\left\{\mathcal{A}_1\int_0^{c_h}\psi_1^{-1}\left(\int_s^{c_h}h(\tau)d\tau\right)ds, \mathcal{A}_2\int_{c_h}^1\psi_1^{-1}\left(\int_{c_h}^s h(\tau)d\tau\right)ds\right\} > 0$. Then $\lambda \geq w_0 M_1^{-1}\psi_1(C_0^{-1}) =: \underline{\lambda}$. Consequently, the problem (1) has no positive solutions for all $\lambda \in (0, \underline{\lambda})$.

$(ii)$ Since $f_0 = \infty$, by (2) and (9), $R_i(m) \to 0$ as $m \to 0$ for $i = 1, 2$. From the definition of $R_2$ and (2), it follows that

$$\lim_{m\to\infty} R_2(m) \leq \lim_{m\to\infty}\frac{\varphi(\frac{m}{\mathcal{C}_2})}{f(m)} \leq \lim_{m\to\infty}\frac{\varphi(m)}{f(m)}\psi_2(\frac{1}{\mathcal{C}_2}) = \frac{1}{f_\infty}\psi_2(\frac{1}{\mathcal{C}_2}) > 0.$$

Thus, there exists $\lambda^* := \sup\{R_2(m) : m \in (0,\infty)\} \in (0,\infty)$. For any $\lambda \in (0,\lambda^*)$, there exist $m_1^\lambda$ and $m_2^\lambda$ such that $0 < m_1^\lambda < m_2^\lambda$ and $R_1(m_1^\lambda) < \lambda < R_2(m_2^\lambda)$. By Theorem 2.3, there exists a positive solution $u_\lambda$ to problem (1) satisfying $m_1^\lambda < \|u_\lambda\|_\infty < m_2^\lambda$. Moreover, since $R_i(m) \to \infty$ as $m \to 0$ for $i = 1, 2$, we may choose $m_1^\lambda$ and $m_2^\lambda$ satisfying $0 < m_1^\lambda < m_2^\lambda$ and $m_2^\lambda \to 0$ as $\lambda \to \infty$. Consequently, we can choose positive solutions $u_\lambda$ to the problem (1) for all small $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0$.

Now we prove the nonexistence of positive solutions to the problem (1). Let $\lambda > 0$ be a constant for which the problem (1) has a positive solution $u$. Since $f_0 = \infty$ and $f_\infty \in (0,\infty)$, there exists $M_2 > 0$ such that

$$f(s) > M_2\varphi(s) \text{ for } s \in \mathbb{R}_+. \tag{15}$$

Let $\sigma$ be a constant satisfying $u(\sigma) = \|u\|_\infty$. We only consider the case $\sigma \geq c_h$, since the proof for the case $\sigma < c_h$ is similar. Since $u$ is monotone increasing on $[0,\sigma]$, $u(t) \geq u(c_h^1)$ for $t \in [c_h^1, \sigma]$. By (15), $f(u(t)) > M_2\varphi(u(c_h^1))$ for $t \in [c_h^1, c_h]$. Then, by (3),

$$\begin{aligned}
u(c_h^1) &\geq \int_0^{c_h^1}\varphi^{-1}\left(\frac{1}{w(s)}\int_s^\sigma \lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\geq \int_0^{c_h^1}\varphi^{-1}\left(\int_{c_h^1}^{c_h}h(\tau)d\tau\|w\|_\infty^{-1}\lambda M_2\varphi(u(c_h^1))\right)ds \\
&\geq \int_0^{c_h^1}\psi_2^{-1}\left(\int_{c_h^1}^{c_h}h(\tau)d\tau\right)ds\varphi^{-1}(\|w\|_\infty^{-1}\lambda M_2\varphi(u(c_h^1))) \\
&\geq C_1\psi_2^{-1}(\|w\|_\infty^{-1}\lambda M_2)u(c_h^1),
\end{aligned}$$

where $C_1 := \min\left\{\int_0^{c_h^1}\psi_2^{-1}\left(\int_{c_h^1}^{c_h}h(\tau)d\tau\right)ds, \int_{c_h^2}^1\psi_2^{-1}\left(\int_{c_h}^{c_h^2}h(\tau)d\tau\right)ds\right\} > 0$. Then $\lambda \leq \|w\|_\infty M_2^{-1}\psi_2(C_1^{-1}) =: \overline{\lambda}$. Consequently, the problem (1) has no positive solutions for $\lambda \in (\overline{\lambda}, \infty)$. $\square$

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