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A CHARACTERIZATION OF AUTOMORPHISMS OF THE UNIT DISC BY THE POINCARÉ METRIC

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Abstract. Non-trivial automorphisms of the unit disc in the complex plane can be classified by three classes; elliptic, parabolic and hyperbolic automorphisms. This classification is due to a representation in the projective special linear group of the real field, or in terms of fixed points on the closure of the unit disc. In this paper, we will characterize this classification by the distance function of the Poincaré metric on the interior of the unit disc.

1. Introduction

By the uniformization of Riemann surfaces ([10, 6]), most Riemann surface can be generated by the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ as a quotient space. More precisely, a compact Riemann surface X of genus $g \geq 2$ is conformally equivalent to a quotient space $\Gamma\backslash\Delta$ where Γ is a discrete group of the holomorphic (orientation-preserving conformal) automorphism group $\mathrm{Aut}(\Delta)$ of Δ. Therefore it has been important to study the group structure of $\mathrm{Aut}(\Delta)$ and its action on Δ.

As a Lie group, $\mathrm{Aut}(\Delta)$ is isomorphic to the projective special linear group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acting on the upper half-plane of $\mathbb{C}$ by fractional linear transformations. The characteristic polynomial of $\mathfrak{H} \in \mathrm{SL}(2,\mathbb{R})$ is of the form

$$\lambda^2 - \mathrm{Tr}(\mathfrak{H})\lambda + \det(\mathfrak{H}) = \lambda^2 - \mathrm{Tr}(\mathfrak{H})\lambda + 1 ,$$

so the value $\mathrm{Tr}(\mathfrak{H})^2 - 4$ gives eigenvalues of $\mathfrak{H}$ by

$$\lambda = \frac{\mathrm{Tr}(\mathfrak{H}) \pm \sqrt{\mathrm{Tr}(\mathfrak{H})^2 - 4}}{2} .$$

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This gives a typical classification of each $\mathfrak{H} \in \mathrm{PSL}(2,\mathbb{R})$ into three types; $\mathfrak{H}$ is called *elliptic*, *parabolic* and *hyperbolic* if and only if $|\mathrm{Tr}(\mathfrak{H})| < 2$, $= 2$ or > 2, with respectively (see [7] and [3]).

Thus using the representation into $\mathrm{PSL}(2,\mathbb{R})$, we have the same classification of $\mathrm{Aut}(\Delta)$ and $\mathrm{Aut}(S)$ of a Riemann surface S conformally equivalent to the unit disc. For a nontrivial automorphism f of Δ and its representation $\mathfrak{H}_f \in \mathrm{PSL}(2,\mathbb{R})$, the type of f (so eigenvalues of $\mathfrak{H}_f$) completely describes its action on $\overline{\Delta}$, as following.

(1) f has a fixed point in Δ if and only if f is elliptic;
(2) f has only one fixed point in the boundary $\partial\Delta$ if and only if f is parabolic;
(3) f has only two fixed points in the boundary $\partial\Delta$ if and only if f is hyperbolic.

The main question of this research is *how one can determine the type of* $f \in \mathrm{Aut}(\Delta)$ *not using the representation into* $\mathrm{PSL}(2,\mathbb{R})$ *and not using boundary extension of* $\mathrm{Aut}(\Delta)$. Namely, *how one can distinguish automorphism types by actions on the inside of* Δ*?*. For the intrinsic approach, it should be natural to employ the *Poincaré metric*,

$$ds^2_\Delta = \frac{1}{\left(1-|z|^2\right)^2}\,|dz|^2 \ ,$$

of the unit disc which is a complete hermitian metric of a negative constant curvature and the unique, holomorphically invariant metric of Δ (see [4]).

1.1. Previous works

In the previous works ([8, 9]), we gave methods to distinguish parabolic and hyperbolic automorphisms in terms of the Poincaré metric.

For the intrinsic argument, we employed, as a base space, a simply connected Riemann surface S admitting a complete hermitian metric g of constant curvature $\kappa \equiv -4$. By the uniformization theorem, this (S, g) is conformally equivalent to the Poincaré disc model (Δ, ds^2_Δ). We can define a type of a nontrivial automorphism f of S by a type of a corresponding automorphism $\tilde{f} \in \mathrm{Aut}(\Delta)$. Here $\tilde{f}$ is given by

$$\tilde{f} = F^{-1} \circ f \circ F$$

for some biholomorphism $F : \Delta \to S$. A corresponding $\tilde{f}$ is not uniquely determined, but a type of $\tilde{f}$ is always same. Thus a type of f is well-defined.

In order to distinguish types, we employed a test function intrinsically given by the metric g. Given base point $p \in S$, a *test function* $\varphi_p : S \to \mathbb{R}$ is the negatively valued function defined by

$$\varphi_{g,p}(s) = \tanh^2(d_g(p, s)) - 1 \tag{1}$$

for $s \in S$ where d_g is the distance function of (S, g). Let $f \in \mathrm{Aut}(S)$ and denote the n-th iteration of f by $f^{(n)}$, i.e. $f^{(1)} = f$ and $f^{(n+1)} = f^{(n)} \circ f$. When we consider the sequence $\{\varphi_n\}$ of functions defined by

$$\varphi_n = \frac{\varphi_{g,p} \circ f^{(n)}}{\left(\varphi_{g,p} \circ f^{(n)}\right)(p)} \;, \tag{2}$$

this sequence converges to some positive function $\hat{\varphi} : S \to \mathbb{R}$. If f has no fixed point in S, equivalently f is parabolic or hyperbolic, the function $\hat{\varphi}$ can determine a type of f.

Theorem 1.1 (Lee [8]). *For an automorphism f of S, the sequence $\{\varphi_n\}$ of functions in* (2) *converges to a positive function $\hat{\varphi}$ and*

$$\hat{\varphi} \circ f \equiv c\hat{\varphi}$$

for some positive c. Suppose that f has no fixed point on S. Then f is parabolic or hyperbolic if and only if $c = 1$ or $c \neq 1$, with respectively.

Note that if f is elliptic, then the constant c of the theorem is always 1. Thus this approach can not distinguish elliptic automorphisms from others.

The function φ_p in (2) is a purely geometric quantity of $S \simeq \Delta$, that means φ_p is computed by an intrinsic way. But by the limit procedure involved in the theorem, types cannot be determined in the finite number of calculation.

In [9], we studied the same problem to distinguish the hyperbolicity and the parabolicity without a limiting procedure. Let us go back to the Riemann surface (S, g) and consider two numbers c_1 and c_2 which are given by $p \in S$ and $f \in \mathrm{Aut}(S)$:

$$c_n = \frac{\left(\varphi_{g,p} \circ f^{(n+1)}\right)(p)}{\left(\varphi_{g,p} \circ f^{(n)}\right)(p)} = \frac{\tanh^2\left(d_g\left(p, f^{(n+1)}(p)\right)\right) - 1}{\tanh^2\left(d_g\left(p, f^{(n)}(p)\right)\right) - 1} \;. \tag{3}$$

Note that $c_n = \varphi_n(f(p))$ for any n, and the numbers c_1 and c_2 are defined by distances from p to its images $f(p)$, $f^{(2)}(p)$, $f^{(3)}(p)$; so can be calculated in a finite step of iterations. A relation between c_1 and c_2 now determines a type of f.

Theorem 1.2 (Lee [9]). *For $f \in \mathrm{Aut}(S)$ without fixed point, f is parabolic or hyperbolic if and only if $8c_2 - 5c_1c_2 - 3 = 0$ or < 0, with respectively.*

In case of elliptic f and $p \in S$ with $f(p) \neq p$, we can also consider the value $8c_2 - 5c_1c_2 - 3$. But this value can be positive, negative or zero by a choice of f.

1.2. The main result

In this paper, we will give a method to intrinsically distinguish types of automorphisms even the case of elliptic automorphisms.

Let

$$a_n = (\varphi_{g,p} \circ f^{(n)})(p) = \tanh^2(d_g(p, f^{(n)}(p))) - 1 \;. \tag{4}$$

Then the number c_n in (3) can be written by

$$c_n = \frac{a_{n+1}}{a_n} \ ,$$

so the quantity $8c_2 - 5c_1c_2 - 3$ is calculated by a_1, a_2, a_3. In this paper, we shall consider only first two constants a_1 and a_2 which give a complete characterization of automorphism types as following.

Theorem 1.3. *Let S be a simply connected Riemann surface admitting a hermitian metric g of constant curvature -4. For a point $p \in S$ with $f(p) \neq p$, let $\varphi_{g,p} : S \to \mathbb{R}$ be a negative function defined as* (1) *and let*

$$\delta(f, p) = a_1 - 4a_2 - 3a_1a_2 \ .$$

where a_1 and a_2 are as in (4)*. Then f is elliptic, parabolic or hyperbolic if and only if $\delta(f, p) > 0$, $= 0$ or < 0, with respectively.*

This says that we can determine the type of $f \in \mathrm{Aut}(S)$ by the quantity $\delta(f, p)$ calculated intrinsically by distances from p to $f(p)$ and $f^{(2)}(p) = f(f(p))$. If $f(p) = p$, then f is the identity or $f(p') \neq p'$ for any $p' \neq p$; thus taking any point $p' \neq p$, we can determine whether f is trivial or elliptic.

In Section 2, we will show Theorem 1.3 for the Poincaré disc (Δ, ds^2_Δ) and some model automorphisms. Then the theorem for general Riemann surface S will be proved in Section 3.

2. Testing the type of automorphisms for the unit disc

In this section, we will prove Theorem 1.3 for $S = \Delta$ and model automorphisms of Δ for each types. In this case, the hermitian metric g of the theorem is the Poincaré metric ds^2_Δ whose distance function $d_\Delta = d_{ds^2_\Delta}$ is given by

$$d_\Delta(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - \bar{w}z} \right|$$

for $z, w \in \Delta$ (see [5, 2]). Given point $p \in \Delta$, the test function in (1) can be written by

$$\varphi_p(z) := \varphi_{ds^2_\Delta, p}(z) = \tanh^2(d_\Delta(p, z)) - 1 = \left| \frac{z - p}{1 - \bar{p}z} \right|^2 - 1 \ . \tag{5}$$

For cases of parabolic and hyperbolic automorphisms, we will consider the case of $p = 0$. Then the test function is now

$$\varphi_0(z) = \tanh^2(d_{ds^2}(0, z)) - 1 = |z|^2 - 1 \ . \tag{6}$$

In order to prove Theorem 1.3 in Section 3, we will see

$$\delta(f, p) = a_1 - 4a_2 - 3a_1a_2 \quad \text{where} \quad a_n = \tanh^2(d_\Delta(p, f^{(n)}(p)) - 1$$

for *model automorphisms* $f \in \mathrm{Aut}(\Delta)$ of the following form.

(1) an *elliptic model* f if the origin $0 \in \Delta$ is the only fixed point of f;

(2) a *parabolic model* f if the boundary point $1 \in \partial\Delta$ is the only fixed point of f;

(3) a *hyperbolic model* f if boundary points $1, -1 \in \partial\Delta$ are only fixed points of f.

2.1. Elliptic automorphisms leaving 0 fixed

Let f be an elliptic automorphism of Δ whose fixed point is the origin 0. Then the Schwarz lemma implies that

$$f(z) = \mathcal{R}_\theta(z) = e^{i\theta} z \tag{7}$$

for some $\theta \in (0, 2\pi)$. Take a real number r with $0 < r < 1$ and consider it as an interior point of Δ, i.e. $r \in \Delta$. We will see

$$a_n = \left(\varphi_r \circ f^{(n)}\right)(r) = \varphi_r\left(\mathcal{R}_{n\theta}(r)\right) \quad \text{and} \quad \delta(\mathcal{R}_\theta, r) = a_1 - 4a_2 - 3a_1a_2$$

where φ_r of (5) is written by

$$\varphi_r(z) = \varphi_{ds^2_\Delta, r}(z) = \left|\frac{z-r}{1-rz}\right|^2 - 1 = (|z|^2 - 1)\frac{1-r^2}{|1-rz|^2} = \varphi_0(z)\frac{1-r^2}{|1-rz|^2} .$$

Since

$$\varphi_0(\mathcal{R}_{n\theta}(z)) = \left|e^{in\theta}z\right|^2 - 1 = |z|^2 - 1 ,$$

we have

$$\begin{aligned} a_n = \varphi_r\left(\mathcal{R}_{n\theta}(r)\right) = \varphi_r\left(e^{in\theta}r\right) &= \frac{-(1-r^2)^2}{|1-e^{in\theta}r^2|^2} \\ &= \frac{-(1-r^2)^2}{|1-r^2\cos n\theta - ir^2\sin n\theta|^2} = \frac{-(1-r^2)^2}{1+r^4-2r^2\cos n\theta} \\ &= \frac{(1-r^2)^2}{-1-r^4+2r^2\cos n\theta} . \end{aligned}$$

Since

$$\delta(\mathcal{R}_\theta, r) = a_1 - 4a_2 - 3a_1a_2 = a_1a_2\left(\frac{1}{a_2} - \frac{4}{a_1} - 3\right)$$

and $a_1a_2 > 0$, it suffice to compute the second factor to determine the sign of $\delta(\mathcal{R}_\theta, r)$. By a straightforward computation, we have

$$\frac{1}{a_2} - \frac{4}{a_1} - 3 = \frac{-1-r^4+2r^2\cos 2\theta}{(1-r^2)^2} + \frac{4+4r^4-8r^2\cos\theta}{(1-r^2)^2} - 3$$

$$= \frac{2r^2\cos 2\theta - 8r^2\cos\theta + 6r^2}{(1-r^2)^2} = 2r^2\frac{2\cos^2\theta - 4\cos\theta + 2}{(1-r^2)^2} = \frac{4r^2(\cos\theta - 1)^2}{(1-r^2)^2} .$$

By choice of $\theta \in (0, 2\pi)$, we can conclude that

Proposition 2.1. *If f is an elliptic automorphism of Δ leaving the origin 0 fixed, then*

$$\delta(f, r) > 0$$

for any $r \in \Delta$ with $0 < r < 1$.

2.2. Parabolic automorphisms leaving 1 fixed

Let f be a parabolic automorphism of Δ leaving $1 \in \partial\Delta$ fixed. Then from Proposition 2.3 in [8], there is a unique $t \in \mathbb{R}$ such that

$$f \equiv \mathcal{P}_t$$

where

$$\mathcal{P}_t(z) = \frac{(2+it)z - it}{itz + (2-it)} \ . \tag{8}$$

For the sake of simplicity, let

$$p_t = \frac{-it}{2+it} \ .$$

Then

$$\mathcal{P}_t(0) = -\bar{p}_t = \frac{-it}{2-it} \quad \text{and} \quad \mathcal{P}_t(z) = \frac{1+\bar{p}_t}{1+p_t} \frac{z+p_t}{1+\bar{p}_t z} \ .$$

As we showed in [8], the family $\mathcal{P} = \{\mathcal{P}_t : t \in \mathbb{R}\}$ is an 1-parameter family, i.e.

$$\mathcal{P}_t \circ \mathcal{P}_s = \mathcal{P}_{t+s} \ .$$

Therefore, $f^{(n)} = \mathcal{P}_{nt}$ and

$$f^{(n)}(0) = -\bar{p}_{nt} = \frac{-int}{2-int} \ .$$

When we choose $p = 0$ as a base point for Theorem 1.3 and (5), the test function is φ_0 as in (6) and

$$a_n = \varphi_0\left(f^{(n)}(0)\right) = \varphi_0(-\bar{p}_{nt}) = \left|\frac{-int}{2-int}\right|^2 - 1 = \frac{n^2t^2}{4+n^2t^2} - 1 = \frac{-4}{4+n^2t^2} \ .$$

Thus

$$\begin{aligned} \delta(f,p) = \delta(\mathcal{P}_t, 0) &= a_1 - 4a_2 - 3a_1a_2 \\ &= -4\left(\frac{1}{4+t^2} + \frac{-4}{4+4t^2} + \frac{12}{(4+t^2)(4+4t^2)}\right) \\ &= \frac{-4}{(4+t^2)(4+4t^2)}\left(4+4t^2 - 4(4+t^2) + 12\right) = 0 \ . \end{aligned}$$

This implies

Proposition 2.2. *If f is a parabolic automorphism of Δ leaving 1 fixed, then*

$$\delta(f, 0) = \delta(\mathcal{P}_t, 0) = 0 \ .$$

Next, we will see $\delta(f,0)<0$ in the case of hyperbolic automorphism f leaving specifically 1 and -1 fixed. But this is not sufficient to prove Theorem 1.3. We have to consider more general hyperbolic case; especially f has 1 and a general point $q\in\partial\Delta\setminus\{1\}$ as its fixed points. We will use the transitivity of the action $\mathcal{P}$ on $\partial\Delta\setminus\{1\}$.

Remark 1. The image of -1 under $\mathcal{P}_t$ is

$$\mathcal{P}_t(-1)=\frac{t^2-1}{t^2+1}+i\frac{2t}{t^2+1}\ .$$

Let q be a boundary point of Δ with $q\neq 1$, i.e. $q\in\partial\Delta\setminus\{1\}$. Taking θ and t satisfying $0<\theta<2\pi$, $q=e^{i\theta}$ and $t=\cot(\theta/2)$, we can show that $\mathcal{P}_t(-1)=q$ so $\mathcal{P}_{-t}(q)=-1$ (see Remark 2.1 in [9]). Therefore the parabolic subgroup $\mathcal{P}=\{\mathcal{P}_t : t\in\mathbb{R}\}$ acts on $\partial\Delta\setminus\{1\}$ transitively.

2.3. Hyperbolic automorphisms leaving 1 and -1 fixed

For each $t\in\mathbb{R}$, we can define the holomorphic function $\mathcal{H}_t$ on Δ by

$$\mathcal{H}_t(z)=\frac{z+h_t}{1+h_t z}\quad\text{where}\quad h_t=\tanh\frac{t}{2}=\frac{e^t-1}{e^t+1}\ . \tag{9}$$

It is a hyperbolic automorphism of Δ leaving 1 and -1 fixed and $\mathcal{H}=\{\mathcal{H}_t : t\in\mathbb{R}\}$ is an 1-parameter family, i.e.

$$\mathcal{H}_t\circ\mathcal{H}_s=\mathcal{H}_{t+s}\ .$$

Conversely, given nontrivial hyperbolic automorphism f of Δ with fixed point 1 and -1, we have a unique $t\neq 0$ such that

$$f\equiv\mathcal{H}_t$$

(see Section 2 of [9]). Simultaneously,

$$f^{(n)}=\mathcal{H}_{nt}\quad\text{and}\quad f^{(n)}(0)=h_{nt}=\frac{e^{nt}-1}{e^{nt}+1}\ .$$

As the same way for the parabolic case in Section 2.2, for the base point $p=0$, we have

$$a_n=\varphi_0\left(f^{(n)}(0)\right)=\varphi_0(h_{nt})=\left(\frac{e^{nt}-1}{e^{nt}+1}\right)^2-1=\frac{-4e^{nt}}{(e^{nt}+1)^2}\ .$$

Thus

$$\begin{aligned}\delta(f,p)&=\delta(\mathcal{H}_t,0)=a_1-4a_2-3a_1a_2\\&=-4\left(\frac{e^t}{(e^t+1)^2}+\frac{-4e^{2t}}{(e^{2t}+1)^2}+\frac{12e^{3t}}{(e^t+1)^2(e^{2t}+1)^2}\right)\\&=\frac{-4}{(e^t+1)^2(e^{2t}+1)^2}\left(e^{5t}-4e^{4t}+6e^{3t}-4e^{2t}+e^t\right)=\frac{-4e^t(e^t-1)^4}{(e^t+1)^2(e^{2t}+1)^2}\ .\end{aligned}$$

Since we only consider $t \neq 0$, i.e. $e^t \neq 1$, we can conclude that

$$\delta(f, 0) = \delta(\mathcal{H}_t, 0) < 0 \ .$$

In the next section, we will deal with hyperbolic automorphism under a weak assumption to boundary fixed points.

2.4. Hyperbolic automorphisms in general

Continuing from the previous subsection, we consider the hyperbolic automorphism $f = \mathcal{H}_t$ of (9) for some nonzero t. Here we will consider a point p in Δ and the sequence of numbers

$$a_n = \left(\varphi_p \circ f^{(n)}\right)(p) = \varphi_p\left(\mathcal{H}_{nt}(p)\right) \tag{10}$$

where

$$\varphi_p(z) = \varphi_{ds^2_\Delta, p}(z) = \left|\frac{z-p}{1-\bar{p}z}\right|^2 - 1 = (|z|^2-1)\frac{1-|p|^2}{|1-\bar{p}z|^2} = \varphi_0(z)\frac{1-|p|^2}{|1-\bar{p}z|^2} \ .$$

Since

$$(\varphi_0 \circ \mathcal{H}_t)(z) = \frac{(|z|^2-1)(1-h_t^2)}{|1+h_t z|^2} \ ,$$

it follows that

$$a_n = \frac{(|p|^2-1)(1-h_{nt}^2)}{|1+h_{nt}p|^2}\frac{1-|p|^2}{|1-\bar{p}\mathcal{H}_{nt}(p)|^2} = \frac{(h_{nt}^2-1)(1-|p|^2)^2}{|1+h_{nt}p|^2\,|1-\bar{p}\mathcal{H}_{nt}(p)|^2} \ .$$

We will calculate the value a_n of (10) for the point p specifically given by

$$p = \frac{-is}{2-is} \quad (s \in \mathbb{R}) \tag{11}$$

which belongs to Δ. Let us consider

$$1 - \bar{p}\mathcal{H}_{nt}(p) = 1 - \frac{is}{2+is}\frac{p+h_{nt}}{1+h_{nt}p} = \frac{(2+is)(1+h_{nt}p) - is(p+h_{nt})}{(2+is)(1+h_{nt}p)} \ .$$

Applying (11) to the numerator $(2+is)(1+h_{nt}p) - is(p+h_{nt})$ only, we can get

$$1 - \bar{p}\mathcal{H}_{nt}(p) = \frac{1}{(2+is)(1+h_{nt}p)}\frac{4(1-ish_{nt})}{2-is} = \frac{4(1-ish_{nt})}{(4+s^2)(1+h_{nt}p)} \ .$$

Now a_n is written by

$$\begin{aligned} a_n &= \frac{(h_{nt}^2-1)(1-|p|^2)^2}{|1+h_{nt}p|^2}\frac{(4+s^2)^2\,|1+h_{nt}p|^2}{16\,|1-ish_{nt}|^2} \\ &= \frac{(4+s^2)^2(1-|p|^2)^2}{16}\frac{(h_{nt}^2-1)}{1+s^2h_{nt}^2} \ . \end{aligned}$$

Since $|p|^2 = s^2/(4+s^2)$ from (11), the first factor above is simply written by

$$\frac{(4+s^2)^2(1-|p|^2)^2}{16} = \frac{(4+s^2)^2}{16}\left(1-\frac{s^2}{4+s^2}\right)^2 = 1\ .$$

Let us compute $\delta(\mathcal{H}_t, p)$:

$$\begin{aligned} a_1 - 4a_2 - 3a_1a_2 &= \frac{(h_t^2-1)}{1+s^2h_t^2} - 4\frac{(h_{2t}^2-1)}{1+s^2h_{2t}^2} - 3\frac{(h_t^2-1)}{1+s^2h_t^2}\frac{(h_{2t}^2-1)}{1+s^2h_{2t}^2} \\ &= \frac{(1+s^2)(4h_t+h_{2t})(h_t-h_{2t})}{(1+s^2h_t^2)(1+s^2h_{2t}^2)}\ . \end{aligned}$$

Applying $h_{2t} = \tanh(t) = 2\tanh(t/2)/(1+\tanh^2(t/2)) = 2h_t/(1+h_t^2)$, it follows that

$$\begin{aligned} a_1 - 4a_2 - 3a_1a_2 &= \frac{(1+s^2)}{(1+s^2h_t^2)(1+s^2h_{2t}^2)}\left(4h_t + \frac{2h_t}{1+h_t^2}\right)\left(h_t - \frac{2h_t}{1+h_t^2}\right) \\ &= \frac{(1+s^2)h_t^2(4h_t^2+6)(h_t^2-1)}{(1+s^2h_t^2)(1+s^2h_{2t}^2)(1+h_t^2)^2}\ . \end{aligned}$$

The only negative factor of the last term is $h_t^2 - 1 = \tanh^2(t/2) - 1 < 0$. Thus we have

$$\delta(\mathcal{H}_t, p) = a_1 - 4a_2 - 3a_1a_2 < 0 \tag{12}$$

for (10) and (11). Using this, we can give the following conclusion.

Proposition 2.3. *If f is a hyperbolic automorphism of Δ leaving 1 fixed, then*

$$\delta(f, 0) < 0\ .$$

Proof. Let f be a hyperbolic automorphism leaving 1 fixed and let $q \in \partial\Delta \setminus \{1\}$ be another fixed point of f. As mentioned in Remark 1, we have a suitable s such that the parabolic model $\mathcal{P}_s$ in (8) satisfies

$$\mathcal{P}_s(q) = -1\ .$$

Since $\mathcal{P}_s$ always leaves 1 fixed, the automorphism

$$\tilde{f} = \mathcal{P}_s \circ f \circ \mathcal{P}_s^{-1} = \mathcal{P}_s \circ f \circ \mathcal{P}_{-s}$$

satisfies $\tilde{f}(1) = 1$ and $\tilde{f}(-1) = -1$. Thus $\tilde{f}$ is a hyperbolic automorphism leaving both 1 and -1 fixed, so $\tilde{f} = \mathcal{H}_t$ for some nonzero t. Moreover iterations of $\tilde{f}$ is simply written by

$$\tilde{f}^{(n)} = \mathcal{H}_{nt} = (\mathcal{P}_s \circ f \circ \mathcal{P}_{-s})^{(n)} = \mathcal{P}_s \circ f^{(n)} \circ \mathcal{P}_{-s}$$

for any positive integer n. As we showed in Section 2.2, the image $p = \mathcal{P}_s(0)$ is

$$p = \mathcal{P}_s(0) = \frac{1+\bar{p}_s}{1+p_s}p_s = \frac{-is}{2-is}$$

which coincides with (11). Note that $0 = \mathcal{P}_{-s}(p)$ so

$$f^{(n)}(0) = f^{(n)}(\mathcal{P}_{-s}(p)) = (f^{(n)} \circ \mathcal{P}_{-s})(p)$$

by the definition of p. Since $\mathcal{P}_s$ is an isometry of the Poincaré metric, we get

$$\begin{aligned} d_\Delta(0, f^{(n)}(0)) &= d_\Delta\left(\mathcal{P}_{-s}(p), (f^{(n)} \circ \mathcal{P}_{-s})(p)\right) \\ &= d_\Delta\left((\mathcal{P}_s \circ \mathcal{P}_{-s})(p), (\mathcal{P}_s \circ f^{(n)} \circ \mathcal{P}_{-s})(p)\right) \\ &= d_\Delta(p, \mathcal{H}_{nt}(p)) = d_\Delta(p, \tilde{f}^{(n)}(p)) . \end{aligned}$$

This means that $\delta(f, 0) = \delta(\mathcal{H}_t, p) < 0$ from (12), so it completes the proof. □

3. Proof of Theorem 1.3

Let S be a simply connected Riemann surface with a complete hermitian metric g of constant curvature -4. Then there is a holomorphic isometry

$$F : (\Delta, ds^2_\Delta) \to (S, g) \tag{13}$$

by the uniformization theorem ([10, 6]) and the Schwarz-Ahlfors lemma ([1])

Let f be an automorphism of S and let $p \in S$ be a some point of S with $f(p) \neq p$. Then we have the pulling-back automorphism $\tilde{f} = F^* f$ of Δ given by

$$\tilde{f} = F^* f = F^{-1} \circ f \circ F .$$

Let $\tilde{p} = F^{-1}(p) \in \Delta$. Then $F(\tilde{p}) = p$ and constants a_n in (4) corresponding to pairs (f, p) and $(\tilde{f}, \tilde{p})$ coincide with each others since

$$\begin{aligned} d_g\left(p, f^{(n)}(p)\right) &= d_g\left(F(\tilde{p}), f^{(n)}(F(\tilde{p}))\right) \\ &= d_\Delta\left(F^{-1}(F(\tilde{p})), F^{-1}(f^{(n)}(F(\tilde{p})))\right) = d_\Delta\left(\tilde{p}, \tilde{f}^{(n)}(\tilde{p})\right) , \end{aligned}$$

where $d_\Delta = d_{ds^2_\Delta}$ is the distance of (Δ, ds^2_Δ). This implies that

$$\delta(f, p) = \delta(\tilde{f}, \tilde{p}) .$$

As we mentioned in Section 1.1, the type of f is the same as that of $\tilde{f}$ and $\delta(\tilde{f}, \tilde{p})$ is not changed any choice of F. Therefore it suffices to determine sign of $\delta(\tilde{f}, \tilde{p})$ taking suitable F (so $\tilde{f}$).

Suppose that f is hyperbolic or parabolic, then $\tilde{f}$ has no fixed point inside Δ. Since Δ is homogeneous, we may assume that

$$\tilde{p} = F^{-1}(p) = 0 .$$

Since $\tilde{f} \in \mathrm{Aut}(\Delta)$ is extended to $\overline{\Delta}$, it has at least one fixed point on the boundary $\partial\Delta$ by the assumption. We can also assume that the extension of $\tilde{f}$ to $\overline{\Delta}$ leaves the point 1 fixed since $\mathcal{R} = \{\mathcal{R}_\theta : \theta \in \mathbb{R}\}$ as in (7) acts on $\partial\Delta$ transitively. Now $\tilde{f}$ is a nontrivial automorphism with

$$\tilde{f}(1) = 1 .$$

By Proposition 2.2 and Proposition 2.3, we can conclude that $\tilde{f}$ is parabolic or hyperbolic if and only if $\delta(\tilde{f}, 0) = 0$ or < 0, with respectively.

Suppose that f is elliptic. Then there is a point $p_0 \in S$ satisfying that $p_0 \neq p$ and $f(p_0) = p_0$. Using the homogeneity of Δ, we can choose a biholomorphism F in (13) with $0 = F^{-1}(p_0) \in \Delta$. By the rotation action $\mathcal{R}$ on Δ, we also assume that $\tilde{p} = F^{-1}(p)$ lies on the positive real line in Δ, i.e. $\tilde{p} = r$ with $0 < r < 1$.

Now $\tilde{f}$ is an elliptic automorphism of Δ with $\tilde{f}(0) = 0$, so

$$\tilde{f} = \mathcal{R}_\theta$$

for some $\theta \neq 2n\pi$. Moreover $\delta(\tilde{f}, \tilde{p}) = \delta(\mathcal{R}_\theta, r)$. Proposition 2.1 implies that $\delta(\tilde{f}, \tilde{p}) > 0$. This completes the proof. □

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