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# SOME RESULTS RELATED TO NON-DEGENERATE LINEAR TRANSFORMATIONS ON EUCLIDEAN JORDAN ALGEBRAS

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Abstract. This article deals with non-degenerate linear transformations on Euclidean Jordan algebras. First, we study non-degenerate for cone invariant, copositive, Lyapunov-like, and relaxation transformations. Further, we study that the non-degenerate is invariant under principal pivotal transformations and algebraic automorphisms.

## 1. Introduction

A matrix $A \in \mathbb{R}^{n\times n}$ is non-degenerate if every principal minors of $A$ is nonzero. It is equivalent to say that $u * (Au) = 0$ implies $u = 0$ (see [8]). The class of non-degenerate matrix has been well studied in literature due to wide applications in linear complementarity problems (see, for example [7]). From given element $q \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n\times n}$, the problem is to find an element $u \in \mathbb{R}^n$ such that

$$u \geq 0,\ v = Au + q \geq 0,\ \langle u, v\rangle = 0.$$

This problem is called the standard linear complementarity problem, LCP$(A, q)$. The solution set in the standard linear complementarity problem [1] is finite if and only if the corresponding matrix is non-degenerate. Different types of matrices have been studied for the existence and uniqueness of solutions of the linear complementarity problem in the literature (see [1]). In the literature linear complementarity problem associated with Euclidean Jordan algebra over a symmetric cone has been studied for the past fifteen years (see [3, 5, 6]).

In this article, we continue our study in the circumstance of Euclidean Jordan algebras about the non-degenerate linear transformations. We focus on copositive transformations, Lyapunav-like transformations, automorphism invariance and relaxation transformations in particular. Especially, our contributions of this paper include:

1. Showing that strict copositivity, $R_0$ property and cone non-degenerate transformations are equivalent for cone invariant transformations.
2. Proving that cone non-degenerate transformations are equivalent to non-degenerate transformations for Lyapunov-like transformation.

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3. Studying that the non-degenerate for some special linear transformations.

A summary of the paper is provided here. Section 2 provides a quick overview of Euclidean Jordan algebras. Section 3 examines the cone non-degenerate for copositive transformations, cone invariant transformations, and the Lyapunov-like transformation. In section 4, we discuss some special linear transformation about non-degenerate.

## 2. Preliminaries

We review a few Euclidean Jordan algebraic notions, properties, and results within this section. Most of these can be found in [3, 4, 6].

Definition 2.1. A Finite dimensional inner product space $(V, \langle .,. \rangle)$ over $\mathbb{R}$ is said to be a Euclidean Jordan algebra if there is a bilinear map from the cartisian product $V \times V$ into $V$ represented by $(u, v) \longmapsto u \circ v$ such that it satisfies the requirements given below:

(i) $u \circ v = v \circ u$ for all $u, v \in V$.
(ii) $u \circ (u^2 \circ v) = u^2 \circ (u \circ v)$ for all $u, v \in V$.
(iii) $\langle u \circ v, w \rangle = \langle v, u \circ w \rangle$ for all $u, v, w \in V$.

Remark 2.2. Suppose, if there exist an element $e \in V$ such that $u \circ e = u$ for all $u \in V$, then we say that $e$ is an unit element of $V$. In the above definition, the product $u \circ v$ is referred to as Jordan product in $V$. In $V$, the set of squares $K := \{u \circ u : u \in V\}$ is a symmetric cone [4]. Also, for any element $u \in V$, we write $u \geq 0$ if and only if $u \in K$. For any element $u \in V$, we write $u = u^+ - u^-$, $u^+, u^- \geq 0$ and $u^+ \circ u^- = 0$.

Definition 2.3. An element $d \in V$ is said to an *idempotent* if $d^2 = d$. We say that a nonzero element $d \in V$ is a *primitive idempotent* if $d$ cannot be represented as the sum of two nonzero idempotents.

Remark 2.4. (i) A *Jordan frame* in $V$ is a collection $\{e_1, e_2, \ldots, e_s\}$ of primitive idempotents having the following properties

$$e_k \circ e_l = 0 \text{ for } k \neq l, \text{ and } \sum_{k=1}^{s} e_k = e.$$

(ii) The Spectral Decomposition (refer to [4]): For any $u \in V$, we can find a real numbers $\lambda_1, \ldots, \lambda_s$ and $\{e_1, e_2, \ldots, e_s\}$ as a Jordan frame in $V$ such that

$$u = \lambda_1 e_1 + \ldots + \lambda_s e_s.$$

In this case the representation $\lambda_1 e_1 + \ldots + \lambda_s e_s$ is called the spectral decomposition of $u$, where $\lambda_i \in \mathbb{R}$ is an eigenvalue of $u$. Throughout this paper, we fix a Jordan frame $\{e_1, e_2, \ldots, e_s\}$ otherwise stated.

(iii) For any $u \in V$, we can define a linear map $L_u : V \longrightarrow V$ as $L_u(w) = u \circ w$. If $L_u L_v = L_v L_u$, then we say $u$ and $v$ operator commute. It is understood that $u$ operator commute with $v$ if and only if $u$ and $v$ have their spectral decompositions with respect to a common Jordan frame ( [4], Lemma X.2.2).

Some standard examples are listed below.

EXAMPLE 2.5. Consider $\mathbb{R}^n$ with the Jordan product and usual inner product respectively which are represented as follows:

$$u \circ v = u * v \text{ and } \langle u, v \rangle = \sum_{i=1}^{n} u_i v_i,$$

where $u * v$ denotes the component wise product of $u$ and $v$. Then $\mathbb{R}^n$ is a Euclidean Jordan algebra with symmetric cone is $\mathbb{R}^n_+$, the nonnegative orthant.

EXAMPLE 2.6. Let $S^n = \{U \in \mathbb{R}^{n\times n} \mid U = U^T\}$. Then, the Euclidean Jordan algebra is the set $S^n$ associated with the Jordan product and inner product respectively given by

$$U \circ V := \frac{1}{2}(UV + VU) \text{ and } \langle U, V \rangle := trace(UV).$$

Moreover, the related symmetric cone is $S^n_+$, the set of all $n \times n$ positive semi-definite matrices.

Now, we recall the proposition from [3] as follows:

PROPOSITION 2.7. *Consider for any $u, v \in V$. Then, the following conditions are equivalent:*

(i) $u \geq 0, v \geq 0$, *and* $\langle u, v \rangle = 0$
(ii) $u \geq 0, v \geq 0$, *and* $u \circ v = 0$.

*In any case, $u$ and $v$ operator commute.*

Throughout this article, we always take $V$ to be a Euclidean Jordan algebra and $K$ to be the associated symmetric cone. Now, we recall some definitions as follows.

DEFINITION 2.8. Let $L : V \to V$ be a linear transformation. We say that $L$ is/has

(i) Strictly copositive on $K$ if $\langle u, L(u) \rangle > 0$ for all $0 \neq u \in K$.
(ii) Copositive on $K$ if $\langle u, L(u) \rangle \geq 0$ for all $u \in K$.
(iii) Non-degenerate if $u \in V$

$$u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) = 0 \Longrightarrow u = 0.$$

(iv) Cone non-degenerate if

$$u \in K,\ u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) = 0 \Longrightarrow u = 0.$$

(v) The strictly semi-monotone(SSM) property if

$$u \in K,\ u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) \leq 0 \Longrightarrow u = 0.$$

(vi) The $R_0$ property if

$$u \in K,\ L(u) \in K \text{ and } u \circ L(u) = 0 \Longrightarrow u = 0.$$

(vii) The globally unique solvable(GUS) property on $K$ if LCP$(L, q)$ has only one solution for all $q \in K$;
(viii) Lyapunov-like transformation if

$$u, v \in K, \text{ and } \langle u, v \rangle = 0 \Longrightarrow \langle L(u), v \rangle = 0.$$

(ix) Cone invariant if $L(K) \subseteq K$.

## 3. Main Results

In this section, we study a non-degenerate transformations on Euclidean Jordan Algebras. This section is divided into two parts. The first one deals with non-degenerate transformations for cone invariant and copositive linear transformations, and the other one deals with non-degenerate on Lyapunov-like transformation; both are related to the Euclidean Jordan algebras.

**3.1. Non-degenerate transformations for cone invariant and copositive linear transformations:** In this subsection, we first characterize the cone non-degenerate transformations for cone invariant transformations.

THEOREM 3.1. *Consider a linear transformation $L : V \to V$ which is cone invariant. Then the following three statements are equivalent:*

(i) *$L$ is cone non-degenerate.*
(ii) *$L$ is strictly copositive on $K$.*
(iii) *$L$ has the $R_0$ property.*

*Proof.* (i)$\Rightarrow$(ii): Suppose $L$ is not strictly copositive on $K$, then we can find a nonzero element $u \in K$ such that $\langle u, L(u)\rangle \leq 0$. However, since $u \geq 0$, $L(u) \geq 0$, we can rule out $\langle u, L(u)\rangle < 0$. By proposition 1, $u \circ L(u) = 0$ and $u$ and $L(u)$ operator commute. By item (i) $u = 0$, which is not possible.

(ii)$\Rightarrow$(i): Let $u \in K$ such that $u$ operator commutes with $L(u)$ and $u \circ L(u) = 0$. Then the Jordan frame will exist as $\{e_1, e_2, ....., e_s\}$ such that

$$u = \sum_{i=1}^{s} u_i e_i \text{ and } L(u) = \sum_{i=1}^{s} v_i e_i.$$

From $u \circ L(u) = 0$, we have $\sum_{i=1}^{s} u_i v_i e_i = 0$. This implies that $u_i v_i = 0$ for all $i$. Now it is enough to prove that $u = 0$. If $u \neq 0$, then $\langle u, L(u)\rangle > 0$ as $L$ is strictly copositive on $K$. This indicates that

$$0 < \langle u, L(u)\rangle = \sum_{i=1}^{s} u_i v_i \|e_i\|^2 = 0$$

which is not possible. Therefore $u = 0$.

(ii)$\Longleftrightarrow$(iii): It follows from proposition 3.1, [9]. □

The following theorem provides that the SSM property is equivalent to the GUS on $K$ and cone non-degenerate for copositive transformation.

THEOREM 3.2. *Let $L : V \to V$ be a linear transformation that is copositive on $K$. Then the following are equivalent:*

(i) *$L$ is cone non-degenerate.*
(ii) *$L$ has the SSM property.*
(iii) *$L$ has the GUS property on $K$.*

*Proof.* (i)⇒(ii): Let us take an element $u \geq 0$ such that $u \circ L(u) \leq 0$ and $u$ operator commutes with $L(u)$. Since $u$ and $L(u)$ operator commute and $u \geq 0$, $L(u) \geq 0$, we have $u \circ L(u) \geq 0$. Hence, $u \circ L(u) = 0$. By item(i), $u = 0$.
(ii)⇒(i): It is obvious.
(i)⇒(iii): Assume that $L$ is cone non-degenerate. Now we want to show that 0 is the only solution to LCP$(L, q)$ for any $q \geq 0$. Suppose $u \in K$ is a solution of LCP$(L, q)$. Then

$$u \geq 0, \;\; L(u) + q \geq 0 \text{ and } \langle u, L(u) + q\rangle = 0.$$

Therefore, by Proposition 1, $u$ operator commutes with $L(u)+q$ and $u\circ(L(u)+q) = 0$. $\langle u, L(u) + q\rangle = 0$ implies that $\langle u, L(u)\rangle + \langle u, q\rangle = 0$. From copositivity of $L$ on $K$, we have $\langle u, q\rangle = 0$ and $\langle u, L(u)\rangle = 0$. Then $u$ operator commutes with $q$ and $u \circ q = 0$ by Proposition 1. This implies that $u \circ L(u) = 0$. Hence $u = 0$ as $L$ is non-degenerate on $K$. This arguments says that LCP$(L, q)$ has the unique solution for all $q \in K$.
(iii)⇒(ii): It is follows from [6]. □

**3.2. Non-degenerate on Lyapunov-like transformation.** Matrix $A \in \mathbb{R}^{n\times n}$ is non-degenerate, then it is invertible. But, the invertible matrix need not be non-degenerate. See the following example:

Example 3.3. Consider a matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$. Det$(A) = -2$. It is invertible. But the principle minors of $A$ are $A_{11} = -3, A_{22} = 0, A_{33} = 1$. Hence $A$ is not non-degenerate matrix.

The following result tells us that for a Lyapunov-like transformation, the invertible transformation is equivalent to the non-degenerate linear transformation.

Theorem 3.4. *A Lyapunov-like transformations $L : V \to V$ is non-degenerate on $V$ if and only if $L$ is invertible .*

*Proof.* It is sufficient to show the reverse part. Suppose that $L$ is invertible. Let us consider an element $u \in V$ such that $u\circ L(u) = 0$ and $u$ operator commutes with $L(u)$. Then we can find a Jordan frame $\{e_1, e_2, \ldots, e_s\}$ such that $u = \sum_{i=1}^{s} u_i e_i$ and $L(u) = \sum_{i=1}^{s} v_i e_i$. This implies that $\sum_{i=1}^{s} u_i L(e_i) = \sum_{i=1}^{s} v_i e_i$. Taking inner product with $e_j$ on both sides, we get $u_j\langle e_j, L(e_j)\rangle = v_j\|e_j\|^2$. Since $u_j v_j = 0$, we have if $u_j \neq 0$, then $v_j = 0$ and if $u_j = 0$, then $v_j\|e_j\|^2 = 0$ implying $v_j = 0$ for all $j$. In any case, $L(u) = 0 \Rightarrow u = 0$ as $L$ is invertible. □

The relationship between cone non-degenerate and the non-degenerate transformations for cone invariant transformation is shown by the results below.

Theorem 3.5. *Let $L : V \to V$ be a Lyapunov-like transformation that is cone invariant. Then the following two statement are equivalent.*

*(i) $L$ is non-degenerate on $K$.*
*(ii) $L$ is non-degenerate on $V$.*

*Proof.* (i) $\Longrightarrow$ (ii). Assume that $L$ is non-degenerate on $K$. Let $u \circ L(u) = 0$ and $u$ operator commute with $L(u)$. Then the Jordan frame $\{e_1, e_2, ....., e_s\}$ will exist such that

$$u = \sum_{i=1}^{s} u_i e_i \text{ and } L(u) = \sum_{i=1}^{s} v_i e_i.$$

Since $u \circ L(u) = 0$, we have $u_i v_i = 0$ for all $i$.
We have

$$\langle u, L(u) \rangle = \sum_{i=1}^{s} u_i v_i \|e_i\|^2 = 0.$$

We know that

$$u = u^+ - u^- \text{ and } L(u) = L(u^+) - L(u^-),$$

which implies that

$$\langle u, L(u) \rangle = \langle u^+ - u^-, L(u^+) - L(u^-) \rangle = 0.$$

This gives that

$$\langle u^+, L(u^+) \rangle + \langle u^-, L(u^-) \rangle - \langle u^+, L(u^-) \rangle - \langle u^-, L(u^+) \rangle = 0.$$

Since $L$ is Lyapunov -like, we have $\langle u^-, L(u^+) \rangle = 0$ and $\langle u^+, L(u^-) \rangle = 0$ which imply $\langle u^+, L(u^+) \rangle = 0$ and $\langle u^-, L(u^-) \rangle = 0$ as $L$ is also cone invariant.

Therefore $u^+ \circ L(u^+) = 0$ and $u^- \circ L(u^-) = 0$. Since $u^+ \geq 0, L(u^+) \geq 0$ and $\langle u^+, L(u^+) \rangle = 0$, $u^+$ and $L(u^+)$ operator commute by Proposition 1. Similarly, $u^- \geq 0, L(u^-) \geq 0$ and $\langle u^-, L(u^-) \rangle = 0$, $u^-$ and $L(u^-)$ operator commute. Hence $u^+ = 0$ and $u^- = 0$ by $L$ is non-degenerate on $K$. This implies that $u = 0$.
(ii) $\Longrightarrow$ (i) is obvious. □

## 4. The non-degenerate on some special transformations

In this segment, we prove some results related to the relaxation transformation and automorphism invariance.

**4.1. The relaxation transformation.** In this subsection, we can define the relaxation transformation $R_D : V \to V$ in a following way. Then for any $u \in V$, the decomposition of Peirce with the Jordan frame $\{e_1, e_2, \ldots, e_s\}$ in $V$ as,

$$u = \sum_{i=1}^{s} u_i e_i + \sum_{i<j} u_{ij}.$$

Then

$$R_D(u) = \sum_{i=1}^{s} v_i e_i + \sum_{i<j} u_{ij},$$

where

$$[v_1, v_2, \ldots, v_s]^T = D[u_1, u_2, \ldots, u_s]^T.$$

The generalized version of this transformation has been established by Gowda and Tao [12]. Gowda and Tao [6, 12] have investigated several relationships between the characteristics of $D$ and $R_D$. We now characterize the non-degeneracy of $R_D$ in terms of the matrix $D$.

THEOREM 4.1. *If $R_D$ is non-degenerate on $V$, then $D$ is non-degenerate matrix.*

*Proof.* Let $u * Du = 0$, where $u^T = [u_1, u_2, \ldots, u_s]$. Let $v = Du = [v_1, v_2, \ldots, v_s]^T$. We claim that $u = 0$. We define $w = \sum_{i=1}^{s} u_i e_i$. Then

$$R_D(w) = \sum_{i=1}^{s} v_i e_i \text{ and } w \circ R_D(w) = \sum_{i=1}^{s} u_i v_i e_i = 0.$$

Note that $w$ and $R_D(w)$ operator commute. Since $R_D(w)$ is non-degenerate, this implies that $w = 0$ and hence $u = 0$. Therefore, $D$ is a non-degenerate matrix. □

By using the following example, the converse of the above theorem need not be true.

EXAMPLE 4.2. Let $V = S^2$ be the set of all $2 \times 2$ real symmetric matrics. Consider elements $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
It is easy to show that $D$ is non-degenerate.
Now we define $R_D$ with respect to the Jordan frame $\{E_1, E_2\}$ as

$$R_D(U) = -aE_1 + (-c)E_2 + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -c \end{bmatrix}$$

Where

$$U = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = aE_1 + cE_2 + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}.$$

And

$$U \circ R_D(U) = \frac{1}{2}[UR_D(U) + R_D(U)U].$$

For $W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} > 0$, $W \circ R_D(W) = 0$ and $W$ operator commutes with $R_D(W)$
Hence $R_D(W)$ is not non-degenerate.

### 4.2. Automorphism Invariance.

DEFINITION 4.3. Let $\Lambda : V \to V$ be an invertible linear transformation. $\Lambda$ is referred to as an algebra automorphism if $\Lambda(u \circ v) = \Lambda(u) \circ \Lambda(v)$ for all $u, v \in V$. Here $Aut(V)$-set of all automorphisms of $V$.

We define transformations $\tilde{L}$ by $\tilde{L} := \Lambda^T L \Lambda$. We say that a property $P$ is invariant under the automorphisms of the algebra if $\tilde{L}$ has property $P$ whenever $L$ has property $P$.

The non-degenerate in Euclidean Jordan algebras is shown in this subsection to be invariant under algebra automorphism. We recall the following result from Proposition 4.2 in [5], which will be used in sequel.

LEMMA 4.4. *Let $\{e_1, e_2, \ldots, e_s\}$ be a Jordan frame in $V$ and $\Lambda \in Aut(V)$. Then $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_s)\}$ is a Jordan frame in $V$ and there exist positive numbers $\theta_1$, $\theta_2$,…,$\theta_s$ such that $(\Lambda^T)^{-1}(e_i) = \theta_i \Lambda(e_i)$ for all $i$.*

Theorem 4.5. *Non-degenerate linear transformations are invariant under algebra automorphism of $V$.*

*Proof.* Let $\Lambda \in Aut(V)$. Suppose $L$ is non-degenerate on $V$. It is enough to show that $\Lambda^T L\Lambda$ is non-degenerate. Let $u$ operator commute with $\Lambda^T L\Lambda(u)$ and $u \circ \Lambda^T L\Lambda(u) = 0$. Then there exist a Jordan frame $\{e_1, e_2, \ldots, e_s\}$ such that

$$u = \sum_{i=1}^{s} u_i e_i \text{ and } \Lambda^T L\Lambda = \sum_{i=1}^{s} v_i e_i.$$

Since $u \circ \Lambda^T L\Lambda(u) = 0$, we have $u_i v_i = 0$. Then we can find positive number $\theta_i$ and a Jordan frame $\{\Lambda(e_1), \Lambda(e_2), \ldots, \Lambda(e_s)\}$ in $V$, such that $(\Lambda^T)^{-1}(e_i) = \theta_i \Lambda(e_i)$ for all $i$ by the above lemma. Thus we have $\Lambda(u) = \sum\limits_{i=1}^{s} u_i \Lambda(e_i)$ and $L(\Lambda(u)) = \sum\limits_{i=1}^{s} v_i \theta_i \Lambda(e_i)$. This means that $\Lambda(u)$ and $L(\Lambda(u))$ operator commute. Since $u_i v_i = 0$ for all $i$, then $\Lambda(u) \circ L(\Lambda(u)) = \sum\limits_{i=1}^{s} u_i v_i \theta_i \Lambda(e_i) = 0$. This implies $\Lambda(u) = 0$ as $L$ is non-degenerate. This implies $u = 0$ as $\Lambda$ is invertible. Hence $\Lambda^T L\Lambda$ is non-degenerate on $V$. □

**4.3. The invariance of principal pivotal transformation.** Let $V_1$ and $V_2$ are two Euclidean Jordan algebras. Then their cartesian product $V = V_1 \times V_2$ is also a Euclidean Jordan algebras. Now let us think about a linear transformation $L$ from $V$ to itself such that $L$ to be expressed in a block form uniquely as

$$L = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

where each entry acts as a linear operator in the following order:

$$P : V_1 \to V_1,\ Q : V_2 \to V_1,\ R : V_1 \to V_2,\ S : V_2 \to V_2.$$

Assuming $P$ is invertible, we can define the principal pivotal transformation [13] of $L$ as

$$L^* = \begin{bmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & L/P \end{bmatrix}.$$

where $L/P = S - RP^{-1}Q$ is the Schur complement of $P$ in $L$.

Note that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}.$$

Theorem 4.6. *$L^*$ holds the non-degenerate whenever $L$ possess the non-degenerate on $V$.*

*Proof.* Suppose

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

operator commute and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \circ L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$

Then $u_i$ operator commutes with $v_i$ for $i = 1, 2$. This implies that

$$\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \text{and} \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}$$

operator commute and

$$\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \circ L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = 0.$$

Since $L$ is non-degenerate, therefore $\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
From $v_1 = 0, u_2 = 0$, we get

$$\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = L \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This indicates that $\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore $L^*$ is non-degenerate. □

## 5. Conclusion

In this paper, we first characterized the cone non-degenerate of co-positive and cone invariant transformations. We proved that non-degenerate transformations of a cone invariant Lyapunav-like transformation coincides with the cone non-degenerate. In addition, we showed that non-degenerate under relaxation transformation. Finally we proved that the non-degenerate is invariant under principal pivotal transformation.

## 6. Author Contribution Statements

For making this article, all the author's contributed equally.

## 7. Competing Interests

All the author's said that they have no competing interests.

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