

## ON BETA PRODUCT OF HESITANCY FUZZY GRAPHS AND INTUITIONISTIC HESITANCY FUZZY GRAPHS

SUNIL.M.P AND J. SURESH KUMAR\*

ABSTRACT. The degree of hesitancy of a vertex in a hesitancy fuzzy graph depends on the degree of membership and non-membership of the vertex. We define a new class of hesitancy fuzzy graph, the intuitionistic hesitancy fuzzy graph in which the degree of hesitancy of a vertex is independent of the degree of its membership and non-membership. We introduce the idea of  $\beta$ -product of a pair of hesitancy fuzzy graphs and intuitionistic hesitancy fuzzy graphs and prove certain results based on this product.

### 1. Introduction

Zadeh [12] put forth the idea of fuzzy set and the concept of fuzzy set brought about revolutionary changes in the area of interdisciplinary research. As Euler pioneered the concept of graph theory, Rosenfeld [8] developed fuzzy graph (FG) theory in 1975. Another innovative study on fuzzy set was made by Atanassov [1] who introduced intuitionistic fuzzy sets. Next breakthrough came when R.Parvathi [5] developed intuitionistic fuzzy graph (IFG) and it was followed by T.Pathinathan [6] who developed the concept of hesitancy fuzzy graph (HFG). Later on, Ch.Chaitanya and T.V. Pradeep Kumar [2] introduced the idea of complete product of FGs. Many perspectives on hesitancy fuzzy sets and HFGs are discussed in [3, 4, 7, 9–11].

We define  $\beta$ -product of a pair of HFGs. In an HFG, the degree of hesitancy ( $\rho_1$ ) of a vertex depends on the degree of membership (MS)  $\lambda_1$  and non-membership (NMS)  $\delta_1$  of the vertex. An HFG is strong if it is  $\lambda$ -strong,  $\delta$ -strong and  $\rho$ -strong. We establish that  $\beta$ -product of a pair of strong HFGs need not be a strong HFG because  $\beta$ -product of a pair of strong HFGs need not be  $\rho$ -strong. We introduce a new class of HFG, the intuitionistic HFG (IHFG) in which  $\rho_1$  is independent of  $\lambda_1$  and  $\delta_1$  and prove that  $\beta$ -product of a pair of strong IHFGs is a strong IHFG. For two complete IHFGs, their  $\beta$ -product is also a complete IHFG. If the  $\beta$ -product of a pair of IHFGs is strong, then at least one of the IHFG will be strong.

---

Received April 10, 2023. Revised October 10, 2023. Accepted October 12, 2023.

2010 Mathematics Subject Classification: 05C72, 05C76.

Key words and phrases: Hesitancy fuzzy graph, Intuitionistic hesitancy fuzzy graph, Strong intuitionistic hesitancy fuzzy graph, Complete intuitionistic hesitancy fuzzy graph,  $\beta$ -product.

\* Corresponding author.

© The Kangwon-Kyungki Mathematical Society, 2023.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

## 2. Preliminaries

DEFINITION 2.1. [5] An IFG is  $G = (V, E, \sigma, \mu)$ ,  $V$  is the vertex set,  $\sigma = (\lambda_1, \delta_1)$ ,  $\mu = (\lambda_2, \delta_2)$  and  $\lambda_1, \delta_1 : V \rightarrow [0, 1]$  represent the degree of MS, NMS of  $v \in V$ ,

$$0 \leq \lambda_1(v) + \delta_1(v) \leq 1.$$

$\lambda_2, \delta_2 : V \times V \rightarrow [0, 1]$  represent the degree of MS, NMS of the edge  $x = (u, v) \in V \times V$ ,

$$\lambda_2(x) \leq \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) \leq \max\{\delta_1(u), \delta_1(v)\}$$

$$0 \leq \lambda_2(x) + \delta_2(x) \leq 1, \forall x \in V \times V$$

DEFINITION 2.2. [6] An HFG is  $G = (V, E, \sigma, \mu)$ ,  $V$  is the vertex set,  $\sigma = (\lambda_1, \delta_1, \rho_1)$ ,  $\mu = (\lambda_2, \delta_2, \rho_2)$  and  $\lambda_1, \delta_1, \rho_1 : V \rightarrow [0, 1]$  represent the degree of MS, NMS and hesitancy of  $v \in V$ ,

$$\lambda_1(v) + \delta_1(v) + \rho_1(v) = 1$$

$$\text{where, } \rho_1(v) = 1 - [\lambda_1(v) + \delta_1(v)].$$

$\lambda_2, \delta_2, \rho_2 : V \times V \rightarrow [0, 1]$  represent the degree of MS, NMS and hesitancy of  $x = (u, v) \in V \times V$ ,

$$\lambda_2(x) \leq \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) \leq \max\{\delta_1(u), \delta_1(v)\}$$

$$\rho_2(x) \leq \min\{\rho_1(u), \rho_1(v)\}$$

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall x \in V \times V$$

## 3. Main Results

In an HFG, the degree of hesitancy ( $\rho_1$ ) of a vertex  $v$  depends on the degree of MS ( $\lambda_1$ ) and NMS ( $\delta_1$ ) of  $v$ . We define a new class of HFG namely, the intuitionistic HFG (IHFG) in which  $\rho_1$  is independent of  $\lambda_1$  and  $\delta_1$ .

DEFINITION 3.1. An IHFG is  $G = (V, E, \sigma, \mu)$ ,  $V$  is the vertex set,  $\lambda_1, \delta_1, \rho_1 : V \rightarrow [0, 1]$  represent the degree of MS, NMS and hesitancy of  $v \in V$ ,

$$0 \leq \lambda_1(v) + \delta_1(v) + \rho_1(v) \leq 1.$$

$\lambda_2, \delta_2, \rho_2 : V \times V \rightarrow [0, 1]$  represent the degree of MS, NMS and hesitancy of  $x = (u, v) \in V \times V$ ,

$$\lambda_2(x) \leq \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) \leq \max\{\delta_1(u), \delta_1(v)\}$$

$$\rho_2(x) \leq \min\{\rho_1(u), \rho_1(v)\}$$

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall x$$

REMARK 3.2. All HFGs are IHFGs but all IHFGs need not be HFGs. In figure 1,  $G_1$  is an HFG since  $\lambda_1(u_i) + \delta_1(u_i) + \rho_1(u_i) = 1, \forall i$  where  $\rho_1(u_i) = 1 - [\lambda_1(u_i) + \delta_1(u_i)]$ .  $G_1$  is also an IHFG.  $G_2$  is an IHFG since  $0 \leq \lambda_1(v_i) + \delta_1(v_i) + \rho_1(v_i) \leq 1, \forall i$ , but  $G_2$  is not a HFG.

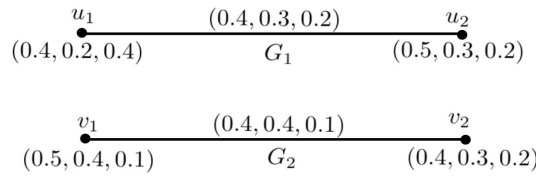


FIGURE 1. Example for HFG and IHFG

DEFINITION 3.3. An HFG  $G_1$  or an IHFG  $G_2$  is

$$\lambda\text{-strong if } \lambda_2(x) = \min\{\lambda_1(u), \lambda_1(v)\}, \forall x = (u, v) \in E$$

EXAMPLE 3.4. Consider the HFG  $G_1$  with vertices  $u_1, u_2, u_3$  and the IHFG  $G_2$  with vertices  $v_1, v_2, v_3$  in figure 2.

$$\begin{aligned} \lambda_2(u_1, u_2) &= 0.3, \lambda_1(u_1) \wedge \lambda_1(u_2) = 0.4 \wedge 0.3 = 0.3, \\ \lambda_2(u_2, u_3) &= 0.3, \lambda_1(u_2) \wedge \lambda_1(u_3) = 0.3 \wedge 0.5 = 0.3 \\ \lambda_2(u_1, u_2) &= \lambda_1(u_1) \wedge \lambda_1(u_2) \\ \lambda_2(u_2, u_3) &= \lambda_1(u_2) \wedge \lambda_1(u_3). \text{ Thus, } G_1 \text{ is a } \lambda\text{-strong HFG.} \\ \lambda_2(v_1, v_2) &= 0.4, \lambda_1(v_1) \wedge \lambda_1(v_2) = 0.4 \wedge 0.5 = 0.4, \\ \lambda_2(v_2, v_3) &= 0.3, \lambda_1(v_2) \wedge \lambda_1(v_3) = 0.5 \wedge 0.3 = 0.3 \\ \lambda_2(v_1, v_2) &= \lambda_1(v_1) \wedge \lambda_1(v_2) \\ \lambda_2(v_2, v_3) &= \lambda_1(v_2) \wedge \lambda_1(v_3). \text{ Thus, } G_2 \text{ is a } \lambda\text{-strong IHFG.} \end{aligned}$$

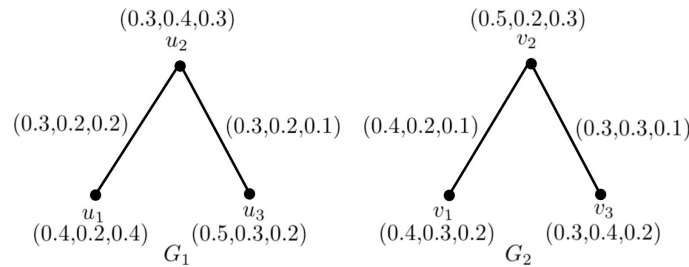


FIGURE 2.  $\lambda$ -strong HFG  $G_1$  and  $\lambda$ -strong IHFG  $G_2$

DEFINITION 3.5. An HFG  $G_1$  or an IHFG  $G_2$  is

$$\delta\text{-strong if } \delta_2(x) = \max\{\delta_1(u), \delta_1(v)\}, \forall x = (u, v) \in E$$

EXAMPLE 3.6. In figure 3,

$$\begin{aligned} \delta_2(u_1, u_2) &= 0.4, \delta_1(u_1) \vee \delta_1(u_2) = 0.2 \vee 0.4 = 0.4, \\ \delta_2(u_2, u_3) &= 0.4, \delta_1(u_2) \vee \delta_1(u_3) = 0.4 \vee 0.3 = 0.4 \\ \delta_2(u_1, u_2) &= \delta_1(u_1) \vee \delta_1(u_2) \\ \delta_2(u_2, u_3) &= \delta_1(u_2) \vee \delta_1(u_3). \text{ Thus, } G_1 \text{ is a } \delta\text{-strong HFG.} \\ \delta_2(v_1, v_2) &= 0.3, \delta_1(v_1) \vee \delta_1(v_2) = 0.3 \vee 0.2 = 0.3, \\ \delta_2(v_2, v_3) &= 0.4, \delta_1(v_2) \vee \delta_1(v_3) = 0.2 \vee 0.4 = 0.4 \\ \delta_2(v_1, v_2) &= \delta_1(v_1) \vee \delta_1(v_2) \\ \delta_2(v_2, v_3) &= \delta_1(v_2) \vee \delta_1(v_3). \text{ Thus, } G_2 \text{ is a } \delta\text{-strong IHFG.} \end{aligned}$$

DEFINITION 3.7. An HFG  $G_1$  or an IHFG  $G_2$  is

$$\rho\text{-strong if } \rho_2(x) = \min\{\rho_1(u), \rho_1(v)\}, \forall x = (u, v) \in E$$

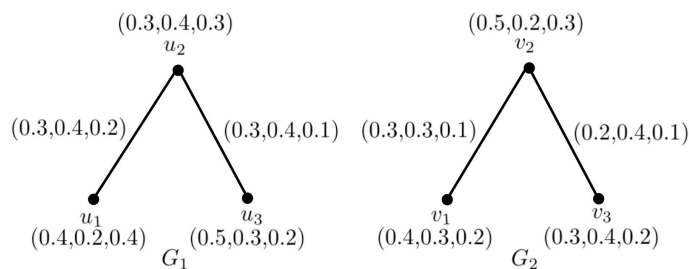


FIGURE 3.  $\delta$ -strong HFG  $G_1$  and  $\delta$ -strong IHFG  $G_2$

EXAMPLE 3.8. In figure 4,  $\rho_2(u_1, u_2) = 0.3, \rho_1(u_1) \wedge \rho_1(u_2) = 0.4 \wedge 0.3 = 0.3,$   
 $\rho_2(u_2, u_3) = 0.2, \rho_1(u_2) \wedge \rho_1(u_3) = 0.3 \wedge 0.2 = 0.2$   
 $\rho_2(u_1, u_2) = \rho_1(u_1) \wedge \rho_1(u_2)$   
 $\rho_2(u_2, u_3) = \rho_1(u_2) \wedge \rho_1(u_3)$ . Thus,  $G_1$  is a  $\rho$ -strong HFG.  
 $\rho_2(v_1, v_2) = 0.2, \rho_1(v_1) \wedge \rho_1(v_2) = 0.2 \wedge 0.3 = 0.2,$   
 $\rho_2(v_2, v_3) = 0.2, \rho_1(v_2) \wedge \rho_1(v_3) = 0.3 \wedge 0.2 = 0.2$   
 $\rho_2(v_1, v_2) = \rho_1(v_1) \wedge \rho_1(v_2)$   
 $\rho_2(v_2, v_3) = \rho_1(v_2) \wedge \rho_1(v_3)$ . Thus,  $G_2$  is  $\rho$ -strong IHFG.

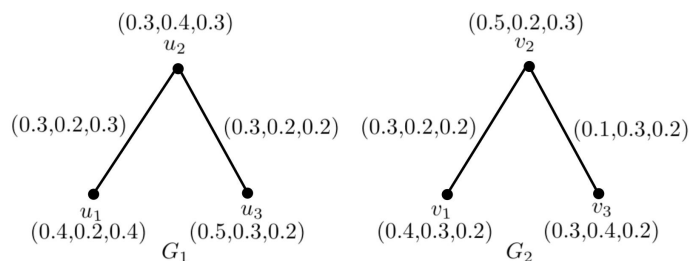


FIGURE 4.  $\rho$ -strong HFG  $G_1$  and  $\rho$ -strong IHFG  $G_2$

DEFINITION 3.9. An HFG  $G_1$  or an IHFG  $G_2$  is strong if it is  $\lambda$ -strong,  $\delta$ -strong and  $\rho$ -strong.

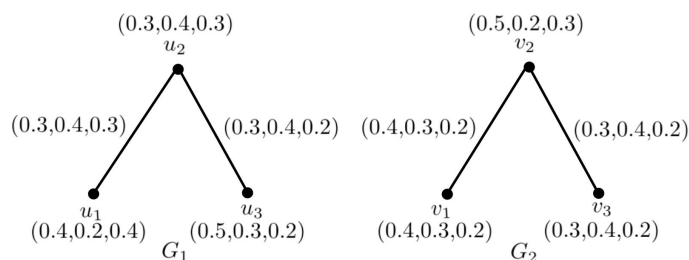


FIGURE 5. strong HFG  $G_1$  and strong IHFG  $G_2$

DEFINITION 3.10. A HFG  $G_1$  or an IHFG  $G_2$  is complete if

$$\lambda_2(x) = \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) = \max\{\delta_1(u), \delta_1(v)\}$$

$$\rho_2(x) = \min\{\rho_1(u), \rho_1(v)\}, \forall u, v \in V.$$

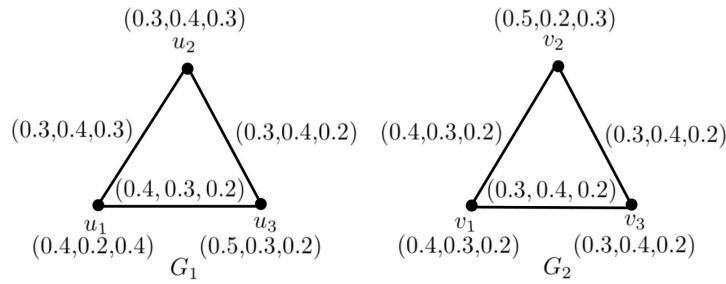


FIGURE 6. complete HFG  $G_1$  and complete IHFG  $G_2$

Now we discuss the  $\beta$ -product of HFG and IHFG.

DEFINITION 3.11. The  $\beta$ -product of two HFGs,  $G_1 = (U, E_U, \sigma, \mu)$ ,  $G_2 = (V, E_V, \sigma', \mu')$  where  $\sigma = (\lambda_1, \delta_1, \rho_1)$ ,  $\mu = (\lambda_2, \delta_2, \rho_2)$ ,  $\sigma' = (\lambda'_1, \delta'_1, \rho'_1)$  and  $\mu' = (\lambda'_2, \delta'_2, \rho'_2)$  is the HFG  $G = G_1 \times_{\beta} G_2 = (U \times V, E, \sigma \times_{\beta} \sigma', \mu \times_{\beta} \mu')$ ,  $E = E_1 \cup E_2 \cup E_3$  where

$$\begin{aligned}
 E_1 &= \{w : w_1 \in E_U, w_2 \in E_V\} \\
 E_2 &= \{w : v_1 \neq v_2, w_1 \in E_U\} \\
 E_3 &= \{w : u_1 \neq u_2, w_2 \in E_V\}, \\
 w &= ((u_1, v_1), (u_2, v_2)), w_1 = (u_1, u_2), w_2 = (v_1, v_2).
 \end{aligned}$$

$$\begin{aligned}
 (\lambda_1 \times_{\beta} \lambda'_1)(x) &= \lambda_1(u) \wedge \lambda'_1(v) \\
 (\delta_1 \times_{\beta} \delta'_1)(x) &= \delta_1(u) \vee \delta'_1(v) \\
 (\rho_1 \times_{\beta} \rho'_1)(x) &= 1 - [\lambda_1(u) \wedge \lambda'_1(v) + \delta_1(u) \vee \delta'_1(v)]
 \end{aligned}$$

$$(1) \quad (\lambda_2 \times_{\beta} \lambda'_2)(w) = \begin{cases} \lambda_2(w_1) \wedge \lambda'_2(w_2), & \text{if } w \in E_1 \\ \lambda'_1(v_1) \wedge \lambda'_1(v_2) \wedge \lambda_2(w_1), & \text{if } w \in E_2 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_2(w_2), & \text{if } w \in E_3 \end{cases}$$

$$(2) \quad (\delta_2 \times_{\beta} \delta'_2)(w) = \begin{cases} \delta_2(w_1) \vee \delta'_2(w_2), & \text{if } w \in E_1 \\ \delta'_1(v_1) \vee \delta'_1(v_2) \vee \delta_2(w_1), & \text{if } w \in E_2 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_2(w_2), & \text{if } w \in E_3 \end{cases}$$

$$(3) \quad (\rho_2 \times_{\beta} \rho'_2)(w) = \begin{cases} \rho_2(w_1) \wedge \rho'_2(w_2), & \text{if } w \in E_1 \\ \rho'_1(v_1) \wedge \rho'_1(v_2) \wedge \rho_2(w_1), & \text{if } w \in E_2 \\ \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_2(w_2), & \text{if } w \in E_3 \end{cases}$$

REMARK 3.12. For two strong HFGs  $G_1, G_2$ , their  $\beta$ -product  $G_1 \times_{\beta} G_2$  need not be  $\rho$ -strong and hence need not be a strong HFG. In figure 7,  $G_1$  and  $G_2$  are two strong HFGs and figure 8 is their  $\beta$ -product  $G_1 \times_{\beta} G_2$ .

$$\begin{aligned}
 (\lambda_1 \times_{\beta} \lambda'_1)(u_1, v_2) &= \lambda_1(u_1) \wedge \lambda'_1(v_2) = 0.3 \wedge 0.5 = 0.3 \\
 (\delta_1 \times_{\beta} \delta'_1)(u_1, v_2) &= \delta_1(u_1) \vee \delta'_1(v_2) = 0.4 \vee 0.3 = 0.4 \\
 (\rho_1 \times_{\beta} \rho'_1)(u_1, v_2) &= 1 - (0.3 + 0.4) = 0.3 \\
 (\lambda_1 \times_{\beta} \lambda'_1)(u_2, v_1) &= \lambda_1(u_2) \wedge \lambda'_1(v_1) = 0.5 \wedge 0.4 = 0.4 \\
 (\delta_1 \times_{\beta} \delta'_1)(u_2, v_1) &= \delta_1(u_2) \vee \delta'_1(v_1) = 0.3 \vee 0.2 = 0.3 \\
 (\rho_1 \times_{\beta} \rho'_1)(u_2, v_1) &= 1 - (0.4 + 0.3) = 0.3
 \end{aligned}$$

Consider the edge  $z = ((u_1, v_2), (u_2, v_1))$ .

Since  $z \in E_1$ ,

$$(\lambda_2 \times_\beta \lambda'_2)(z) = \lambda_2(w_1) \wedge \lambda'_2(w_2) = 0.3 \wedge 0.4 = 0.3$$

$$(\delta_2 \times_\beta \delta'_2)(z) = \delta_2(w_1) \vee \delta'_2(w_2) = 0.4 \vee 0.3 = 0.4$$

$$(\rho_2 \times_\beta \rho'_2)(z) = \rho_2(w_1) \wedge \rho'_2(w_2) = 0.2 \wedge 0.2 = 0.2$$

i.e.,  $(\lambda_2 \times_\beta \lambda'_2)(z) = (\lambda_1 \times_\beta \lambda'_1)(u_1, v_2) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_1)$ .

This is true for all the other edges in  $G_1 \times_\beta G_2$  and hence  $G_1 \times_\beta G_2$  is  $\lambda$ -strong.

$(\delta_2 \times_\beta \delta'_2)(z) = (\delta_1 \times_\beta \delta'_1)(u_1, v_2) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_1)$ , which is true for all the other edges and hence  $G_1 \times_\beta G_2$  is  $\delta$ -strong.

But,  $(\rho_2 \times_\beta \rho'_2)(z) \neq (\rho_1 \times_\beta \rho'_1)(u_1, v_2) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_1)$ . i.e.,  $G_1 \times_\beta G_2$  is not  $\rho$ -strong and hence not a strong HFG.

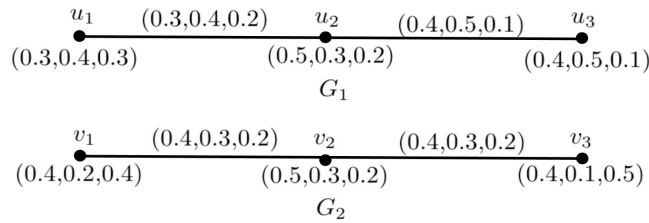


FIGURE 7. Strong HFGs  $G_1$  and  $G_2$

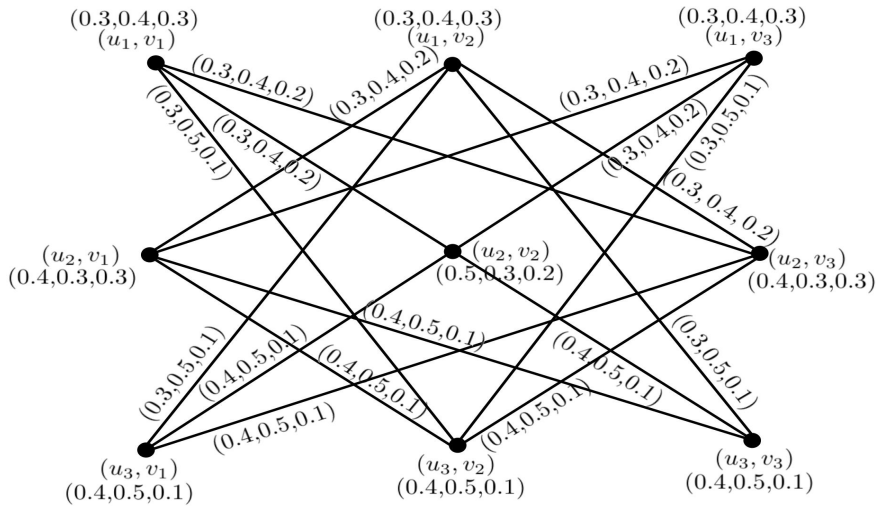


FIGURE 8.  $\beta$ -product of strong HFGs  $G_1$  and  $G_2$

DEFINITION 3.13. The  $\beta$ -product of two IHFGs  $G_1 = (U, E_U, \sigma, \mu)$ ,  $G_2 = (V, E_V, \sigma', \mu')$  where  $\sigma = (\lambda_1, \delta_1, \rho_1)$ ,  $\mu = (\lambda_2, \delta_2, \rho_2)$ ,  $\sigma' = (\lambda'_1, \delta'_1, \rho'_1)$  and  $\mu' = (\lambda'_2, \delta'_2, \rho'_2)$  is the IHFG  $G = G_1 \times_\beta G_2 = (U \times V, E, \sigma \times_\beta \sigma', \mu \times_\beta \mu')$ ,  $E = E_1 \cup E_2 \cup E_3$  where

$$E_1 = \{w : w_1 \in E_U, w_2 \in E_V\}$$

$$E_2 = \{w : v_1 \neq v_2, w_1 \in E_U\}$$

$$E_3 = \{w : u_1 \neq u_2, w_2 \in E_V\}.$$

$$\begin{aligned}
 (\lambda_1 \times_\beta \lambda'_1)(x) &= \lambda_1(u) \wedge \lambda'_1(v) \\
 (\delta_1 \times_\beta \delta'_1)(x) &= \delta_1(u) \vee \delta'_1(v) \\
 (\rho_1 \times_\beta \rho'_1)(x) &= \rho_1(u) \wedge \rho'_1(v)
 \end{aligned}$$

and equations (1), (2) and (3).

**THEOREM 3.14.** *If  $G_1, G_2$  are two strong IHFGs, then their  $\beta$ -product  $G_1 \times_\beta G_2$  is also a strong IHFG.*

*Proof.* Let  $G_1, G_2$  be two strong IHFGs. Then, for  $w_1 \in E_U, w_2 \in E_V$ ,

$$\begin{aligned}
 \lambda_2(w_1) &= \lambda_1(u_1) \wedge \lambda_1(u_2), \\
 \lambda'_2(w_2) &= \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\
 \delta_2(w_1) &= \delta_1(u_1) \vee \delta_1(u_2), \\
 \delta'_2(w_2) &= \delta'_1(v_1) \vee \delta'_1(v_2) \\
 \rho_2(w_1) &= \rho_1(u_1) \wedge \rho_1(u_2), \\
 \rho'_2(w_2) &= \rho'_1(v_1) \wedge \rho'_1(v_2).
 \end{aligned}$$

**Case(i)** When  $w \in E_1$

$$\begin{aligned}
 (\lambda_2 \times_\beta \lambda'_2)(w) &= \lambda_2(w_1) \wedge \lambda'_2(w_2) \\
 &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\
 &= (\lambda_1 \times_\beta \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_2) \\
 (\delta_2 \times_\beta \delta'_2)(w) &= \delta_2(w_1) \vee \delta'_2(w_2) \\
 &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2) \\
 &= (\delta_1 \times_\beta \delta'_1)(u_1, v_1) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_2) \\
 (\rho_2 \times_\beta \rho'_2)(w) &= \rho_2(w_1) \wedge \rho'_2(w_2) \\
 &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_1(v_1) \wedge \rho'_1(v_2) \\
 &= (\rho_1 \times_\beta \rho'_1)(u_1, v_1) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_2)
 \end{aligned}$$

**Case(ii)** When  $w \in E_2$

$$\begin{aligned}
 (\lambda_2 \times_\beta \lambda'_2)(w) &= \lambda'_1(v_1) \wedge \lambda'_1(v_2) \wedge \lambda_2(w_1) \\
 &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\
 &= (\lambda_1 \times_\beta \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_2) \\
 (\delta_2 \times_\beta \delta'_2)(w) &= \delta'_1(v_1) \vee \delta'_1(v_2) \vee \delta_2(w_1) \\
 &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2) \\
 &= (\delta_1 \times_\beta \delta'_1)(u_1, v_1) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_2) \\
 (\rho_2 \times_\beta \rho'_2)(w) &= \rho'_1(v_1) \wedge \rho'_1(v_2) \wedge \rho_2(w_1) \\
 &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_1(v_1) \wedge \rho'_1(v_2) \\
 &= (\rho_1 \times_\beta \rho'_1)(u_1, v_1) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_2)
 \end{aligned}$$

**Case(iii)** When  $w \in E_3$

$$\begin{aligned}
 (\lambda_2 \times_\beta \lambda'_2)(w) &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_2(w_2) \\
 &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\
 &= (\lambda_1 \times_\beta \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_2) \\
 (\delta_2 \times_\beta \delta'_2)(w) &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_2(w_2) \\
 &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2) \\
 &= (\delta_1 \times_\beta \delta'_1)(u_1, v_1) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_2) \\
 (\rho_2 \times_\beta \rho'_2)(w) &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_2(w_2) \\
 &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_1(v_1) \wedge \rho'_1(v_2) \\
 &= (\rho_1 \times_\beta \rho'_1)(u_1, v_1) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_2)
 \end{aligned}$$

Thus,  $G = G_1 \times_\beta G_2$  is a strong IHFG. □

EXAMPLE 3.15. In figure 9,  $G_1$  and  $G_2$  are two strong IHFGs. Their  $\beta$ -product  $G_1 \times_\beta G_2$  in figure 10 is a strong IHFG since all the edges are strong edges.

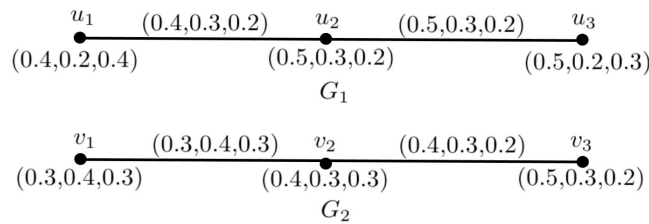


FIGURE 9. Strong IHFGs  $G_1$  and  $G_2$

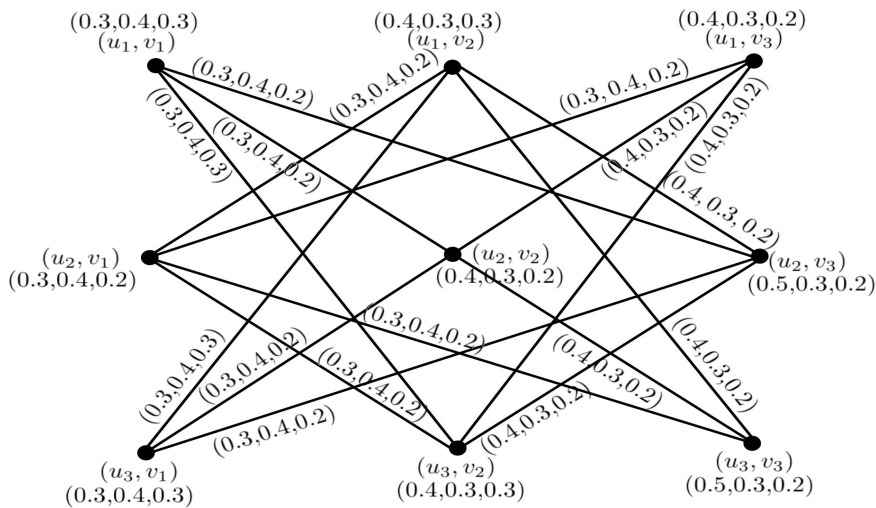


FIGURE 10.  $\beta$ -product of strong IHFGs  $G_1$  and  $G_2$

THEOREM 3.16. If  $G_1$  and  $G_2$  are two complete IHFGs, then their  $\beta$ -product  $G_1 \times_\beta G_2$  is also a complete IHFG .

Proof. Similar to 3.14. □



**THEOREM 3.17.** *If  $G_1, G_2$  are two IHFGs such that  $G_1 \times_\beta G_2$  is strong, then at least one of  $G_1$  or  $G_2$  will be strong.*

*Proof.* Assume that the two IHFGs  $G_1, G_2$  are not strong. Then there exists at least one  $w_1 = (u_1, u_2) \in E_U, w_2 = (v_1, v_2) \in E_V$ , with

$$\begin{aligned} \lambda_2(w_1) &< \lambda_1(u_1) \wedge \lambda_1(u_2), & \lambda'_2(w_2) &< \lambda'_1(v_1) \wedge \lambda'_1(v_2), \\ \delta_2(w_1) &< \delta_1(u_1) \vee \delta_1(u_2), & \delta'_2(w_2) &< \delta'_1(v_1) \vee \delta'_1(v_2), \\ \rho_2(w_1) &< \rho_1(u_1) \wedge \rho_1(u_2), & \rho'_2(w_2) &< \rho'_1(v_1) \wedge \rho'_1(v_2). \end{aligned}$$

Let  $w = ((u_1, v_1), (u_2, v_2)) \in E_1$ . Then,

$$\begin{aligned} (\lambda_2 \times_\beta \lambda'_2)(w) &= \lambda_2(w_1) \wedge \lambda'_2(w_2) \\ &< \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\ \text{i.e., } (\lambda_2 \times_\beta \lambda'_2)(w) &< (\lambda_1 \times_\beta \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_2). \end{aligned}$$

$$\begin{aligned} (\delta_2 \times_\beta \delta'_2)(w) &= \delta_2(w_1) \vee \delta'_2(w_2) \\ &< \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2) \\ \text{i.e., } (\delta_2 \times_\beta \delta'_2)(w) &< (\delta_1 \times_\beta \delta'_1)(u_1, v_1) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_2) \end{aligned}$$

$$\begin{aligned} (\rho_2 \times_\beta \rho'_2)(w) &= \rho_2(w_1) \wedge \rho'_2(w_2) \\ &< \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_1(v_1) \wedge \rho'_1(v_2) \\ \text{i.e., } (\rho_2 \times_\beta \rho'_2)(w) &< (\rho_1 \times_\beta \rho'_1)(u_1, v_1) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_2) \end{aligned}$$

i.e.,  $G_1 \times_\beta G_2$  is not strong, a contradiction. So at least one of  $G_1$  or  $G_2$  will be strong. □

#### 4. Application

IHFGs can be suitably used in real life problems. It can work as a good aid in solving companies' merger problems. Consider two strong networks of IHFGs  $G_1$  and  $G_2$  with vertices indicating distinct companies. The MS degree of the vertices and the edges indicates the market worth of the companies and the market worth of the companies' joint ventures respectively. Since the IHFGs  $G_1$  and  $G_2$  are strong, all the edges in  $G_1$  and  $G_2$  are strong and all the edges in the  $\beta$ -product  $G_1 \times_\beta G_2$  are also strong. That means the joint venture of two strong networks will be strong and the production carried out by the joint venture will be surely successful. Thus, this product is stronger and more reliable and the decision on merger problems based on this result will be more accurate.

For example, consider two strong companies, one which is successful in the production of scooters and another company which is expert in the production of battery. A joint venture, if initiated, will benefit both the companies and expertise of both the companies in their respective production, will be a strong foundation to introduce a new production unit for manufacturing electric scooters and thus produce a new brand of electric scooters. Thus the production carried out by the joint venture will be surely successful and may result in making both the companies involved in the joint venture more stronger.

## 5. Conclusion

IHFGs offers a wide range of uses in the fields of robotics, artificial intelligence and medical diagnosis. We proved that  $\beta$ -product of a pair of strong IHFGs is a strong IHFG and  $\beta$ -product of a pair of complete IHFGs is a complete IHFG. Also we proved that if  $\beta$ -product of a pair of IHFGs is strong, then at least one of the IHFG will be strong. IHFG models provide exact and accurate outcomes for making decisions and resolving merger related problems. Our future work is to broaden the scope of our investigation to study the complement of  $\beta$ -product of IHFG.

## References

- [1] K.T. Atanassov and S. Stoeva, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96.
- [2] Ch. Chaitanya and T.V. Pradeep Kumar, *On the complete product of fuzzy graphs*, South East Asian Journal of Mathematics and Mathematical Sciences (SEAJMMS) **18** (2) (2022), 185–196.
- [3] M. Javaid, A. Kashif and T. Rashid, *Hesitant Fuzzy Graphs and Their Products*, Fuzzy Information and Engineering **12** (2) (2020), 238–252.
- [4] F.Karaaslan, *Hesitant fuzzy graphs and their applications in decision making*, J Intell Fuzzy Syst. **36** (3) (2019), 2729–2741.
- [5] R. Parvathi and M. G. Karunambigai, *Intuitionistic fuzzy graphs*, Computational intelligence, theory and applications, Springer, Berlin, Heidelberg, (2006), 139–150.
- [6] T. Pathinathan, J. Arockiaraj and J.J. Rosline, *Hesitancy fuzzy graph*, Indian Journal of Science and Technology **8** (35) (2015), 1–5
- [7] R. Rajeswari, M. G. Rani, K. H. Sulthana, *Hesitant fuzzy connected and hesitant fuzzy trees*, Int J Pure Appl Math. **118** (10) (2018), 121–134.
- [8] A. Rosenfeld, *Fuzzy graphs*, In: L. A. Zadeh, K. S. Fu, M. Shimura, Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Process, Academic Press, New York,(1975), 77–95.
- [9] J.J. Rosline and T. Pathinathan, *Different types of products on Hesitancy fuzzy graphs*, Int Journal of Pure and Appl Math. **119** (9) (2018), 255–265.
- [10] R. Shakthivel, R. Vikramaprasad and N. Vinothkumar, *Domination in Hesitancy Fuzzy Graphs*, International Journal of Advanced Science and Technology, 28(16) (2019), 1142–1156.
- [11] N. Vinothkumar, G. Geetharamani, *Operations in hesitancy fuzzy graphs*, Int J of Pure Appl Math. **119** (16) (2018), 4325–4338.
- [12] LA. Zadeh, *Fuzzy sets*, Inform Control **8** (1965), 338–356.

### Sunil.M.P

PG and Research Department of Mathematics, N.S.S. Hindu College,  
Changanacherry, Kottayam, Kerala 686102, India  
*E-mail*: sunilmp.mp@gmail.com

### J.Suresh Kumar

PG and Research Department of Mathematics, N.S.S. Hindu College,  
Changanacherry, Kottayam, Kerala 686102, India  
*E-mail*: jsuresh.maths@gmail.com