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LIFTINGS OF A COMPLEMENTED SUBSPACE OF $\mathcal{L}_1$–SPACES

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Abstract. In this article, we prove that an infinite dimensional complemented subspace X of $\mathcal{L}_1$-space Z with unconditional basis (x_n) has the lifting property. Hence we can give an alternative proof that X is isomorphic to ℓ_1 given by Lindenstrauss and Pelczyński.

1. Introduction

Lindenstrauss and Pelczyński [6] proved that ℓ_1 and c_0 have a unique unconditional basis as like ℓ_2. And also they showed that every G.T. Banach space with an unconditional basis $(x_i)_{i\in\Gamma}$ is isomorphic to $\ell_1(\Gamma)$ for some index set Γ; see ([8], pp.114). In ([5] and [4], pp.1726), Lindenstrauss proved the existence of a lifting operator on $\mathcal{L}_1$-space under some more conditions. That is, if the kernel of the quotient map is a complemented subspace in its second dual, then every bounded linear operator has a lifting operator. In this direction of research, main questions are arising under what conditions do we find the lifting property on a subspace of $\mathcal{L}_1$-space. The purpose of this paper is to show that if an infinite dimensional complemented subspace X of an $\mathcal{L}_1$-space Z has an unconditional basis (x_n), then X has the lifting property. From this, we can give an alternative approach of Lindenstrauss and Pelczyński's Theorem by using the lifting property. For this direction of study, we need to begin our studying of several well-known facts concerning with the corresponding bounded linear operators between Banach spaces.

Definition 1.1. Let X, Y and Z be Banach spaces and $\pi : Z \to Y$ be a surjective linear map of Z onto Y. It is called that a Banach space X has *the lifting property* if $T : X \to Y$ is a bounded linear operator, then there is a bounded linear operator $\tilde{T} : X \to Z$ such that $\pi \circ \tilde{T} = T$ and $\|\tilde{T}\| \leq \lambda\|T\|$ and such that the following diagram commutes ;

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \tilde{T}\downarrow & \nearrow \pi & \\ Z & & \end{array} \tag{1.1}$$

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As the well-known fact, Köthe characterized the space ℓ_1 with the lifting property including nonseparable case. He proved that for every index set Γ, the space $\ell_1(\Gamma)$ has the lifting property. Also he proved that the converse of lifting property of $\ell_1(\Gamma)$ as following in [3];

THEOREM 1.2. [3] *If X is a Banach space with the lifting property, then X is isomorphic to $\ell_1(\Gamma)$-space, for some index set Γ.*

Our discussion will be focus on the absolutely 1-summing operator on a Banach space.

DEFINITION 1.3. A continuous linear operator $T : X \to Y$ is called an *absolutely 1-summing* if there is a constant $K > 0$ such that for any finite subset (x_i) in X, we can have

$$(\sum_{i=1}^{n} \|Tx_i\|) \leq K \sup\{(\sum_{i=1}^{n} |x^*(x_i)|) : x^* \in B_{X^*}\}. \tag{1.2}$$

This definition (1.2) says that those operators $T : X \to Y$ take unconditionally summable sequences (x_n) in X to absolutely summable sequences (Tx_n) in Y. We will define $\pi_1(T)$ the smallest constant K satisfying (1.2). Also the set of all absolutely 1-summing operator $T : X \to Y$ will be denoted by $\Pi_1(X, Y)$. It is known that $\Pi_1(X, Y)$ is a Banach space with a norm $\pi_1(T)$. We will give a fundamental and fine result of Grothendieck's theorem to motivate our study.

THEOREM 1.4. **(Grothendieck's Theorem)** *Every bounded linear operator $\ell_1 \to \ell_2$ is absolutely 1-summing operator.*

Latter on, Lindenstrauss and Pelzyński proved an extension of Grothendieck's theorem 1.4. as following;

THEOREM 1.5. ([5] and [1], pp. 60) *If X is an $\mathcal{L}_{1,\lambda}$-space and Y is an $\mathcal{L}_{2,\lambda'}$-space, then every operator $T : X \to Y$ is absolutely 1-summing with $\pi_1(T) \leq \lambda \cdot \lambda' \cdot K_G\|T\|$ where K_G is the Grothendieck's constant.*

2. Main results

In this article, our main questions are based on the fact that a Banach space X with the lifting property is isomorphic to ℓ_1 space. Especially Lindenstrauss and Pelczyński proved that every G.T. Banach space with unconditional basis is isomorphic to $\ell_1(\Gamma)$, for some index set Γ. In this paper, by seeking the lifting of subspace of $\mathcal{L}_1$, we can give an alternative approach that if an infinite dimensional complemented subspace X of an $\mathcal{L}_1$-space Z has an unconditional basis (x_n), then (x_n) is equivalent to unit vector basis of ℓ_1 and so X must be isomorphic to ℓ_1 in [1].

For the purpose of our research. we need several definitions and well known theorems.

DEFINITION 2.1. We say that a Banach space satisfies *Grothendieck's theorem* (in short G.T.) if

$$B(X, \ell_2) = \Pi_1(X, \ell_2) \tag{2.1}$$

where $B(X, \ell_2)$ is the set of all bounded linear operators on X into ℓ_2. From this definition, we will say that X is a *G.T. Banach space.*

DEFINITION 2.2. A basis (x_n) for a Banach space X is *unconditional* if there is a constant $C > 0$ such that

$$\|\sum_{k=1}^{n} \epsilon_k a_k x_k\| \leq C \cdot \|\sum_{k=1}^{n} a_k x_k\| \tag{2.2}$$

for any choice of finite sets $\{a_1, a_2, \cdots, a_n\}$ of scalars and $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ of $\pm$'s. Hence if (x_n^*) is the corresponding biorthogonal sequence in X^*, then $\sum_n < x_n^*, x > x_n$ converges unconditionally for each $x \in X$.

LEMMA 2.3. *Let X be a complemented subspace of $\mathcal{L}_1$-space Z with unconditional basis (x_n). If $S : X \to \ell_2$ is a bounded linear operator given by $S(x_n) = e_n$ where (e_n) is an unit vector basis of ℓ_2, then for any finite set of scalars $\{a_1, a_2, \cdots, a_n\}$, we have*

$$\sum_{k=1}^{n} |a_k| \leq C \cdot \pi_1(S) \cdot \|\sum_{k=1}^{n} a_k x_k\| \tag{2.3}$$

where $C > 0$ is constant and $\pi_1(S)$ is an absolutely summing norm of S.

Proof. Let (x_n) be an unconditional basis of X. Then without loss of generality, by normalizing we can assume that for each n, $\|x_n\| = 1$. Now if an operator $S : X \to \ell_2$ defined by $S(x) = \sum_n < x_n^*, x > e_n$ where (x_n^*) is the corresponding biorthogonal sequence in X^*. Since Z is a $\mathcal{L}_1$-space and ℓ_2 is a $\mathcal{L}_2$-space and X is a complemented subspace of Z, Grothendieck's Theorem 1.5 says that for any finite set of scalars $\{a_1, a_2, \cdots, a_n\}$, a bounded linear operator $S : X \to \ell_2$ defined by $S(\sum_{k=1}^n a_k x_k) = \sum_{k=1}^n a_k e_k$ is an absolutely 1-summing operator. Hence, for any finite set of scalars $\{a_1, a_2, \cdots, a_n\}$ and the basis constant $C > 0$ from (2.2), we can have

$$\begin{aligned}
\sum_{k=1}^{n} |a_k| &= \sum_{k=1}^{n} \|a_k e_k\|_{\ell_2} \\
&= \sum_{k=1}^{n} \|S(a_k x_k)\|_{\ell_2} \\
&\leq \pi_1(S) \cdot \{\|\sum_{k=1}^{n} \epsilon_k a_k x_k\| : \epsilon_1, \cdots, \epsilon_n = \pm 1\} \\
&\leq C \cdot \pi_1(S) \cdot \|\sum_{k=1}^{n} a_k x_k\|, \quad \text{by definition of unconditional basis} \qquad (2.4)
\end{aligned}$$

This prove the lemma. □

Now we can prove the main result that a complemented subspace of $\mathcal{L}_1$-space with unconditional basis has the lifting property.

THEOREM 2.4. *If an infinite dimensional complemented subspace X of an $\mathcal{L}_1$-space Z has an unconditional basis (x_n), then X has the lifting property.*

Proof. Let X be a complemented subspace of $\mathcal{L}_1$ -space with unconditional basis (x_n) where $C > 0$ is the unconditional basis constant in (2.2). Again by normalizing we may assume that $\|x_n\| = 1$, for all n. Now for each n, define $T : X \to Y$ by $T(x_n) = y_n$. To show the lifting property of X, let $\pi : W \to Y$ be a surjective linear map from a Banach space W onto Y. Then by the open mapping theorem, for each n, there is $w_n \in W$ such that $\pi(w_n) = y_n$ with $\|w_n\| \leq \lambda\|y_n\|$ for $\lambda > 0$. Now we define a lifting $\tilde{T} : X \to W$ of T by $\tilde{T}(x_n) = w_n$, for each n. Then we need to show that this operator is a well defined bounded linear operator. Hence for any finite sequence of scalars $\{a_1, a_2, \cdots, a_n\}$, we can say that

$$\begin{aligned}
\|\tilde{T}(\sum_{k=1}^{n} a_k x_k)\| &= \|\sum_{k=1}^{n} a_k w_k\| \\
&\leq \sum_{k=1}^{n} |a_k|\|w_k\| \\
&\leq \lambda \cdot \sum_{k=1}^{n} |a_k|\|y_k\| \\
&\leq \lambda \cdot \|T\| \cdot \sum_{k=1}^{n} |a_k| \quad (\because \|y_n\| = \|T(x_n)\| \leq \|T\|) \\
&\leq \lambda \cdot C \cdot \|T\| \cdot \pi_1(S) \cdot \|\sum_{k=1}^{n} a_k x_k\| \quad \text{by lemma 2.3.} \qquad (2.5)
\end{aligned}$$

Here we can apply above lemma 2.3 since $S : X \to \ell_2$ is an absolutely summing operator. This showed that $\|\tilde{T}\| \leq \lambda \cdot C \cdot \pi_1(S) \cdot \|T\|$. And so $\tilde{T}$ is a bounded linear operator on X into W.

Finally, we need to show that $\tilde{T}$ is a desired lifting of T such that $\pi \circ \tilde{T} = T$. For any finite span $x = \sum_{k=1}^{n} a_k x_k$ which are dense in X, we can have

$$\begin{aligned}
\pi \circ \tilde{T}(\sum_{k=1}^{n} a_k x_k) &= \pi(\sum_{k=1}^{n} a_k w_k) \\
&= \sum_{k=1}^{n} a_k y_k \\
&= \sum_{k=1}^{n} a_k T(x_k) = T(\sum_{k=1}^{n} a_k x_k) \qquad (2.6)
\end{aligned}$$

Hence we prove that $\tilde{T}$ is a desired lifting of T by extending $\tilde{T}$ on the whole X. □

Now by applying theorem 2.4 and Köthe's theorem 1.2 [3], we can give another approach of proof which is given by Lindenstrauss and Pelczynski's theorem in [6] and ([1], p68), as following;

Corollary 2.5. [1,6] *If an infinite dimensional complemented subspace X of $\mathcal{L}_1$-space X has an unconditional basis (x_n), then (x_n) is equivalent to the unit vector basis of ℓ_1 and so X must be isomorphic to ℓ_1.*

Proof. Let X be an infinite dimensional complemented subspace of an $\mathcal{L}_1$-space Z with an unconditional basis (x_n). Then by above theorem 2.4, X has the lifting property which may not norm preserving. Hence by theorem 1.2, X is isomorphic to ℓ_1. This prove the corollary. □

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