Inner Automorphisms of an Abelian Extension of a Quandle

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Abstract. The inner automorphism groups of quandles are related to the classification problem of quandles. The inner automorphism group of a quandle is generated by inner automorphisms which are presented by columns in the operation table of the quandle.

In this paper, we describe inner automorphisms of an abelian extension of a quandle by expressing columns of the operation table of the extended quandle as columns of the operation table of the original quandle. Such a description will be helpful in studying inner automorphism groups of abelian extensions of quandles.

1. Introduction and Preliminaries

Quandle is an algebraic structure motivated from knot theory. It consists of a set and a binary operation that satisfy three axioms corresponding to three Reidemeister moves. In fact, a *quandle* is defined as a pair of a set Q and a binary operation $* : Q \times Q \to Q$ satisfying

(i) $x * x = x$ for all $x \in Q$,

(ii) for any $x, y \in Q$, there uniquely exists $z \in Q$ such that $z * x = y$, and

(iii) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

In this paper, we consider only finite quandle, so that we can think a finite quandle $Q = \{x_1, \cdots, x_m\}$ as an $m \times m$ matrix $T = [x_i * x_j] \in \mathcal{M}_{m \times m}(Q)$, called

the *operation matrix* of $(Q, *)$. The operation matrix of a quandle is widely used to study the structure of the quandle. For example, Ho and Nelson[6] classified finite quandles up to isomorphism and computed their automorphism groups by using the operation matrices. Also, Murillo, Nelson and Thompson[8] showed some finite quandles are isomorphic to Alexander quandles by using their matrices.

For two quandles $(Q, *)$ and $(Q', *')$, a *quandle homomorphism* is a function $f : Q \to Q'$ provided $f(x * y) = f(x) *' f(y)$ for all $x, y \in Q$. A bijective quandle homomorphism is called a *quandle isomorphism* and a quandle isomorphism from a quandle to itself is called a *quandle automorphism*.

By the definition of a quandle, the map $(- * x) : Q \to Q$ given by $y \mapsto y * x$ is a quandle automorphism. We call the automorphism $(- * x)$ an *inner automorphism*. Also, the set of all quandle automorphisms on $(Q, *)$ forms a group under the composition. We call the group the *automorphism group* and denote it $\text{Aut}(Q, *)$. The subgroup $\text{Inn}(Q, *)$ of $\text{Aut}(Q, *)$ generated by $(- * x)$ for each $x \in Q$ is called the *inner automorphism group* of $(Q, *)$.

The studies on automorphism groups and inner automorphism groups of quandles are related to the classification problem of quandles. Elhamdadi, Macquarrie and Restrepo[4] studied the automorphism groups of dihedral quandles and Hou[7] described the automorphism groups of Alexander quandles. Also, Ferman, Nowik and Teicher[5] introduced the generators of an Alexander quandle by using its automorphism group and Bardakov, Dey and Singh[2] studied the automorphism groups of quandles obtained from groups.

Now, we review abelian extensions of quandles.

Definition 1.1. [3] Let $(Q, *)$ be a quandle and A an abelian group. A *quandle 2-cocycle* of $(Q, *)$ by A is a function $\phi : Q \times Q \to A$ satisfying

(i) $\phi(x, x) = 0$ for all $x \in Q$,

(ii) $\phi(x, y) + \phi(x * y, z) = \phi(x, z) + \phi(x * z, y * z)$ for all $x, y, z \in Q$.

For a finite quandle $Q = \{x_1, x_2, \cdots, x_m\}$, a quandle 2-cocycle can be thought as an $m \times m$ matrix $M = [\phi(x_i, x_j)] \in \mathcal{M}_{m \times m}(A)$., called the 2-cocycle matrix.

Example 1.2.

(1) For any quandle $(Q, *)$, the zero function $\phi_0 : Q \times Q \to A$ defined by $\phi_0(x, y) = 0$ for all $x, y \in Q$ is a quandle 2-cocycle which is called the *trivial 2-cocycle* of $(Q, *)$ by A. The 2-cocycle matrix for ϕ_0 is the zero matrix of $|Q| \times |Q|$.

(2) Let $(Q, *_0)$ denote the trivial quandle($x *_0 y = x$ for all $x, y \in Q$) and A an abelian group. Then any function $\phi : Q \times Q \to A$ satisfies the second condition for quandle 2-cocycle. Hence $\phi : Q \times Q \to A$ is a quandle 2-cocycle if and only if $\phi(x, x) = 0$ for all $x \in Q$. In other words, every $|Q| \times |Q|$ matrix with 0 diagonal entries can be a 2-cocycle matrix for $(Q, *_0)$.

(3) Consider the dihedral quandle $(\mathbb{Z}_3, *)$ with the operation $x * y = 2y - x$ in $\mathbb{Z}_3$. Let A be an abelian group. Then the quandle 2-cocycle matrix M of $(\mathbb{Z}_3, *)$ has the form

$$M = \begin{bmatrix} 0 & a & a - b \\ b & 0 & b - a \\ -b & -a & 0 \end{bmatrix} \in \mathcal{M}_{3 \times 3}(A).$$

For, if $\phi : \mathbb{Z}_3 \times \mathbb{Z}_3 \to A$ is a quandle 2-cocycle of $(\mathbb{Z}_3, *)$ by A, then ϕ satisfies that $\phi(x, x) = 0$ and $\phi(x, y) + \phi(x * y, z) = \phi(x, z) + \phi(x * z, y * z)$ for all $x, y, z \in \mathbb{Z}_3$. Since $x * y = z$ for mutually different $x, y, z \in \mathbb{Z}_3$, the quandle 2-cocycle condition for ϕ is changed into $\phi(x, y) - \phi(y, x) = \phi(x, z)$, which gives the equalities:

$$\phi(0, 1) = \phi(0, 2) - \phi(2, 0), \quad \phi(0, 2) = \phi(0, 1) - \phi(1, 0)$$
$$\phi(1, 0) = \phi(1, 2) - \phi(2, 1), \quad \phi(1, 2) = \phi(1, 0) - \phi(0, 1)$$
$$\phi(2, 0) = \phi(2, 1) - \phi(1, 2), \quad \phi(2, 1) = \phi(2, 0) - \phi(0, 2).$$

Hence, we get the result by putting $a = \phi(0, 1)$ and $b = \phi(1, 0)$.

Definition 1.3. [3] Let $(Q, *)$ be a quandle and A an abelian group. If $\phi : Q \times Q \to A$ be a quandle 2-cocycle, then $Q \times A$ is a quandle under the binary operation $\widetilde{*} : (Q \times A) \times (Q \times A) \to Q \times A$ defined by $(x, a) \widetilde{*} (y, b) = (x * y, a + \phi(x, y))$. The quandle $(Q \times A, \widetilde{*})$ is called the *abelian extension* of $(Q, *)$ by A with ϕ and denoted by $E(Q, A, \phi)$.

For a finite quandle and a finite abelian group, we can express the operation matrix for $E(Q, A, \phi)$ by the operation matrix for Q and the 2-cocycle matrix for $\phi : Q \times Q \to A$.

Remark 1.4. Let $(Q, *)$ be a finite quandle on $Q = \{x_1, x_2, \cdots, x_m\}$ with the operation matrix $T = [x_i * x_j] \in \mathcal{M}_{m \times m}(Q)$. Let $A = \{a_1, a_2, \cdots, a_n\}$ be a finite abelian group and let $\phi : Q \times Q \to A$ be a quandle 2-cocycle with the 2-cocycle matrix $M = [\phi(x_i, x_j)] \in \mathcal{M}_{m \times m}(A)$. Then the operation matrix for the abelian extension $E(Q, A, \phi)$ is the $mn \times mn$-matrix of the form $T = [T \times (a_i + M)]_{i \in \{1, \cdots, n\}}$. Here, $T \times M$ means the matrix of the form $[(a_{ij}, b_{ij})]$ for two same size matrices $T = [a_{ij}]$ and $M = [b_{ij}]$, while $a + M$ means the matrix $[a + m_{ij}]$ for $a \in A$ and $M = [m_{ij}] \in \mathcal{M}_{m \times m}(A)$.

This expression of the operation matrix for $E(Q, A, \phi)$ is given by a anti-lexicographic list of the elements of $E(Q, A, \phi)$:

$$(x_1, a_1), (x_2, a_1), \cdots, (x_m, a_1), \quad \cdots, (x_1, a_n), (x_2, a_n), \cdots, (x_m, a_n)$$

because $(x_i, a_p) \widetilde{*} (x_j, a_q) = (x_i * x_j, a_p + \phi(x_i, x_j))$ for all q, the $((i, p), (j, q))$-th entry of the matrix is $(x_i * x_j, a_p + \phi(x_i, x_j))$.

Example 1.5.

(1) Consider the dihedral quandle $(\mathbb{Z}_3, *)$ with the operation matrix

$$T = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

If $\phi_0 : Q \times Q \to \mathbb{Z}_2$ is the trivial 2-cocycle of $(Q, *_0)$, then the abelian extension $E(Q, \mathbb{Z}_2, \phi_0)$ is the cartesian product of $(Q, *_0)$ and the trivial quandle on $\mathbb{Z}_2$ with the operation matrix:

$$\left[\begin{array}{ccc|ccc} (0,0) & (2,0) & (1,0) & (0,0) & (2,0) & (1,0) \\ (2,0) & (1,0) & (0,0) & (2,0) & (1,0) & (0,0) \\ (1,0) & (0,0) & (2,0) & (1,0) & (0,0) & (2,0) \\ \hline (0,1) & (2,1) & (1,1) & (0,1) & (2,1) & (1,1) \\ (2,1) & (1,1) & (0,1) & (2,1) & (1,1) & (0,1) \\ (1,1) & (0,1) & (2,1) & (1,1) & (0,1) & (2,1) \end{array} \right].$$

(2) We provide non-trivial quandle 2-cocycle ϕ of the dihedral quandle $(\mathbb{Z}_3, *)$ with the following 2-cocycle matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The operation matrix for the abelian extension $E(\mathbb{Z}_3, \mathbb{Z}_2, \phi)$ is given by

$$\left[\begin{array}{ccc|ccc} (0,0) & (2,1) & (1,0) & (0,0) & (2,1) & (1,0) \\ (2,1) & (1,0) & (0,0) & (2,1) & (1,0) & (0,0) \\ (1,1) & (0,1) & (2,0) & (1,1) & (0,1) & (2,0) \\ \hline (0,1) & (2,0) & (1,1) & (0,1) & (2,0) & (1,1) \\ (2,0) & (1,1) & (0,1) & (2,0) & (1,1) & (0,1) \\ (1,0) & (0,0) & (2,1) & (1,0) & (0,0) & (2,1) \end{array} \right].$$

2. Inner Automorphisms of Abelian Extensions

Now, we observe inner automorphisms of an abelian extension by comparing to inner automorphisms of the original quandle. Each inner automorphism $(- * x)$ is the permutation on Q which is given as the x-column vector of the operation matrix, so that Remark 1.4. can be used to study inner automorphisms of abelian extensions. Indeed, since the inner automorphism group of Q is generated by each inner automorphism $(- * x)$, this observation will be helpful to study the inner automorphism group of abelian extensions.

Let $(Q, *)$ be a quandle. We denote

$$x *^n y = \underbrace{((x * y) * \cdots) * y}_{n\text{-times}}$$

for a non-negative integer n. For $x, y \in Q$, if there exists a non-negative integer k such that $x *^k y = x$ and if k is the smallest such integer, then one can get a cycle of length k on Q;

$$\sigma = [x \to x * y \to x *^2 y \to \cdots \to x *^{k-1} y].$$

Indeed, one can note that this cycle σ is a factor of the inner automorphism $(- * y)$.

For a finite quandle $(Q, *)$, there exists a non-negative integer k satisfying $x *^k y = x$ for all $x, y \in Q$, so that we always get a closed cycle. Hence, for $x \in Q$, the inner automorphism $(- * x) : Q \to Q$ defined by $y \mapsto y * x$ is a permutation on Q and presented as a product of disjoint cycles. The following lemma details the cycle structure of inner automorphisms.

Lemma 2.1. *For $x \in Q$, there exist $x_1, x_2, \cdots, x_p$ in Q satisfying*

(i) $x_j *^{k_j} x = x_j$ *for some non-negative integer k_j,*

(ii) $\sigma_{x_j} = \left[x_j \to x_j * x \to x_j *^2 x \to \cdots \to x_j *^{k_j - 1} x\right]$ *are mutually disjoint cycles for $j =, 1 \cdots, p$, and*

(iii) *the inner automorphism $(- * x)$ is the product of cycles $\sigma_{x_1}, \sigma_{x_2}, \cdots, \sigma_{x_p}$:*

$$(- * x) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}.$$

We call $\sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$ the cycle presentation of $(- * x)$.

Proof. Choose an element x_1 in Q. Since Q is finite, there is the smallest non-negative integer $k_1 \in \mathbb{N} \cup \{0\}$ such that $x_1 *^{k_1} x = x_1$, so that we have a closed cycle $\sigma_{x_1} = \left[x_1 \to x_1 * x \to x_1 *^2 x \to \cdots \to x_1 *^{k_1 - 1} x\right]$ on Q.

If $\{x_1 *^k x \mid k = 0, \cdots, k_1 - 1\} = Q$, then $(- * x) = \sigma_{x_1}$. If $\{x_1 *^k x \mid k = 0, \cdots, k_1 - 1\} \neq Q$, choose $x_2 \in Q - \{x_1 *^k x \mid k = 0, \cdots, k_1 - 1\}$. There exists the smallest non-negative integer $k_2 \in \mathbb{N} \cup \{0\}$ such that $x_2 *^{k_2} x = x_2$, and hence we have a closed cycle $\sigma_{x_2} = \left[x_2 \to x_2 * x \to x_2 *^2 x \to \cdots \to x_2 *^{k_2 - 1} x\right]$ on Q.

By the second condition of the definition of quandle, $\{x_1 *^k x \mid k = 1, \cdots, k_1 - 1\} \cap \{x_2 *^k x \mid k = 1, \cdots, k_2 - 1\} \neq \emptyset$ if and only if $x_1 *^i x = x_2 *^j x$, that is $x_2 = x_1 *^{i-j} x$ or $x_1 = x_2 *^{j-i} x$ for some i, j. That is σ_{x_1} and σ_{x_2} are disjoint cycles on Q.

If $\{x_1 *^k x \mid k = 0, \cdots, k_1 - 1\} \cup \{x_2 *^k x \mid k = 0, \cdots, k_2 - 1\} = Q$, then $(- * x) = \sigma_{x_1} \sigma_{x_2}$. Otherwise, one can repeat the same process to get the result.

In fact, the product $\sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$ maps y to $y * x$ for all $y \in Q$, i.e. $(- * x) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$. $\square$

Example 2.2.

(1) If $(Q, *_0)$ is the trivial quandle on $Q = \{x_1, x_2, \cdots, x_n\}$, then $(- *_0 x)$ is the identity function for all $x \in Q$. Indeed, $(- *_0 x) = (x_1)(x_2) \cdots (x_n) = \mathrm{id}_Q$.

(2) Consider the dihedral quandle on $\mathbb{Z}_3$ in Example 1.2(3). The inner automorphisms are given as $(- * 0) = (0)(12) = (12)$, $(- * 1) = (02)(1) = (02)$ and $(- * 2) = (01)(2) = (01)$. Indeed, one can see each inner automorphism in each column vector of the operation matrix given in Example 1.5(1).

Now, we consider inner automorphisms of abelian extensions.

Let A be an abelian group and $\phi : Q \times Q \to A$ a quandle 2-cocycle. Let $E(Q, A, \phi)$ denote the abelian extension of Q by A with ϕ. Suppose that k is the smallest non-negative integer satisfying $x *^k y = x$ for $x, y \in Q$. For $a, b \in A$, the sequence $[(x, a) \to (x, a)\widetilde{*}(y, b) \to (x, a)\widetilde{*}^2(y, b) \to \cdots \to (x, a)\widetilde{*}^{k-1}(y, b)]$ does not form a cycle on $E(Q, A, \phi)$ in general, because $(x, a)\widetilde{*}^k(y, b) = (x *^k y, a + \phi(x, y) + \phi(x * y, y) + \cdots + \phi(x *^{k-1} y, y))$ and $\phi(x, y) + \phi(x * y, y) + \cdots + \phi(x *^{k-1} y, y) \neq 0$. Indeed, if l is the order of $\phi(x, y) + \phi(x * y, y) + \cdots + \phi(x *^{k-1} y, y)$ in A, then

$$\widetilde{\sigma}_{(x,a)} = [(x, a) \to (x, a)\widetilde{*}(y, b) \to (x, a)\widetilde{*}^2(y, b) \to \cdots \to (x, a)\widetilde{*}^{kl-1}(y, b)]$$

is a closed cycle on $E(Q, A, \phi)$ of length kl.

Fix $(x, a) \in E(Q, A, \phi)$. By Lemma 2.1, there exist $x_1, x_2, \cdots, x_p$ in Q such that $(- * x) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$, where $\sigma_{x_j} = \left[x_j \to x_j * x \to x_j *^2 x \to \cdots \to x_j *^{k_j - 1} x \right]$ for $x_j *^{k_j} x = x_j$. Let l denote the order of $\sum_{i=0}^{k_j - 1} \phi(x_j *^i x, x)$ in A. Then, for any $b \in A$,

(a)
$$\widetilde{\sigma}_{(x_j,b)} = \left[(x_j, b) \to (x_j, b)\widetilde{*}(x, a) \to (x_j, b)\widetilde{*}^2(x, a) \to \cdots \to (x_j, b)\widetilde{*}^{k_j l - 1}(x, a) \right]$$

is a closed cycle of length $k_j l$ in $E(Q, A, \phi)$ starting at (x_j, b). If we choose $b' \neq b \in A$, then $\widetilde{\sigma}_{(x_j, b)}$ and $\widetilde{\sigma}_{(x_j, b')}$ are either disjoint or the same. Since A is finite, $\{(x_j, b)\widetilde{*}^i(x, a) \mid i \in \mathbb{N} \cup \{0\}, b \in A\}$ is also finite. From the above observation, there exist $b_1, \cdots, b_{\frac{|A|}{l}} \in A$ such that

$$\{(x_j, b)\widetilde{*}^i(x, a) \mid i \in \mathbb{N} \cup \{0\}, b \in A\} = \bigcup_{k=1}^{\frac{|A|}{l}} \widetilde{\sigma}_{(x_j, b_k)}$$

and each $\widetilde{\sigma}_{(x_j, b_k)}$ are disjoint.

We now describe inner automorphisms of an abelian extension of a quandle using the cycle presentation.

Theorem 2.3. *Let $E(Q, A, \phi)$ be an abelian extension of a quandle Q by a abelian group A via a quandle 2-cocycle ϕ. Take $x \in Q$ and $a \in A$. Suppose that the inner automorphism $(- * x)$ has the cycle presentation $\sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$ on Q. Let*

$\widetilde{\sigma}_{x_j} = \prod_{k=1}^{\frac{|A|}{l}} \widetilde{\sigma}_{(x_j, b_k)}$ *denote the product of cycles* $\widetilde{\sigma}_{(x_j, b_k)}$ *given in the equation* (a). *Then* $\widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ *is a cycle presentation of the inner automorphism* $(- \; \widetilde{*} \; (x, a))$ *of* $E(Q, A, \phi)$. *That is,* $(- \; \widetilde{*} \; (x, a)) = \widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$.

Proof. To show that $\widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ is the cycle presentation for $(- \; \widetilde{*} \; (x, a))$, it suffice to check $\widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ maps $(y, b) \in E(Q, A, \phi)$ to $(y, b) \; \widetilde{*} \; (x, a)$ for all $(y, b) \in E(Q, A, \phi)$. By the cycle presentation of Q, there uniquely exists $j \in \{1, 2, \cdots, p\}$ such that y is appeared in $\sigma_{x_j} = \left[x_j \to x_j * x \to x_j *^2 x \to \cdots \to x_j *^{k_j - 1} x \right]$, i.e., $y = x_j *^q x = y$ for some $q \in \{0, \cdots, k_j - 1\}$.

Consider the element $(x_j, b - \sum_{i=1}^{q-1} \phi(x_j *^i x, x)) \in E(Q, A, \phi)$. Since $\widetilde{\sigma}_{x_j} = \prod_{k=1}^{\frac{|A|}{l}} \widetilde{\sigma}_{(x_j, b_k)}$ and since $\widetilde{\sigma}_{(x_j, b)}$ and $\widetilde{\sigma}_{(x_j, b')}$ are either disjoint or the same, there uniquely exists $b_{k_0} \in A$ such that $(x_j, b - \sum_{i=1}^{q-1} \phi(x_j *^i x, x)) \; \widetilde{*}^q \; (x, a))$ is appeared in $\widetilde{\sigma}_{(x_j, b_{k_0})}$. Denote the order of $\sum_{i=1}^{q-1} \phi(x_j *^i x, x)$ by l. Since $\widetilde{\sigma}_{(x_j, b_{k_0})} = \left[(x_j, b_{k_0}) \to (x_j, b_{k_0}) \widetilde{*}(x, a) \to (x_j, b_{k_0}) \widetilde{*}^2(x, a) \to \cdots \to (x_j, b_{k_0}) \widetilde{*}^{(k_j l - 1)}(x, a) \right]$ is a closed cycle, there exist $r \in \{1, \cdots, \frac{|A|}{l}\}$ such that

$$(x_j, b_{k_0}) \; \widetilde{*}^r \; (x, a) = (x_j, b - \sum_{i=1}^{q-1} \phi(x_j *^i x, x)) \; \widetilde{*}^q \; (x, a).$$

Since $(x_j, b - \sum_{i=1}^{q-1} \phi(x_j *^i x, x)) \; \widetilde{*}^q \; (x, a) = (y, b)$, (y, b) is appeared in the cycle $\widetilde{\sigma}_{(x_j, b_{k_0})}$ so that $\widetilde{\sigma}_{(x_j, b_{k_0})}$ maps $(x_j, b - \sum_{i=1}^{q-1} \phi(x_j *^i x, x)) \; \widetilde{*}^q \; (x, a)$ to

$$[(x_j, b - \sum_{i=1}^{q-1} \phi(x^j *^i x, x)) \; \widetilde{*}^q \; (x, a)] \; \widetilde{*} \; (x, a) = (y, b) \; \widetilde{*} \; (x, a).$$

Hence $\widetilde{\sigma}_{x_j}$ maps (y, b) to $(y, b) \; \widetilde{*} \; (x, a)$. Therefore, $\widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ maps (y, b) to $(y, b) \; \widetilde{*} \; (x, a)$, since the cycles are disjoint. $\qquad \square$

Example 2.4. Let $(Q, *)$ be a quandle. Consider the trivial 2-cocycle $\phi_0 : Q \times Q \to A$ defined in Example 1.2. Let $x \in Q$ with $(- * x) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$. Suppose that $\sigma_{x_j} = \left[x_j \to x_j * x \to \cdots \to x_j *^{k-1} x \right]$ is a cycle of length k. Since ϕ_0 is trivial, the order of $\sum_{i=0}^{k-1} \phi_0(x_j *^i x, x)$ in A is zero. Therefore,

$$\left[(x_j, b) \to (x_j, b) \widetilde{*}(x, a) \to (x_j, b) \widetilde{*}^2(x, a) \to \cdots \to (x_j, b) \widetilde{*}^{k-1}(x, a) \right]$$
$$= \left[(x_j, b) \to (x_j * x, b) \to (x_j *^2 x, b) \cdots \to (x_j *^{k-1}, b) \right]$$

forms a cycle on $E(Q, A, \phi)$ for $a, b \in A$. Hence, each $\widetilde{\sigma}_{x_i}$ is the product of such $|A|$ cycles and $(-\widetilde{*}(x, a)) = \widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ consists of $p|A|$ disjoint cycles and these $p|A|$ disjoint cycles are just $|A|$ copies of σ_{x_j}.

Example 2.5. Let $(Q, *)$ be a finite quandle and A an finite abelian group. Let $\phi : Q \times Q \to A$ be a quandle 2-cocycle with 2-cocycle matrix $M = [\phi(x, y)] \in$

$\mathcal{M}_{|Q| \times |Q|}(A)$. Fix $x \in Q$ and suppose the cycle presentation $(-*x) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$, where $\sigma_{x_j} = [x_j \rightarrow x_j * x \rightarrow x_j *^2 x \rightarrow \cdots \rightarrow x_j *^{k-1} x]$. Since 2-cocycle matrix and the operation matrix of Q has the same size, here we call x-column of M by the column of the same order as x-column of the operation matrix of Q. Assume that the sum of all entries of x-column of M is zero in A, that is $\sum_{x_j \in Q} \phi(x_j, x) = 0$. Then, for any $a \in A$,

$$\left[(x_j, b) \rightarrow (x_j, b) \widetilde{*}(x, a) \rightarrow (x_j, b) \widetilde{*}^2(x, a) \rightarrow \cdots \rightarrow (x_j, b) \widetilde{*}^{k-1}(x, a) \right]$$

forms a closed cycle on $E(Q, A, \phi)$ for all $b \in A$, because the order of $\sum_{x_j \in Q} \phi(x_j, x)$ $= \sum_{i=0}^{|Q|} \phi(x_0 *^i x, x)$ for some $x_0 \in Q$ is zero in A. Therefore,

$$\widetilde{\sigma}_{x_j} = \prod_{b \in A} \left[(x_j, b) \rightarrow (x_j, b) \widetilde{*}(x, a) \rightarrow (x_j, b) \widetilde{*}^2(x, a) \rightarrow \cdots \rightarrow (x_j, b) \widetilde{*}^{k-1}(x, a) \right]$$

and $(- \widetilde{*} (x, a)) = \widetilde{\sigma}_{x_1} \widetilde{\sigma}_{x_2} \cdots \widetilde{\sigma}_{x_p}$ consists of $p|A|$.

For example, consider the quandle $Q(4, 1)$ on $\{0, 1, 2, 3\}$ with the following operation matrix.

$$T = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Here, we consider the 0-column and $(- * 0) = (123)$.

We give a 2-cocycle matrix M of a quandle 2-cocycle ϕ of $Q(4, 1)$ by $\mathbb{Z}_2$ and the operation matrix of the abelian extension $E(Q(4, 1), \mathbb{Z}_2, \phi)$.

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} (0,0) & (3,1) & (1,1) & (2,0) & (0,0) & (3,1) & (1,1) & (2,0) \\ (2,1) & (1,0) & (3,1) & (0,0) & (2,1) & (1,0) & (3,1) & (0,0) \\ (3,1) & (0,1) & (2,0) & (1,0) & (3,1) & (0,1) & (2,0) & (1,0) \\ (1,0) & (2,0) & (0,0) & (3,0) & (1,0) & (2,0) & (0,0) & (3,0) \\ \hline (0,1) & (3,0) & (1,0) & (2,1) & (0,1) & (3,0) & (1,0) & (2,1) \\ (2,0) & (1,1) & (3,0) & (0,1) & (2,0) & (1,1) & (3,0) & (0,1) \\ (3,0) & (0,0) & (2,1) & (1,1) & (3,0) & (0,0) & (2,1) & (1,1) \\ (1,1) & (2,1) & (0,1) & (3,1) & (1,1) & (2,1) & (0,1) & (3,1) \end{bmatrix}$$

One can see that each column vector of M satisfies the sum of values is zero in $\mathbb{Z}_2$. The inner automorphism

$$(- \widetilde{*} (0, 0)) = [(1, 0) \rightarrow (2, 1) \rightarrow (3, 0)] [(1, 1) \rightarrow (2, 0) \rightarrow (3, 1)].$$

Remark 2.6. The cycle presentation of inner automorphisms of an abelian extension of a quandle can be used to study the inner automorphism group of an abelian extension of a quandle. For the trivial extension shown in Example 2.4, the cycle presentation of inner automorphisms is given as $|A|$ copies of the cycle presentation of the underlying quandle. Indeed, one can see that two elements $(x, a), (y, b) \in E(Q, A, \phi_0)$ can be in the same cycle if and only if $a = b$. Therefore,

we get the result that the inner automorphism group of the trivial extension of a quandle coincides with the inner automorphism group of the underlying quandle, i.e. $\mathrm{Inn}E(Q, A, \phi) = \mathrm{Inn}Q$.

For example, we know that the dihedral quandle on $\mathbb{Z}_{2n}$ is isomorphic to the trivial extension of the dihedral quandle on $\mathbb{Z}_n$ (denoted R_n) for odd number n, so that $\mathrm{Inn}R_{2n} = \mathrm{Inn}R_n$, n:odd.

However, the inner automorphism group of the abelian extension $E(Q(4, 1), \mathbb{Z}_2, \phi)$, described in Example 2.5, is distinct from the inner automorphism group of $Q(4, 1)$ even though each inner automorphism has $p|A|$ disjoint cycles. In fact, it is known that $\mathrm{Inn}Q(4, 1) = A_4$ and $\mathrm{Inn}E(Q(4, 1), \mathbb{Z}_2, \phi) = SL_2(\mathbb{Z}_3)$.

Example 2.7. Finally, we provide an exercise in determining which quandles will be abelian extensions of other quandles by observing the cycle presentations of inner automorphisms.

In [1], the authors introduced sufficient conditions for partitioned tables to be quandle operation matrices. Here, we determine whether some quandle operation matrices introduced in [1] are abelian extensions or not.

1. Let Q be a quandle with the operation matrix T. Consider the block matrix whose four blocks are isomorphic to T up to translation. Obviously, it is the operation matrix for the abelian extension of Q by $\mathbb{Z}_2$ with the trivial 2-cocycle. In Example , there is the operation matrix for the trivial extension of the dihedral quandle $(\mathbb{Z}_3, *)$ by $\mathbb{Z}_2$ with the trivial 2-cocycle and one can see that its four blocks are isomorphic to the operation matrix of $(\mathbb{Z}_3, *)$.

2. Consider the partitioned matrix whose all sub-matrices consist of identity permutations, but one off-diagonal. By [1], it is a quandle operation matrix, for the non-trivial off-diagonal sub-matrix is a sequence of commutative permutations.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 4 & 4 \\ 4 & 4 & 4 & 3 & 4 & 3 & 3 \\ \hline 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

If it is an abelian extension of another quandle, we have cycle presentations containing some copies of cycles. However, we see that the matrix above is not an abelian extension of another quandle, as it has a non-trivial permutation $(012)(34)$ that cannot be the product of two copies.

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