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The Geometry of δ-Ricci-Yamabe Almost Solitons on Paracontact Metric Manifolds

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Abstract. In this article we study a δ-Ricci-Yamabe almost soliton within the framework of paracontact metric manifolds. In particular we study δ-Ricci-Yamabe almost soliton and gradient δ-Ricci-Yamabe almost soliton on K-paracontact and para-Sasakian manifolds. We prove that if a K-paracontact metric g represents a δ-Ricci-Yamabe almost soliton with the non-zero potential vector field V parallel to ξ, then g is Einstein with Einstein constant $-2n$. We also show that there are no para-Sasakian manifolds that admit a gradient δ-Ricci-Yamabe almost soliton. We demonstrate a δ-Ricci-Yamabe almost soliton on a (κ, μ)-paracontact manifold.

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1. Introduction

Paracontact geometry methods plays an important role in modern mathematics. In the sam way that almost contact manifolds extend almost Hermitian manifolds, the geometry of almost paracontact manifolds is a natural extension of almost para-Hermitian geometry. Over the last few years, the study of paracontact geometry has evolved from the mathematical formalism of classical mechanics (see [13, 21]). The concept of Ricci flow, is an evolution equation for metrics defined on connected almost contact metric manifolds whose automorphism groups have maximal dimensions.

Very recently, in [14], Güler and Crasmareanu studied Ricci-Yamabe flow of the type (α, β). A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton (abbreviated to RYS) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the Riemannain manifold (M^n, g), $n > 2$ is said to admit a (α, β)-Ricci-Yamabe soliton or simply a Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta)$ if it satisfies the equation

$$\mathcal{L}_V g + 2\alpha Ric_g + (2\lambda - \beta r)g = 0,$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along the vector field V, Ric_g is the Ricci tensor, r is the scalar curvature and λ, α, β are real scalars.

In [8], Dey et al. defined a δ-Ricci-Yamabe soliton (in short δ-RYS). A complete Riemannian manifold (M^n, g) is said to be a δ-Ricci-Yamabe almost soliton, denoted by $(M^n, g, V, \delta, \lambda)$, if there exists smooth vector field V on M^n, a soliton function $\lambda \in C^\infty(M^n)$ and a non-zero real valued function δ on M^n such that

$$\delta\mathcal{L}_V g + 2\alpha Ric_g + (2\lambda - \beta r)g = 0. \tag{1.1}$$

This soliton is called shrinking, steady or expanding according as λ is negative, zero or positive respectively. If the potential vector field V can be written as a gradient of a smooth function u on M^n, then the δ-Ricci-Yamabe almost soliton is called a gradient δ-Ricci-Yamabe almost soliton. In this case, (1.1) can be expressed as

$$\delta\nabla^2 u + \alpha Ric_g + (\lambda - \frac{1}{2}\beta r)g = 0, \tag{1.2}$$

where $\nabla^2 u$ be the Hessian of u. We denote this as (M^n, g, Du, λ). Now, the identity (1.2) can be written as

$$\delta Hess f + \alpha Ric_g + (\lambda - \frac{1}{2}\beta r)g = 0. \tag{1.3}$$

There are many papers that prove the existence of Ricci solitons and gradient Ricci solitons on paracontact manifolds. In particular, Calvaruso et al. [3] exhibited Ricci solitons on 3-dimensional almost paracontact manifolds. Ricci solitons and their generalizations have been well studied within the framework of contact and paracontact metric manifolds. See [1] for fundamental background, and say, [9] which list many recent related papers in it extensive references. Recently, Erken [11] demonstrated Yamabe solitons on 3-dimensional para-cosymplectic manifold and proved, for example, that the manifold is either η-Einstein or Ricci flat.

In [17, 19], Patra gave answers to the following important questions associated to almost Ricci, and almost Ricci–Bourguignon, solitons: *Under which conditions is a* (*gradient*) *Ricci almost soliton Einstein*? *...trivial*? and *Under which conditions is a* (*gradient*) *Ricci–Bourguignon almost soliton Einstein* (*trivial*) *on a paracontact metric manifold*?. It is natural to ask the same questions about more general solitons.

Question. *Under which conditions is a* (*gradient*) *δ-Ricci-Yamabe almost soliton on a paracontact metric manifold Einstein* (*trivial*)?

We find sufficient conditions under which a paracontact metric manifold admitting a δ-Ricci-Yamabe almost soliton or a gradient δ-Ricci-Yamabe almost soliton is Einstein (trivial). We prove the following.

Lemma 1. If a K-paracontact metric g is a δ-Ricci-Yamabe almost soliton, then

$$(\mathcal{L}_V\eta)(\xi) = -\eta(\mathcal{L}_V\xi) = \frac{1}{\delta}\{4n\alpha - (2\lambda - \beta r)\}. \tag{1.4}$$

Patra [17] proved that *"if a paracontact metric manifold endows a Ricci soliton with nonzero potential vector field V parallel to the Reeb vector field ξ and the Ricci operator commutes with paracontact structure ϕ, then the manifold is Einstein with Einstein constant $-2n$"*. Here, we generalize this result for δ-Ricci-Yamabe almost soliton and, removing the commutativity condition, prove that the potential vector field V being parallel to ξ is a sufficient condition under which a K-paracontact manifold admitting a δ-Ricci-Yamabe almost soliton is Einstein (trivial). So, we have the following.

Theorem 1. If K-paracontact metric g endows a δ-Ricci-Yamabe almost soliton with the non-zero potential vector field V is parallel to ξ, then g is Einstein with Einstein constant $-2n$. Moreover, V is a constant multiple of ξ.

After Theorem 1, we prove the following result.

Proposition 1. Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a para-Sasakian manifold. If the metric g represents a δ-Ricci-Yamabe almost soliton with the potential vector field V, then the following relation holds:

$$\begin{aligned}(\nabla_\xi\mathcal{L}_V\nabla)(\xi,\xi) &= (\beta r - 2\lambda + 4n\alpha)\eta(\nabla_\xi D\delta) + \beta\{\eta(\nabla_\xi Dr) + \xi(\xi(r))\xi \\ &- \xi(r)D\delta - \nabla_\xi Dr\} - 2\{\eta(\nabla_\xi D\lambda) + \xi(\xi(\lambda))\xi - \nabla_\xi D\lambda \\ &- \xi(\lambda)D\delta\} + (2\lambda - \beta r - 4n\alpha)\nabla_\xi D\delta.\end{aligned}$$

Next, we get results on K-paracontact manifold and para-Sasakian manifold whose metric endows a gradient δ-Ricci-Yamabe almost soliton. We state this as follows.

Theorem 2. Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a K-paracontact manifold. If the metric g represents a gradient δ-Ricci-Yamabe almost soliton, then M^{2n+1} satisfies either

$$(\nabla_\xi Q)V_1 + 2\phi QV_1 + 4n\phi V_1 = 0 \tag{1.5}$$

or $\alpha = 0$, that is, it becomes a gradient δ-Yamabe almost soliton, provided $\beta = 2$.

In [12], Ghosh proved that if a K-contact manifold endows a gradient Ricci almost soliton, then it is of constant scalar curvature. Recently, Patra [18] generalized this result and proved that if a K-contact manifold admits a non-trivial gradient Ricci almost soliton, then the manifold becomes an Einstein metric with constant scalar curvature $2n(2n+1)$. Here, we prove the nonexistence of a para-Sasakian metric g admitting a gradient Ricci–Yamabe almost soliton with a Ricci operator Q which commutes with a paracontact metric structure ϕ.

Theorem 3. There does not exist a para-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ with gradient δ-Ricci-Yamabe almost soliton.

As every para-Sasakian manifold is always K-paracontacto, this theorem also holds for K-paracontact manifolds.

Now, we turn our attention to a gradient δ-Ricci-Yamabe almost soliton on a (κ, μ)-paracontact manifold, and state the following results.

Lemma 5. If a (κ, μ)-paracontact manifold (dimension (2n+1)) with $\kappa > -1$ endows a gradient δ-Ricci-Yamabe almost soliton, then we have

$$\kappa(2 - \mu) = \mu(n + 1). \tag{1.6}$$

By virtue of Lemma 5 and Theorem 3, we can assert the following:

Theorem 4. If a (κ, μ)-paracontact manifold (dimension (2n+1)) with $\kappa > -1$ admits a gradient δ-Ricci-Yamabe almost soliton, then the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature -4.

The structure of this paper is the following. In Section 2, after a brief introduction, we discuss some preliminaries of paracontact metric manifolds. In Section 3, we examine δ-Ricci-Yamabe almost solitons on K-paracontact and para-Sasakian manifolds. Also, we show that if K-paracontact metric g represents δ-Ricci-Yamabe almost soliton with the non-zero potential vector field V is parallel to ξ, then g is Einstein with Einstein constant $-2n$. Section 4 deals with a gradient δ-Ricci-Yamabe almost solitons on K-paracontact and para-Sasakian manifolds, and proves that there does not exist such a manifold. In the last section, we study δ-Ricci-Yamabe almost solitons within the framework of (κ, μ)-paracontact manifold. Here, we prove that if a (κ, μ)-paracontact manifold with $\kappa > -1$ admits a gradient δ-Ricci-Yamabe almost soliton, then the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and a n-dimensional manifold of negative constant curvature -4.

2. Preliminaries

In this section, we discuss some definitions and identities of paracontact metric manifolds (for more details see [4, 5, 15, 23]). A dimensional smooth manifold M is said to be an *almost paracontact structure* (ϕ, ξ, η) if it endows a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(V_1) = V_1 - \eta(V_1)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0 \tag{2.1}$$

and there is a paracontact distribution $\mathcal{D} : q \in M \to \mathcal{D}_q \subset T_qM : \mathcal{D}_q = Ker(\eta) = \{x \in T_qM : \eta(x) = 0\}$ generated by η. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi V_1, \phi V_2) = -g(V_1, V_2) + \eta(V_1)\eta(V_2) \tag{2.2}$$

for all V_1, V_2 on M, then M has an almost paracontact metric structure (ϕ, ξ, η, g) and g is called a compatible metric. Notice that, since Eq. (2.2) holds any compatible metric g has signature $(n+1, n)$. The fundamental 2-form Φ of an almost paracontact metric structure (ϕ, ξ, η, g) is defined by $\Phi(V_1, V_2) = g(V_1, \phi V_2)$ for all vector fields V_1, V_2 on M. The manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called paracontact metric manifold, if $\Phi = d\eta$. Here, η is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$, ξ is its Reeb vector field and M is a contact manifold (see [4]). We define self-adjoint operators $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $l = R(., \xi)\xi$, where $\mathcal{L}_\xi$ is the Lie-derivative along ξ and R is the Riemannian curvature tensor of g on a paracontact metric manifold. The operators h and l satisfy [23]:

$$Tr_g h = 0, \quad Tr_g(h\phi) = 0, \quad h\xi = 0, \quad l\xi = 0, \quad h\phi = -\phi h. \tag{2.3}$$

The following results hold on a paracontact metric manifold [23]:

$$\nabla_{V_1}\xi = -\phi V_1 + \phi h V_1, \quad \nabla_\xi \xi = 0, \quad V_1 \in \chi(M), \tag{2.4}$$

$$\nabla_\xi h = -\phi + \phi h^2 - \phi l, \tag{2.5}$$

$$Ric_g(\xi, \xi) = g(Q\xi, \xi) = Trl = Tr(h^2) - 2n \tag{2.6}$$

$$(\nabla_{\phi V_1}\phi)\phi V_2 - (\nabla_{V_1}\phi)V_2 = 2g(V_1, V_2) - \eta(V_2)(V_1 - hV_1 + \eta(V_1)\xi) \tag{2.7}$$

for all V_1, V_2 on M^{2n+1}, where ∇ is the operator of covariant differentiation of g and Q denotes the Ricci operator given by $Ric_g(V_1, V_2) = g(\phi V_1, V_2)$ $\forall$ V_1, V_2 on M^{2n+1}. M is said to be a K-paracontact manifold if the vector field ξ is a killing (equivalently $h = 0$). On a K-paracontact manifold the following formula holds [23]:

$$\nabla_{V_1}\xi = -\phi V_1, \quad (\nabla_{V_1}\eta)V_2 = g(V_1, V_2) - \eta(V_1)\eta(V_2), \tag{2.8}$$

$$R(V_1, \xi)\xi = -V_1 + \eta(V_1)\xi, \tag{2.9}$$

$$Q\xi = -2n\xi \tag{2.10}$$

for any vector fields V_1, V_2 on M^{2n+1}. Moreover, from [23] we have $(\mathcal{L}_\xi g)(V_1, V_2) = 2g(V_1, \phi h V_2)$ and therefore, M is K-paracontact if and only if $\phi h = 0$.

A paracontact metric structure on M is said to be normal if the almost para-complex structure on $M \times R$ defined by

$$J(V_1, f d/dt) = (\phi V_1 + f\xi, \eta(V_1) d/dt),$$

where f is a real function on $M \times R$, is integrable. A normal paracontact metric manifold is said to be para-Sasakian. A para-Sasakian manifold is always a K-paracontact manifold. A 3-dimensional K-paracontact manifold is a para-Sasakian manifold [2], which may not be true in higher dimensions [16]. Equivalently, a paracontact metric manifold is said to para-Sasakian if [23]:

$$(\nabla_{V_1}\phi)V_2 = -g(V_1, V_2)\xi + \eta(V_2)V_1 \tag{2.11}$$

for any vector fields V_1, V_2 on M^{2n+1}. Further, on any para-Sasakian manifold [23]:

$$R(V_1, V_2)\xi = \eta(V_1)V_2 - \eta(V_2)V_1, \tag{2.12}$$

$$R(V_1, \xi)\xi = -V_1 + \eta(V_1)\xi \tag{2.13}$$

for any vector fields V_1, V_2 on M^{2n+1}.

We recall the following commutation formula from [22]

$$\begin{aligned}(\mathcal{L}_V \nabla_{V_3} g - \nabla_{V_3} \mathcal{L}_V g - \nabla_{[V, V_3]} g)(V_1, V_2) &= -g((\mathcal{L}_V \nabla)(V_3, V_1), V_2) \\ &\quad -g((\mathcal{L}_V \nabla)(V_3, V_2), V_1)\end{aligned}$$

for all vector fields V_1, V_2 on M^{2n+1}. By virtue of parallelism of the pseudo-Riemannian metric g, this formula yields

$$(\nabla_{V_3} \mathcal{L}_V g)(V_1, V_2) = g((\mathcal{L}_V \nabla)(V_3, V_1), V_2) + g((\mathcal{L}_V \nabla)(V_3, V_2), V_1) \tag{2.14}$$

for all vector fields V_1, V_2 on M^{2n+1}. We also recall the following from [10, p. 39]

$$(\mathcal{L}_V \nabla)(V_1, V_2) = \nabla_{V_1}\nabla_{V_2} V - \nabla_{\nabla_{V_1} V_2} V + R(V, V_1)V_2 \tag{2.15}$$

for any vector fields V_1, V_2, V on M^{2n+1}.

Let R be the Riemannian curvature tensor of the Levi-Civita connection ∇ of g, given by

$$R(V_1, V_2) = \nabla_{V_1}\nabla_{V_2} - \nabla_{V_2}\nabla_{V_1} - \nabla_{[V_1, V_2]}, \qquad V_1, V_2 \in \chi(M). \tag{2.16}$$

where $\chi(M)$ is the set of all vectors fields on M. On a paracontact metric manifold the following formula holds

$$\nabla_{V_1}\xi = V_1 - \eta(V_1)\xi - \phi h V_1 (\nabla_\xi \xi = 0) \tag{2.17}$$

for any V_1, $V_2 \in \chi(M)$,

$$R(V_1, V_2)\xi = \eta(V_1)(V_2 - \phi h V_2) - \eta(V_2)(V_1 - \phi h V_1) + (\nabla_{V_2}\phi h)V_1 - (\nabla_{V_1}\phi h)V_2 \tag{2.18}$$

for any vector fields V_1, $V_2 \in \chi(M)$.

The reading of nullity conditions on paracontact geometry is an attractive topic in paracontact geometry. In [6], Cappelletti-Montano et al. initiated the notion of (κ, μ)-paracontact structure. They defined a (κ, μ)-paracontact manifold as a paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(V_1, V_2)\xi = \kappa\{\eta(V_2)V_1 - \eta(V_1)V_2\} + \mu\{\eta(V_2)h'V_1 - \eta(V_1)h'V_2\} \tag{2.19}$$

for some real numbers (κ, μ). Many geometers have studied (κ, μ)-paracontact manifolds and attained several significant properties of these manifold (see [7, 20]). On a (κ, μ)-paracontact manifold one has [6]

$$h = 0 \Leftrightarrow h' = 0, h'^2 V_1 = (k+1)\phi^2 V_1, \tag{2.20}$$

$$h^2(V_1) = -(\kappa+1)[V_1 - \eta(V_1)\xi] \tag{2.21}$$

for $V_1 \in \chi(M)$. And also we have the following

$$R(\xi, V_1)V_2 = \kappa\{g(V_1, V_2)\xi - \eta(V_2)V_1\} - 2\{g(h'V_1, V_2)\xi - \eta(V_2)h'V_1\}, \tag{2.22}$$

$$QV_1 = -2nV_1 + 2n(\kappa+1)\eta(V_1)\xi - 2nh'(V_1), \tag{2.23}$$

$$r = 2n(\kappa - 2n), \tag{2.24}$$

$$(\nabla_{V_1}\eta)V_2 = g(V_1, V_2) - \eta(V_1)\eta(V_2) + g(h'V_1, V_2), \tag{2.25}$$

where V_1 and V_2 are any vector fields on M.

3. On δ-Ricci-Yamabe Almost Solitons

In this section, we prove the results we stated about δ-Ricci-Yamabe almost solitons on K-paracontact and para-Sasakian manifolds. We begin with the following.

Proof of the Lemma 1. In light of identity (2.10), the soliton equation (1.1) gives

$$(\mathcal{L}_V g)(V_1, \xi) = \frac{1}{\delta}\{4n\alpha - (2\lambda - \beta r)\}\eta(V_1). \tag{3.1}$$

Taking the Lie differentiation of $\eta(V_1) = g(V_1, \xi)$ by the vector field V, we achieve $(\mathcal{L}_V \eta)(V_1) - g(\mathcal{L}_V \xi, V_1) = (\mathcal{L}_V g)(V_1, \xi)$. By using (3.1), we acquire

$$(\mathcal{L}_V \eta)(V_1) - g(\mathcal{L}_V \xi, V_1) = \frac{1}{\delta}\{4n\alpha - (2\lambda - \beta r)\}\eta(V_1). \tag{3.2}$$

The result then follows using (3.2) with $g(\xi, \xi) = 1$. □

Lemma 2. [17] *On a K-paracontact manifold $M^{2n+1}(\phi, \xi, \eta, g)$, we have*

$$(i)(\nabla_{V_1} Q)\xi = Q\phi V_1 + 2n\phi V_1,$$

$$(ii)(\nabla_\xi Q)V_1 = Q\phi V_1 - \phi Q V_1$$

for all vector fields V_1 on $M^{2n+1}(\phi, \xi, \eta, g)$.

Proof of the Theorem 1. Since the potential vector field V is parallel to ξ, i.e., $V = \sigma\xi$ for a non-zero smooth function σ on M, we acquire $\nabla_{V_1} V = V_1(\sigma)\xi - \sigma(\phi V_1)$ by the derivative of $V = \sigma\xi$ covariantly by $V_1 \in \chi(M)$ and using the identity (2.8). Thus, the equation (1.1) reduces to

$$\delta\{V_1(\sigma)\eta(V_2) + V_2(\sigma)\eta(V_1)\} + 2\alpha Ric_g(V_1, V_2) + (2\lambda - \beta r)g(V_1, V_2) = 0 \tag{3.3}$$

for all $V_1, V_2 \in \chi(M)$. Now, we insert $V_1 = V_2 = \xi$ into (3.3) and use fact (2.10) to infer $\xi(\sigma) = \frac{1}{2\delta}\{4n\alpha - (2\lambda - \beta r)\}$. Setting $V_2 = \xi$ in (3.3) and recalling (2.10), we get

$$V_1(\sigma) = \xi(\sigma)\eta(V_1), \qquad V_1 \in \chi(M)$$

and therefore, by (2.8), get

$$Hess_\sigma(V_1, V_2) = V_1(\xi(\sigma))\eta(V_2) - \xi(\sigma)g(\phi V_1, V_2), \quad V_1, V_2 \in \chi(M). \tag{3.4}$$

Since $Hess_\sigma$ is symmetric and ϕ is skew-symmetric, by (2.1) and (3.4), we get

$$\xi(\sigma)d\eta(V_1, V_2) = 0 \quad \forall V_1, V_2 \perp \xi,$$

as $d\eta(V_1, V_2) = g(V_1, \phi V_2)$. This exposes that $\xi(\sigma) = 0$, as $d\eta$ is a non-zero on M, hence, $\nabla\sigma = 0$. Hence, σ is constant on M. This simplifies the equation (3.4) to

$$2\alpha Ric_g(V_1, V_2) = -(2\lambda - \beta r)g(V_1, V_2) = -4n\alpha g(V_1, V_2), \quad V_1, V_2 \in \chi(M),$$

using $Q\xi = -2n\xi$ and hence (M, g) is an Einstein with Einstein constant $-2n$. This finishes the proof. □

Proof of the Proposition 1. Now, we use identities (1.1) and (2.14) to acquire

$$g((\mathcal{L}_V \nabla)(V_3, V_1), V_2) + g((\mathcal{L}_V \nabla)(V_3, V_2), V_1) =$$
$$-\tfrac{1}{\delta}[V_3(\delta)(\mathcal{L}_V g)(V_1, V_2) + 2\alpha(\nabla_{V_3} Ric_g)(V_1, V_2) - \{2V_3(\lambda) - \beta V_3(r)\}g(V_1, V_2)]$$

for all vector fields V_1, V_2, V_3 on M^{2n+1}. Interchanging cyclicly the roles of V_1, V_2 and V_3 in the upstairs equalization and with the straight enumeration we gain

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1,V_2),V_3) &= -\frac{1}{\delta}[2\alpha\{(\nabla_{V_1}Ric_g)(V_2,V_3)+(\nabla_{V_2}Ric_g)(V_1,V_3) \\ &- (\nabla_{V_3}Ric_g)(V_1,V_2)\}+V_1(\delta)(\mathcal{L}_Vg)(V_2,V_3) \\ &+ V_2(\delta)(\mathcal{L}_Vg)(V_1,V_3)-V_3(\delta)(\mathcal{L}_Vg)(V_1,V_2) \\ &+ \{2V_3(\lambda)-\beta V_3(r)\}g(V_1,V_2)-\{2V_1(\lambda) \\ &- \beta V_1(r)\}g(V_2,V_3)-\{2V_2(\lambda)-\beta V_2(r)\}g(V_1,V_3)] \end{aligned}$$

$\forall$ V_1, V_2, V_3 on M^{2n+1}. Recall the following from [23, Lemma 3.15]:

$$\begin{aligned} (\nabla_{V_3}Ric_g)(V_1,V_2) &= (\nabla_{V_1}Ric_g)(V_2,V_3)-(\nabla_{\phi V_2}Ric_g)(\phi V_1,V_3) \\ &- \eta(V_1)Ric_g(V_2,V_3)-2\eta(V_2)Ric_g(\phi V_1,V_3) \\ &- 2n\eta(V_1)g(\phi V_2,V_3)-4n\eta(V_2)g(\phi V_1,V_3). \end{aligned} \tag{3.5}$$

Using $(\nabla_{V_3}Ric_g)(V_1,V_2)=g((\nabla_{V_3}Q)V_1,V_2)$ and the identity (2.1) of Lemma 2, we find $\nabla_\xi Q = Q\phi-\phi Q = 2n\eta\otimes\xi$ after putting $V_3=\xi$ into (3.5). With this, Lemma 2 and substituting V_2 by ξ in (3.5) we can get

$$\begin{aligned} (\mathcal{L}_V\nabla)(V_1,\xi) &= -\frac{2\alpha}{\delta}(2n\eta(V_1)+4n\phi V_1)+\{\beta V_1(r)-2V_1(\lambda)\}\xi \\ &- \{(2\lambda-\beta r)\xi-4n\alpha\}V_1(\delta)+\{\beta\xi(r)-2\xi(\lambda)\}V_1 \\ &- \{2\alpha QV_1+(2\lambda-\beta r)V_1\}\xi(\delta)+\{(2\lambda-\beta r-4n\alpha)D\delta \\ &+ 2D\lambda-\beta Dr\}\eta(V_1) \end{aligned} \tag{3.6}$$

for all V_1 on M. Now, we taking the covariant differentiation of (3.6) by a vector field V_2 on M^{2n+1} and applying (2.8), (2.10) and (2.11) we obtain

$$\begin{aligned} (\nabla_{V_2}\mathcal{L}_V\nabla)(V_1,\xi) &+ (\mathcal{L}_V\nabla)(V_1,V_2)-\eta(V_2)(\mathcal{L}_V\nabla)(V_1,\xi) \\ &= -\frac{2\alpha}{\delta}\{2n(\nabla_{V_2}\eta)V_1+4n(\nabla_{V_2}\phi)V_1\} \\ &+ 2\alpha V_2(\delta)(2Q\phi V_1+4n\phi V_1)+\{\beta g(V_1,\nabla_{V_2}Dr) \\ &- 2g(V_1,\nabla_{V_2}D\lambda)\}-\{\beta V_1(r)-2V_1(\lambda)\}\phi^2V_2 \\ &+ V_1(\delta)\{2\lambda-\beta r-4n\alpha\}\phi^2V_2-(2\lambda-\beta r-4n\alpha) \\ &\quad g(V_1,\nabla_{V_2}D\delta)+\{\beta V_2(\xi(r))-2V_2(\xi(\lambda))\}V_1 \\ &+ \{(2V_2(\lambda)-\beta V_2(r))D\delta+(2\lambda-\beta r-4n\alpha)\nabla_{V_2}D\delta \\ &+ 2\nabla_{V_2}D\lambda-\beta\nabla_{V_2}Dr\}\eta(V_1)+\{(2\lambda-\beta r-4n\alpha)D\delta \\ &+ 2D\lambda-\beta Dr\}(g(V_1,V_2)-\eta(V_1)\eta(V_2)). \end{aligned} \tag{3.7}$$

Now, we plug $V_1=\xi$, $V_2=\xi$ in the equation (3.7) and using (2.1), (2.8) and (2.11) to achieve

$$\begin{aligned}(\nabla_\xi \mathcal{L}_V \nabla)(\xi,\xi) &= (\beta r - 2\lambda + 4n\alpha)\eta(\nabla_\xi D\delta) + \beta\{\eta(\nabla_\xi Dr) + \xi(\xi(r))\xi \\ &- \xi(r)D\delta - \nabla_\xi Dr\} - 2\{\eta(\nabla_\xi D\lambda) + \xi(\xi(\lambda))\xi - \nabla_\xi D\lambda \\ &- \xi(\lambda)D\delta\} + (2\lambda - \beta r - 4n\alpha)\nabla_\xi D\delta.\end{aligned}$$

This completes the proof. □

4. On Gradient Almost δ-Ricci-Yamabe Solitons

In this section, we take into account the gradient almost δ-Ricci-Yamabe solitons on paracontact metric manifolds.
Let a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ admits a gradient almost δ-Ricci-Yamabe soliton. Then the soliton equation (1.2) can be demonstrated as

$$\delta\nabla_{V_1}\nabla u + \alpha QV_1 + (\lambda - \frac{\beta r}{2})V_1 = 0 \tag{4.1}$$

for all $V_1 \in \chi(M)$ and hence the curvature tensor gained from (4.1) and (2.16) satisfies

$$\begin{aligned}\delta R(V_1,V_2)\nabla u &= \alpha\{(\nabla_{V_2}Q)V_1 - (\nabla_{V_1}Q)V_2\} + V_1(\lambda)V_2 \\ &- V_2(\lambda)V_1 - \frac{\beta}{2}\{V_1(r)V_2 - V_2(r)V_1\}.\end{aligned} \tag{4.2}$$

Proof of the Theorem 2. First, we take the covariant differentiation of (2.10) through the vector field $V_1 \in \chi(M)$, and apply (2.8) to yield

$$(\nabla_{V_1}Q)\xi = Q\phi V_1 + 2n\phi V_1. \tag{4.3}$$

Since ξ is killing, we have

$$\begin{aligned}0 &= (\mathcal{L}_\xi Q)V_1 \\ &= \mathcal{L}_\xi(QV_1) - Q(\mathcal{L}_\xi V_1) \\ &= [\xi, QV_1] - Q([\xi, V_1]) \\ &= \nabla_\xi(QV_1) - \nabla_{QV_1}\xi - Q(\nabla_\xi V_1 - \nabla_{V_1}\xi) \\ &= (\nabla_\xi Q)V_1 - \nabla_{QV_1}\xi + Q(\nabla_{V_1}\xi)\end{aligned}$$

It follows from (2.8) that $\nabla_\xi Q = Q\phi - \phi Q$. Now, we replace V_1 by ξ into identity (4.2) and then replace the scalar product with $V_1 \in \chi(M)$ to yield

$$\begin{aligned}\delta g(R(\xi,V_2)\nabla u, V_1) &= \alpha\{g(\phi QV_2,V_1) + 2ng(\phi V_2,V_1) + 2n\eta(V_1)\eta(V_2)\} \\ &+ \{\xi(\lambda) - \frac{\beta}{2}\xi(r)\}g(V_1,V_2) \\ &- \{V_2(\lambda) - \frac{\beta}{2}V_2(r)\}\eta(V_1).\end{aligned} \tag{4.4}$$

Now, by identity (4.3) and equation (2.8) we get

$$g((\nabla_{V_1}\phi)V_2, V_3) - g((\nabla_{V_2}\phi)V_1, V_3) = g(R(V_1, V_2)V_3, \xi).$$

Using the Bianchis's first identity, we achieve

$$g(R(\xi, V_3)V_2, V_1) = g((\nabla_{V_3}\phi)V_2, V_1), \qquad V_1, V_2, V_3 \in \chi(M).$$

We insert the above identity into (4.4) and $\xi(r) = 0$ (which holds since $\nabla_\xi Q = Q\phi - \phi Q$) to get

$$\begin{aligned}&\delta g((\nabla_{V_2}\phi)V_1, \nabla u) + \alpha\{g(\phi QV_2, V_1) + 2ng(\phi V_2, V_1) + 2n\eta(V_1)\eta(V_2)\}\\ &+\xi(\lambda)g(V_1, V_2) - \{V_2(\lambda) - \frac{\beta}{2}V_2(r)\}\eta(V_1) = 0. \qquad (4.5)\end{aligned}$$

Now setting $V_1 = \phi V_1$, and $V_2 = \phi V_2$ in (4.5) and eliminating (4.5) from the results expression, we get

$$\begin{aligned}&\delta\{g((\nabla_{\phi V_2}\phi)\phi V_1, \nabla u) - g((\nabla_{V_2}\phi)V_1, \nabla u)\} - \alpha g(Q\phi V_2 + \phi QV_2, V_1)\\ &-2\xi(\lambda)g(V_1, V_2) + V_2(\lambda - \frac{\beta}{2}r)\eta(V_1) + \xi(\lambda)\eta(V_1)\eta(V_2)\\ &-4n\alpha g(\phi V_2, V_1) = 0. \qquad (4.6)\end{aligned}$$

Herre we also used (2.10). The following formula for paracontact metric manifolds is from [23, Lemma 2.7]:

$$(\nabla_{\phi V_2}\phi)\phi V_1 - (\nabla_{V_2}\phi)V_1 = 2g(V_1, V_2)\xi - \eta(V_1)\{V_2 - hV_2 + \eta(V_2)\xi\}. \qquad (4.7)$$

Using (4.6) and (4.7), we can infer that

$$\begin{aligned}&2\xi(\delta u - \lambda)g(V_1, V_2) + V_2(\lambda - \frac{\beta r}{2} - \delta u)\eta(V_1)\\ &-\xi(\delta u - \lambda)\eta(V_1)\eta(V_2) = \alpha g(Q\phi V_2 + \phi QV_2, V_1) + 4n\alpha g(\phi V_2, V_1), \qquad (4.8)\end{aligned}$$

since $h = 0$ for K-paracontact manifold. At this point, placing V_2 by ξ in (4.2), we get

$$\begin{aligned}\delta R(V_1, \xi)\nabla u &= \alpha\{(\nabla_\xi Q)V_1 - (\nabla_{V_1}Q)\xi\} + V_1(\lambda)\xi\\ &- \xi(\lambda)V_1 - \frac{\beta}{2}\{V_1(r)\xi - \xi(r)V_1\},\end{aligned}$$

replacing the scalar product in the above result with ξ and using (2.9) and (4.3) we get

$$V_1(\lambda - u\delta - \frac{\beta r}{2}) = \xi(\lambda - u\delta)\eta(V_1), \qquad (4.9)$$

by $\nabla_\xi Q = Q\phi - \phi Q$. Let $\sigma = \lambda - u\delta - \frac{\beta r}{2}$. Equation (4.9) becomes $V_1(\sigma) = \xi(\sigma)\eta(V_1)$, for $V_1 \in \chi(M)$ as $\xi(r) = 0$. In this manner, by the argument in Section 3, we get that $\sigma = \lambda - u\delta - \frac{\beta r}{2}$ is constant on M. Using $\nabla_\xi Q = Q\phi - \phi Q$ which follows from (4.8), we get

$$\alpha\{g((\nabla_\xi Q)V_2, V_1) + 2g(\phi QV_2, V_1) + 4ng(\phi V_2, V_1)\} = 0.$$

This implies that $\alpha\{g((\nabla_\xi Q)V_2 + 2\phi QV_2 + 4n\phi V_2, V_1)\} = 0$. So either $\alpha = 0$ or $g((\nabla_\xi Q)V_2 + 2\phi QV_2 + 4n\phi V_2, V_1) = 0$, which completes the proof. □

Proof of the Theorem 3. On a para-Saskian manifold, a Ricci operator satisfies the following (see [23, Lemma 3.15])

$$QV_1 = \phi Q\phi V_1 - 2n\eta(V_1)\xi. \quad V_1 \in \chi(M) \tag{4.10}$$

With this and (2.1) we see that the Ricci operator Q deflects the paracontact structure ϕ.

On the other hand, para-Sasakian manifolds are K-paracontact. So one has $\nabla_\xi Q = Q\phi - \phi Q = 2n\eta \otimes \xi$, which it implies the contradiction $\eta \otimes \xi = 0$. This finishes the proof. □

By virtue of this formula $\nabla_\xi Q = Q\phi - \phi Q = 2n\eta \otimes \xi$, Theorem 3 gives a non-existence theorem.

5. δ-Ricci-Yamabe Almost Solitons on (κ, μ)-Paracontact Manifold

In this last section, we discuss the nullity agreements on paracontact geometry. In [5], Cappelletti-Montano et al. introduced the notion of (κ, μ)-paracontact structures. According to them a (κ, μ)-paracontact manifold is a paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(V_1, V_2)\xi = \kappa\{\eta(V_2)V_1 - \eta(V_1)V_2\} + \mu\{\eta(V_2)hV_1 - \eta(V_1)hV_2\} \tag{5.1}$$

for all $V_1, V_2 \in \chi(M)$ and for some real numbers (κ, μ). Equivalently, this equation can be written as

$$R(V_1, \xi)V_2 = \kappa\{\eta(V_2)V_1 - g(V_1, V_2)\xi\} + \mu\{\eta(V_2)hV_1 - g(hV_1, V_2)\xi\} \tag{5.2}$$

for all $V_1, V_2 \in \chi(M)$. On a (κ, μ)-paracontact manifold one has [5]:

$$h^2 = (\kappa + 1)\phi^2 \tag{5.3}$$

$$Q\xi = 2n\kappa\xi \tag{5.4}$$

Lemma 3. (see [5]). *In any (κ, μ)-paracontact manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the Ricci operator Q of M can be written as*

$$\begin{aligned} QV_1 &= [2(1-n) + n\mu]V_1 + [2(n-1) + \mu]hV_1 \\ &+ [2(n-1) + n(2\kappa - \mu)]\eta(V_1)\xi, \quad for \quad \kappa > -1 \end{aligned} \tag{5.5}$$

for any vector field V_1 on M^{2n+1}. Moreover, the scalar curvature of M is $2n(2(1-n) + \kappa + n\mu)$.

Lemma 4. (see [5]). *On a (κ, μ)-paracontact manifold $M^{2n+1}(\phi, \xi, \eta, g)$, we have*

$$(\nabla_\xi h)V_1 = -\mu\phi h V_1 \tag{5.6}$$

for any vector field V_1 in M^{2n+1}.

Proof of the Lemma 5. First, taking the covariant derivative of (5.4) through an arbitrary vector field V_1 on M and applying (2.4) we get

$$(\nabla_{V_1} Q)\xi = Q(\phi - \phi h)V_1 - 2n\kappa(\phi - \phi h)V_1. \tag{5.7}$$

With (1.2) this can be written as

$$\delta\nabla_{V_2} Du + \alpha Q V_2 + (\lambda - \frac{\beta r}{2})g = 0 \tag{5.8}$$

for all vector fields V_2 on M^{2n+1}. Using the foregoing equation in the famous manifestation of the curvature tensor $R(V_1, V_2) = [\nabla_{V_1}, \nabla_{V_2}] - \nabla_{[V_1, V_2]}$, we can easily derive

$$\delta R(V_1, V_2)Du = \alpha\{(\nabla_{V_2} Q)V_1 - (\nabla_{V_2} Q)V_1\} \tag{5.9}$$

for all V_1 and V_2 on M^{2n+1}. In this manner, taking the scalar product of (5.9) along ξ and making use of (5.3) and (5.7) leaves that

$$\begin{aligned} \delta g(R(V_1, V_2)Du, \xi) &= \alpha\{g((Q\phi + \phi Q)V_2, V_1) \\ &- g((Q\phi h + h\phi Q)V_2, V_1) - 4n\kappa g(\phi V_2, V_1)\}. \end{aligned} \tag{5.10}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (5.10) and noting that $R(\phi V_1, \phi V_2)\xi = 0$ (from (5.1)) and (2.1), we get

$$Q\phi V_1 + \phi Q V_1 + \phi Q h V_1 + hQ\phi V_1 - 4n\kappa\phi V_1 = 0. \tag{5.11}$$

Now, we put $V_1 = \phi V_1$ into (5.5) and use $\phi\xi = 0$ to obtain

$$Q\phi V_1 = [2(1-n) + n\mu]\phi V_1 + [2(n-1) + \mu]h\phi V_1.$$

Dy acting h on the last equation and making use of (2.1), (5.3) and $h\xi = 0$ leaves

$$hQ\phi V_1 = [2(1-n) + n\mu]h\phi V_1 + (\kappa + 1)[2(n-1) + \mu]\phi V_1.$$

In addition, operating ϕ on (5.5) and using $\phi\xi = 0$, we get

$$\phi Q V_1 = [2(1-n) + n\mu]\phi V_1 + [2(n-1) + \mu]\phi h V_1.$$

Now, we replace V_1 by hV_1 in the foregoing equation and use (5.3) to yield

$$\phi Q h V_1 = [2(1-n) + n\mu]\phi h V_1 + (\kappa + 1)[2(n-1) + \mu]\phi V_1.$$

Applying the last four equations in (5.9) and also using $\phi h = -h\phi$ we obtain (1.6). This completes the proof. □

Proof of the Theorem 4. First, we substitute ξ for V_1 in (5.10) and use the identity (5.4) and $h\xi = \phi\xi = 0$ to acquire $\delta g(R(\xi, V_2)\xi, Du) = 0$. With this and (5.3) we get

$$\kappa\{Du - (\xi u)\xi\} + \mu h Du = 0, \tag{5.12}$$

where we have used $g(V_1, Du) = V_1 u$. Now, we take a covariant differentiation of the equation (5.12) by ξ and use the relation (5.6), $\nabla_\xi \xi = 0$ to achieve

$$\kappa\{\nabla_\xi Du - \xi(\xi u)\xi\} + \mu\{\mu h\phi Du + h(\nabla_\xi Du)\} = 0. \tag{5.13}$$

By equations (5.5) and (5.8), we have

$$\delta \nabla_\xi D(\lambda - \frac{\beta r}{2}) = (2n\kappa\alpha + \lambda - \frac{\beta r}{2})\xi \tag{5.14}$$

and also

$$\delta \xi(\xi u) = 2n\kappa\alpha + (\lambda - \frac{\beta r}{2}). \tag{5.15}$$

Making use of (5.14) and (5.15) in (5.13) and using $\phi\xi = 0$, we have $\mu^2 h\phi Du = 0$. Applying this to ϕ and using (2.1) we get $\mu^2 h Du = 0$. By the operation of h and the use of (2.1) and (5.3) we get

$$\mu^2(\kappa + 1)(Du - (\xi u)\xi) = 0.$$

For $\kappa > -1$, either (i) $\mu = 0$ or (ii) $\mu \neq 0$.

Case (i). In this case, as $\kappa > -1$ it follows from (1.6) that $\kappa = 0$. Hence $R(V_1, V_2)\xi = 0$ for any vector fields $V_1, V_1 \in \chi(M)$, and therefore M is the product of a flat $(n+1)$-dimensional manifold of negative constant curvature -4 (see [24, Theorem 3.3]).

Case (ii). This case yields $Du = (\xi u)\xi$. We differentiate this along with an arbitary vector field V_1 together with (2.1) to acquire

$$\nabla_{V_1} Du = V_1(\xi u)\xi - (\xi u)(\phi V_1 - \phi h V_1).$$

As $g(\nabla_{V_1} D(\lambda - \frac{\beta r}{2}), V_2) = g(\nabla_{V_2} D(\lambda - \frac{\beta r}{2}), V_1)$, the last equation gives

$$V_1(\xi u)\eta(V_2) - V_2(\xi u)\eta(V_1) + (\xi u)d\eta(V_1, V_2) = 0.$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 and using $\phi\xi = 0$ we find $\xi u = 0$. We apply $d\eta \neq 0$ on M^{2n+1}. Then, $Du = 0$, i.e., u is constant and consequently (5.8), (5.14) and (5.15) yield $Ric_g = (\lambda - \frac{\beta r}{2})g = 2n\kappa\alpha$, i.e., M^{2n+1} is an Einstein. This gives

$r = 2n\kappa\alpha(2n+1)$. In addition Lemma 3 yields $r = 2n\{2(1-n) + \kappa\alpha + n\mu\}$. Combining both we have

$$n\mu = 2(n\kappa\alpha + n - 1). \tag{5.16}$$

Now, using (5.16) and $Ric_g = 2n\kappa\alpha g$ in (5.5) we get $2(n-1) + \mu = 0$. Thus (1.6) yields $\kappa = \frac{1-n^2}{n}$, a contradiciton. This finishes the proof. □

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