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On the Characterization of Conformally Flat Weakly Einstein Finsler Metrics

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Abstract. In this paper, we prove that every weakly Einstein slope metric, which is conformally flat on a manifold M of dimension $n \geq 3$, is either a locally Minkowski metric or a Riemannian metric. We also prove the same result for conformally flat weakly Einstein Kropina metrics.

1. Introduction

The study of conformal properties has proven to be highly significant, encompassing both physical and geometrical domains. The theory of conformal transformations deserves extra attention because of Weyl's famous theorem [13] which indicates that the conformal and projective properties of a Finsler metric determine its metric relationships uniquely. Along with the geometrical importance in the general theory of relativity, it has been observed that conformal transformations preserve structure up to time orientation, and light-like geodesics up to parameterization. In Riemannian conformal geometry, there are several significant local and global discoveries that contribute to a deeper understanding of Riemann manifolds. For example, Poincar*é* metric on $\mathbb{B}^n$ is a conformally flat Riemannian metric of curvature $K = -1$.

Finsler geometry explores the impact of external forces, such as wind and current, and has led to extensive investigations on killing, homothetic, and conformal vector fields. Conformal transformation on M gives rise to conformal vector fields. The conformal transformations between two Finsler metrics preserve Ricci curvature, Landsberg curvature, mean Landsberg curvature, and S-curvature were

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established by S. Bacso [3]. Kang [11] affirmed that any conformally flat Randers metric of scalar flag curvature is projectively flat and such metrics were characterized entirely. Recently, in [14], authors characterized the conformal transformations between two non-Riemannian general (α, β)-metrics.

Two Finsler metrics F and $\tilde{F}$ are said to be conformally related, if they satisfy $F = e^{k(x)}\tilde{F}$, where $k(x)$ is a scalar function on M. Also, if $\tilde{F}$ is a Minkowski metric then F is called a conformally flat Finsler metric. Conformal fields on Riemann-Finsler spaces contain all killing fields and homothetic fields. Conformal changes are more significant, whenever the conformal factor is constant, i.e., homothetic because this leaves geodesics invariant. As a matter of fact, for dimension, $n \geq 3$, the Beltrami theorem states that the necessary and sufficient condition for Riemannian metric to be projectively flat is that it is of constant curvature. This implies conformal flatness, whereas its Finsler analogous does not hold.

In [2] Asanov proved that tangent Minkowski spaces of special-relativistic Finsler space $\mathcal{F}_{SR}$ are conformally flat. This motivated an attractive way to propose Finslerian extension of electromagnetic field equations. Many Finsler geometers have explored the theory of conformal transformation of the Finsler metrics. In [9] Hashuiguchi initiated the study of Finsler conformal metrics and gave geometrical meaning to it. In [10] Hashuiguchi and Ichijyō introduced a conformally invariant linear connection for an (α, β)-metric, and based on their connection, they provided a condition that Randers metric is conformally flat.

A Finsler metric F on a manifold M is called weakly Einstein metric, if its Ricci curvature satisfies $\mathbf{Ric}= (n-1)(3\theta/F+\sigma)F^2$, where $\theta = t_i(x)y^i$ is a 1-form and $\sigma(x)$ is a scalar function. A Finsler metric F is known as an Einstein metric, if $\theta = 0$. Every Riemannian surface is Einstein, although not necessarily of Ricci constant, according to the definition of Einstein metrics. In [5], the Schur-type lemma assures that every Einstein Riemannian metric on a manifold of dimension $n \geq 3$, is Ricci constant. Precisely, for $n = 3$, necessary and sufficient for a Riemannian metric to be Einstein is that its sectional curvature is constant. Chen and Cheng [6] established a significant result stating that conformally flat weakly Einstein (α, β)-metrics can only be either Riemannian or locally Minkowski metrics. Recently, conformally flat weakly Einstein exponential metric, cubic metric, and the fourth root have been studied in [17, 18, 19].

In this paper, firstly we prove an important result that a weakly Einstein conformally flat slope metric is Ricci flat. Consequently, this result is used to prove that any weakly Einstein conformally flat slope metric is either a Riemannian or a locally Minkowski metric. In section 4, we prove similar results for weakly Einstein conformally flat Kropina metric.

2. Preliminaries

In this section, we give some definitions and results related to Finsler spaces. For more elaborate understanding related to Finsler spaces and conformal geometry, refer [4, 8, 15].

Definition 2.1. ([15]) Let M be a connected (smooth) manifold. A Finsler metric on M is a non-negative function on tangent bundle $F : TM \to [0, \infty)$ which satisfies:

1. Positive homogeneity, i.e., $F(x, \lambda y) = \lambda F(x, y),\ \forall \lambda > 0$;
2. F is smooth on slit tangent bundle $TM \backslash \{0\}$;
3. Strong convexity: The $n \times n$ Hessian matrix formed by

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) = \left[\frac{1}{2} F^2\right]_{y^i y^j},$$

is positive definite on slit tangent bundle $TM/\{0\}$.

The space (M, F) is called Finsler space. Finsler geometry encompasses several important quantities that vanish in the Riemannian case, such as Cartan torsion, S-curvature, Landsberg curvature, etc. Let $y \in T_xM$, be a non-zero vector, the the Cartan torsion $C_y = C_{ijk} dx^i \otimes dx^j \otimes dx^k : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{4} [F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Suppose $y \in T_xM_0$, define $I_y : T_xM \to \mathbb{R}$ by

$$I_y(u) = \sum_{i=1}^{n} g^{ij} C_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is standard basis for T_xM at $x \in M$. The family $I := \{I_y\}_{y \in TM_0}$ is called mean Cartan torsion. Mean Cartan torsion for an (α, β)-metric is given in Lemma 3.2.

The class of (α, β)-metrics forms a rich class of Finsler metrics. It has also been discovered that (α, β)-metrics have vital applications in physics, biology, and ecology [1]. The study of (α, β)-metrics can help us better explore Finsler's geometric qualities. Hence, these metrics are worth studying deeply.

Definition 2.2. ([1]) Let $F = \alpha\phi(s); s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $\beta = b_i(x) y^i$ is a 1-form on an n-dimensional manifold with $||\beta|| < b_0$. Then F is Finsler metric if and only if following conditions are satisfied:

$$\phi(s) > 0,\ \phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0\ \forall\ |s| \leq b < b_0.$$

Now, we discuss the geometrical motivation behind slope metric and Kropina metric. Suppose a person is walking on the surface S with speed v and gravity of magnitude g along a path making an angle ϵ with sea level. S is considered to be embedded in Euclidean space $\mathbb{R}^3$ with parametrization $(x, y) \to (x, y, z = f(x, y))$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Using Okubo's method [1] we get slope metric as

$$F(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{v\alpha - \frac{g}{2}\beta},$$

where

$$\alpha = \sqrt{(1+{f_x}^2)\dot{x}^2 + 2f_xf_y\dot{x}\dot{y} + (1+{f_y}^2)\dot{y}^2},$$

$$\beta = f_x\dot{x} + f_y\dot{y}.$$

For the sake of simplicity, we take $v = \frac{g}{2}$, and obtain the usual form of slope metric $F = \frac{\alpha^2}{\alpha-\beta}$. See [16] for more details for slope metric.

Kropina metrics are non regular (α,β)-metrics where $\phi(s) = \frac{1}{s}$, i.e., $F = \frac{\alpha^2}{\beta}$. The concept is proposed by Russian physicist V. K. Kropina [12]. Despite having singularities ($\beta = 0$), it is useful in the Lagrangian function's representation of the general dynamic system. Suppose an open sea is represented as a Riemannian manifold (M,h) under the influence of wind $W = W^i\frac{\partial}{\partial x^i}$ such that $h(W,W) = 1$. The singular solution of Zermelo navigation problem in this case was found to be geodesics of Kropina metric [20]. Since both metrics serve as vital tools for analyzing time-minimization problems, further study on them is highly worthwhile.

The fundamental tensor of $F = \alpha\phi(s)$ is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

where,

$$\alpha_i := \alpha^{-1}a_{ij}y^j, \quad \rho := \phi(\phi - s\phi^{'}), \quad \rho_0 := \phi\phi^{''} + \phi^{'}\phi^{'},$$

$$\rho_1 := -s(\phi\phi^{''} + \phi^{'}\phi^{'}) + \phi\phi^{'}, \quad \rho_2 := s\{s(\phi\phi^{''} + \phi^{'}\phi^{'}) - \phi^{'}\phi^{'}\}.$$

Let us suppose

$$\delta := \frac{\rho_0 - \epsilon^2\rho_2}{\rho}, \quad \epsilon := \frac{\rho_1}{\rho_2}, \quad \tau := \frac{\delta}{1+\delta b^2},$$

By some computations, we get

$$g^{ij} := \rho^{-1}\{a^{ij} - \tau b^i b^j - \eta Y^i Y^j\},$$

where

$$\eta := \frac{\mu}{1+Y^2\mu}, \;\; \mu := \frac{\rho_2}{\rho}, \;\; Y := \sqrt{1+(\lambda+\epsilon)s+\lambda\epsilon b^2}, \;\; Y^i := \frac{y^i}{\alpha}+\lambda b^i, \;\; \lambda := \frac{\epsilon-\delta s}{1+\delta b^2}.$$

Next, geodesic coefficients of an (α,β)-metric is defined as follows:

Definition 2.3. ([1]) The general formula for geodesic spray coefficients of an (α,β)-metric F is

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \Theta\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i,$$

where G^i_α denote the spray coefficients of α.

$$Q := \frac{\phi'}{\phi - s\phi'}, \Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}, \Psi := \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}.$$

Let us define some useful notations related to Levi-Civita connection of α. For an (α, β)-metric, define $b_{i;j}\theta^j := db_j - b_j\theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form α. Let us put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

$$r_j := b^i r_{ij}, \quad r := b^i b^j r_{ij}, \quad r_0 := r_j y^j, \quad s_j := b^i s_{ij}, \quad s_0 := s_j y^j$$

$$r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^i y^j, \quad s_{i0} := s_{ij}y^j, \quad s^i_j := a^{im}s_{mj}, \quad r^i_j := a^{im}r_{mj},$$

$$q_{ij} := r_{im}s^m_j, \quad t_{ij} := s_{im}s^m_j, \quad q_j := b^i q_{ij}, \quad t_j := b^i t_{ij}.$$

Definition 2.4. Two Finsler metrics F and $\tilde{F}$ on a manifold M are said to be conformally related if $\tilde{F} = e^{c(x)}F$, where $c := c(x)$ is a scalar function on M, called the conformal factor.

Remark 2.5. A Finsler metric which is conformally related to Minkowski metric is called conformally flat Finsler metric.

Definition 2.6. Let (M, F) be an n-dimensional smooth Finsler manifold. Let ϕ be a diffeomorphism on M. Let $\phi_* : T_xM \to T_{\phi(x)}M$ be tangent map at point x. Then ϕ is called conformal transformation on M, if it satisfies

$$F(\phi(x), \phi_*(y)) = e^{2c(x)}F(x, y),$$

where $y \in T_xM$ and $c = c(x)$ is a function on M, called conformal factor. Now, we define a linear transformation $\mathbf{R}_y : T_xM \to T_xM$ defined as $\mathbf{R}_y(u) := R^i_k(y)u^k \frac{\partial}{\partial x^i}$ and

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Also, $\mathbf{R}_y$ satisfies homogeneity condition as $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y, \quad \forall\lambda > 0$.

This family $\mathbf{R} := \{\mathbf{R_y}\}_{\mathbf{y \in TM_0}}$ is known as Riemann curvature. The Ricci curvature is defined as the trace of Riemann curvature i.e., $\mathbf{Ric}(x, y) := R^m_m(x, y)$.

A Finsler metric F on an n-dimensional manifold M is called weakly Einstein metric, if Ricci curvature satisfies:

$$\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2, \tag{2.1}$$

where $\theta = t_i(x)y^i$ is a 1-form and $\sigma = \sigma(x)$ is a scalar function on M. If $\theta = 0$, then F is known as Einstein metric. Also, if $\mathbf{Ric} = 0$, we say F is Ricci flat.

3. On Weakly Einstein Slope Metric

Theorem 3.1. *Let $F = F(x, y)$ be slope metric which is conformally flat on a manifold M of dimension $n \geq 3$. Suppose that F is weakly Einstein metric, i.e.,*

$$Ric = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2, \tag{3.1}$$

where $\theta = t_i(x)y^i$ is 1-form and $\sigma = \sigma(x)$ is a scalar function on M. Then, F is either Riemannian metric or a locally Minkowski metric.

In order to prove our main Theorem 3.1, we recall some lemmas. Firsly, we calculate Ricci curvature of F for which we need to compute mean Cartan torsion of an (α, β)-metric given by following:

Lemma 3.2. ([7]) *For an (α, β)-metric $F = \alpha(\beta/\alpha)$, the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi^{'})h_i,$$

where

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ^{'}) - (b^2 - s^2)(1 + sQ)Q^{''},$$

$$\Delta := 1 + sQ + (b^2 - s^2)Q^{'}, h_i := b_i - \alpha^{-1}sy^i$$

and $y_i := a_{ij}y^j$.

Suppose $F = \alpha\phi(s)$, $s = \beta/\alpha$ is a conformally flat (α, β)-metric on a manifold M. Then, there exists a locally Minkowski metric $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$, which satisfy $F = e^k\tilde{F}$, where $k = k(x)$ is a scalar function. We consider the following notations to determine Ricci curvature of a (α, β)-metric.

$$k_i := \frac{\partial k}{\partial x^i}, \quad k_0 := k_i y^i, \quad k_{ij} := \frac{\partial^2 k}{\partial x^i \partial x^j}, \quad k^i_{\tilde{\alpha}} := \tilde{a}^{ij}k_j, \quad k_{00} := k_{ij}y^i y^j,$$

$$||\nabla k||^2_{\tilde{F}} := \tilde{g}^{ij}k_i k_j, \quad \tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j, \quad f := k_i b^i, \quad f_1 := k_{ij}\tilde{b}^i y^j, \quad f_2 := k_{ij}\tilde{b}^i\tilde{b}^j.$$

Next, we recall the Lemma from [6]:

Lemma 3.3. ([6]) *Let $F = \alpha\phi(s), s = \beta/\alpha$ be a conformally flat (α, β)-metric on a manifold M, i.e., there exists a locally Minkowski metric $\tilde{F} = \tilde{\alpha}\phi(\tilde{s}), s = \beta/\alpha$, such that $F = exp(k)\tilde{F}$, where $k = k(x)$ is a scalar function on M. Then, the Ricci curvature of F is given by:*

$$\mathbf{Ric} = c_1||\nabla k||^2_{\tilde{\alpha}}\tilde{\alpha}^2 + c_2 k_0^2 + c_3 k_0 f\tilde{\alpha} + c_4 f^2\tilde{\alpha} + c_5 f_1\tilde{\alpha} + c_6\tilde{\alpha}^2 + c_7 k_{00} \tag{3.2}$$

where,

$$c_6 := c_{61}\tilde{a}^{ij}k_{ij} + c_{62}f_2,$$

$$c_{61} := -\frac{\phi}{\phi - s\phi'}, \quad c_{62} := \frac{\phi\phi^{''}}{(\phi - s\phi')[(\phi - s\phi') + (b^2 - s^2)\phi'']}.$$

All the coefficients $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 are dependent on s only and independent of $\tilde{\alpha}, k_0, k_{00}, f_1, f, f_2$ and $\tilde{a}^{ij}k_{ij}$.

Next, we prove the following lemma:

Lemma 3.4. *Let $F = \alpha\phi(s), s = \beta/\alpha$, be weakly Einstein conformally flat slope metric on a manifold M, of dimension $n \geq 3$. Then,* **Ric**=*0.*

Proof. In order to prove Lemma 3.4, let us suppose F is conformally flat weakly Einstein slope metric on a manifold M. If $\tilde{b} = 0$, then $F = e^k\tilde{\alpha}$ is Riemannian metric. Next, we assume $\tilde{b} \neq 0$, Using equations 3.1 and 3.2, we get that

(3.3)
$$(n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2 = c_1||\nabla k||^2_{\tilde{\alpha}}\tilde{\alpha}^2 + c_2k_0^2 + c_3k_0f\tilde{\alpha} + c_4f^2\tilde{\alpha} + c_5f_1\tilde{\alpha} + c_6\tilde{\alpha}^2 + c_7k_{00}$$

Let us suppose
$$\chi_1 := (2s-1), \quad \chi_2 := (1 - 3s + 2b^2),$$
with the help of Maple, we obtain the coefficients of equation 3.3 as follows:

$$c_1 := -\frac{1}{2}\frac{\tilde{c_1}}{\chi_1^3\chi_2^2\phi}, \quad c_2 := -\frac{1}{4}\frac{\tilde{c_2}}{\chi_1^3\chi_2^4\phi}, \quad c_3 := \frac{\tilde{c_3}}{4\chi_2^4\chi_1^3\phi}, c_4 := \frac{\tilde{c_4}}{4\chi_2^4\chi_1^3\phi}, c_5 := \frac{\tilde{c_5}}{4\chi_2^4\chi_1^3\phi},$$
$$c_6 := \frac{\tilde{c_6}}{\chi_1\chi_2}, c_7 := \frac{\tilde{c_7}}{2\chi_1\chi_2^2}.$$

In order to simplify the calculations, consider an orthonormal basis at x with respect to $\tilde{\alpha}$ such that
$$\tilde{\alpha} = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \tilde{\beta} = \tilde{b}y^1.$$

Next, we take the coordinate transformations as follows: $\Psi : (s, u^A) \to (y^i)$ defined as
$$y^1 = \frac{s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad \text{where, } \bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}.$$

Here, index conventions are $1 \leq i, j, ..., \leq n$ and $2 \leq A, B, ..., \leq n$. Hence, we have
$$\tilde{\alpha} = \frac{\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}, \quad \tilde{\beta} = \frac{\tilde{b}s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}.$$

Thus,
$$F = e^{k(x)}\tilde{\alpha}\phi(s) = e^{k(x)}\frac{\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\tilde{\alpha}\phi(s), \quad s = \frac{\tilde{\beta}}{\tilde{\alpha}}, \tag{3.4}$$

Further, in this coordinate system we have

$$\theta = \frac{t_1 s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha} + \bar{t}_0, \quad f = k_1\tilde{b}, \quad f_1 = \frac{\tilde{b}sk_{11}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha} + \tilde{b}\bar{k}_{10}, \quad f_2 = k_{11}\tilde{b}^2,$$

$$k_0 = \frac{k_1 s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha} + \bar{k}_0, \; k_{00} = \frac{k_{11}s^2}{\tilde{b}^2 - s^2}\bar{\alpha}^2 + \frac{2\bar{k}_{10}s}{\sqrt{\tilde{b}^2 - s^2}}\tilde{\alpha} + \bar{k}_{00},$$

where

$$\bar{t}_0 := \sum_{A=2}^{n} t_A y^A, \quad \bar{k}_0 = \sum_{A=2}^{n} k_A y^A, \quad \bar{k}_{10} := \sum_{A=2}^{n} k_{1A} y^A, \quad \bar{k}_{00} := \sum_{A,B=2}^{n} k_{AB} y^A y^B.$$

Simplying equation 3.3 using 3.4, we get

(3.5)
$$\frac{(n-1)e^k(3t_1\tilde{b}s + \sigma e^k\tilde{b}^2\phi)}{\tilde{b}^2 - s^2} = [c_1\tilde{b}||\nabla k||^2_{\tilde{\alpha}} + c_2k_1^2s^2 + c_3\tilde{b}^2k_1^2s + c_4k_1^2\tilde{b}^4 + c_5k_{11}\tilde{b}^2s$$
$$+(c_{61}\delta^{ij}k_{ij} + c_{62}k_{11}\tilde{b}^2)\tilde{b}^2 + c_7k_{11}s^2]\frac{\tilde{\alpha}^2}{\tilde{b}^2 - s^2} + c_2\bar{k}_0^2 + c_7\bar{k}_{00}$$

(3.6) $$3(n-1)e^k t_A\tilde{b}\phi = (2c_2s + c_3\tilde{b}^2)k_1k_A + (c_5\tilde{b}^2 + 2c_7s)k_{1A}.$$

On taking $\phi(s) := 1/(1-s), \; s = \beta/\alpha$, on using Maple to multiply 3.5, with $4(b^2 - s^2)\chi_1^3\chi_2^4\phi^2$, we get the following equation:

(3.7) $$\mathcal{A}_{14}s^{14} + \mathcal{A}_{13}s^{13} + ... + \mathcal{A}_0,$$

where $\mathcal{A}_i$ are polynomials, especially we have where $\mathcal{A}_{14} = (-2304nk_1\alpha + 3168k_1\alpha)$, $\mathcal{A}_{13} = (-21936k_1\alpha - 4992b^2k_1\alpha - 3456nk_{10}\alpha + 3840nb^2k_1\alpha + 3168k_0\sqrt{b^2 - s^2} + 5184k_{10}\alpha - 2304nk_0\sqrt{b^2 - s^2})$. As the right-hand side of 3.5 is a rational polynomial with respect to s, whereas the left-hand side is an irrational polynomial with respect to s. Hence, we get $\sigma = 0, t_1 = 0$. Similarly, from 3.6, we get $t_A = 0$, which implies $\theta = 0$, consequently, we get from equation 3.1 that $\mathbf{Ric} = 0$. This completes the proof that weakly Einstein slope metrics are Ricci flat. □

Now we are ready to prove Theorem 3.1:

Proof. Suppose F be an (α, β)-metric expressed as $F = \alpha\phi(s), s = \beta/\alpha$ on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Two Finsler metrics F and $\tilde{F}$ are conformal to each other on an n-dimensional manifold M if and only if there exists a scalar function $k = k(x)$ such that $F(x, y) = e^k\tilde{F}(x, y)$. It is simple to check that $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ is also an (α, β)-metric, where $\tilde{\alpha} = e^{-k(x)}\alpha, \tilde{\beta} = e^{-k(x)}\beta$. Take $\tilde{\alpha} = \sqrt{\tilde{a}_{ij}(x)y^iy^j}, \beta = \tilde{b}_i(x)y^i$ with $\tilde{a}_{ij} = e^{-2k(x)}a_{ij}, \; \tilde{b}_i = e^{-k(x)}b_i$.

A Finsler metric $F = F(x, y)$ on a manifold M is called a conformally flat metric if there exists a locally Minkowski metric $\tilde{F}$, such that $F = e^{k(x)}\tilde{F}$, where $k = k(x)$ is a scalar function on M.

If we take $\tilde{b} = 0$, we get that $F = e^k\tilde{\alpha}$ is a Riemannian metric. Without loss of generality, we assume that F is a non-Riemannian metric. Hence, we are going to assume $\tilde{b} \neq 0$.

Using the proof of the above lemma, we have $\sigma = t_i = 0$. Hence, from equation 3.5 there exists a function $\eta := \eta(s)$ such that

$$c_2k_Ak_B + c_7k_{AB} = \eta(s)\delta_{AB}. \tag{3.8}$$

On taking $A \neq B$, equation 3.8 takes form

$$c_2k_Ak_B + c_7k_{AB} = 0. \tag{3.9}$$

On multiplying equation 3.9 with $4\chi_1^3\chi_2^4$, using Maple we get an identity with respect to s in the form

$$\mathcal{P}_9s^9 + \mathcal{P}_8s^8 + ... + \mathcal{P}_1s + \mathcal{P}_0 = 0, \tag{3.10}$$

where coefficients $\mathcal{P}_i$ are collected as follows:

$$\mathcal{P}_9 = (2304n - 3168)k_Ak_B, \tag{3.11}$$

$$\mathcal{P}_8 = -2592k_{AB} + 4992k_Ak_Bb^2 + 15600k_Ak_B - 11136nk_Ak_Bb^2 + 1728nk_{AB} \tag{3.12}$$

and all other coefficients $\mathcal{P}_i (0 \leq i \leq 7)$ are polynomials in k_A, k_B and k_{AB}.

For $n \geq 3$, we can easily get from equation 3.11 that $k_Ak_B = 0$, consequently equation 3.12 implies $k_{AB} = 0$.

In equation 3.6, denote $c_{57} := c_5\tilde{b}^2 + 2c_7s$. It is trivial to see $c_{57} \neq 0$ and as $k_A = 0$, equation 3.6 becomes $c_{57}k_{1A} = 0$, which implies $k_{1A} = 0$. In this case, equation 3.5 reduces to the following:

$$(c_1\tilde{b}^2 + c_2s^2 + c_3\tilde{b}^2s + c_4\tilde{b}^4)k_1^2 + (c_5\tilde{b}^2s + c_{61}\tilde{b}^2 + c_{62}\tilde{b}^4 + c_7s^2)k_{11} = 0 \tag{3.13}$$

Using Maple to multiply equation 3.13 with $4\chi_2^4$, we get the following identity in s:

$$\mathcal{B}_8s^8 + \mathcal{B}_7s^7 + ...\mathcal{B}_1s + \mathcal{B}_0 = 0, \tag{3.14}$$

where coefficients can be precisely collected as

$$\mathcal{B}_8 = (288n - 396)k_1^2, \tag{3.15}$$

$$\mathcal{B}_7 = (1356k_1^2 - 480k_1^2nb^2 - 960k_1^2n + 624k_1^2b^2 - 324k_{11} + 216k_{11}n), \tag{3.16}$$

also the other coefficients $\mathcal{B}_i (0 \leq i \leq 6)$ are polynomials of k_1 and k_{11}.

The identity in 3.14, firstly implies $k_1 = 0$, consequently using this into equation 3.16, we get that $k_{11} = 0$ also (since $n \geq 3$). Hence, $k_i = 0$, which implies $k(x)$ is a constant. And this proves that F is a locally Minkowski metric. $\square$

4. On Weakly Einstein Kropina Metric

Theorem 4.1. *Let F be Kropina metric on a manifold M of dimension $n \geq 3$ such that F is conformally flat. Suppose that F is a weakly Einstein metric then F is either a Riemannian metric or a locally Minkowski metric.*

Before proving Theorem 4.1, we prove the following Lemma:

Lemma 4.2. *Let $F = \alpha\phi(s), s = \beta/\alpha$, be weakly Einstein conformally flat Kropina metric on a manifold M, of dimension $n \geq 3$. Then,* **Ric**=*0.*

Proof. Proceeding similarly as in the last proof Ricci curvature of weakly Einstein Kropina metric takes form 3.3, where constants are as follows:

$$c_1 := -\frac{\tilde{c_1}}{4b^2},\;\; c_2 := \frac{\tilde{c_2}}{4b^4},\;\; c_3 := \frac{\tilde{c_3}}{4b^4},\;\; c_4 := \frac{\tilde{c_4}}{2b^2},\;\; c_5 := \frac{\tilde{c_5}}{2b^2},\;\; c_6 := \frac{-\tilde{c_6}}{2b^2},\;\; c_7 := \frac{-\tilde{c_7}}{2b^2},$$

where $\tilde{c}_i$ are polynomials in s. On taking $\phi(s) := 1/s$, in equation 3.5 and using Maple to multiply this equation with $\Psi := 4(b^2 - s^2)^{\frac{3}{2}} s^2 b^4$, we get an identity with respect to s as follows:

$$\mathcal{C}_9 s^9 + \mathcal{C}_8 s^8 + ... + \mathcal{C}_1 s + \mathcal{C}_0 = 0, \tag{4.1}$$

where some of the coefficients can be collected as:

$$\mathcal{C}_9 = 12(1-n)k_1\alpha,$$

$$\mathcal{C}_8 = 12\sqrt{b^2 - s^2}(1-n)k_0,$$

$$\mathcal{C}_0 = 4e^{2K} b^6 \alpha^2 \sqrt{b^2 - s^2}\sigma(1-n).$$

From equation 4.1, we know that $\mathcal{C}_i = 0, (i = 0, 1, ..., 9)$ which gives $\sigma = 0$, similarly we can get $t_1 = t_A = 0$. This implies $\theta = 0$. Since σ and θ both vanish so from equation 3.1, we can conclude **Ric**=0. This proves the fact that conformally flat weakly Einstein Kropina metric is Ricci-flat. $\square$

Now we are ready to prove our Theorem 4.1.

Proof. For the sake of generalilty, we take $\tilde{b} \neq 0$, since if $\tilde{b} = 0$, F is Riemannian metric. From the above discussion we see that $\sigma = t_i = 0$, which follows from equation 3.5 that there exists is a function $\zeta := \zeta(s)$ satisfying

$$c_2 k_A k_B + c_7 k_{AB} = \zeta(s)\delta_{AB}. \tag{4.2}$$

For $A \neq B$, above equation reduces to

$$c_2 k_A k_B + c_7 k_{AB} = 0. \tag{4.3}$$

Using Maple to multiply above equation by $4b^4$, we get an identity in terms of s as follows:

$$12(n-1)k_Ak_Bs^4 + 4(n-1)k_{AB}b^2s^2 + (3-n)b^4[2k_{AB} - k_Ak_B] = 0. \tag{4.4}$$

It is clear to see that the equation 4.4, gives $k_Ak_B = 0$ and $k_{AB} = 0$. Now consider equation 3.6 and denote $c_{57} := c_5\tilde{b}^2 + 2c_7s$. It is clear to observe that $c_{57} \neq 0$, which implies that equation 3.6 yields $c_{57}k_{1A} = 0$. This gives $k_{1A} = 0$. This shows that equation 3.5 reduces to

$$(c_1\tilde{b}^2 + c_2s^2 + c_3\tilde{b}^2s + c_4\tilde{b}^4)k_1^2 + (c_5\tilde{b}^2s + c_{61}\tilde{b}^2 + c_{62}\tilde{b}^4 + c_7s^2)k_{11} = 0. \tag{4.5}$$

In order to simplfy equation 4.5, we use Maple to multiply it by b^4, we get an equation in s as follows:

$$3(n-1)k_1^2s^6 - (n-1)(3k_1^2b^2 + k_{11}b^2)s^4 = 0. \tag{4.6}$$

From equation 4.6, it is trivial to conclude $k_1 = 0$ and consequently, $k_{11} = 0$, which implies $k_i = 0$, i.e., $k(x)$ is a constant. This proves that F is a locally Minkowski metric. □

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