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A Note on Marcinkiewicz Integral Operators on Product Domains

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Abstract. In this paper we establish the L^p boundedness of Marcinkiewicz integral operators on product domains with rough kernels satisfying a weak size condition. We assume that our kernels are supported on surfaces generated by curves more general than polynomials and convex functions. This generalizes and extends previous results.

1. Introduction

Let $d \geq 2$ ($d = n$ or $d = m$) and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma(y') = 0 \tag{1.1}$$

where $y' = \frac{y}{|y|} \in \mathbb{S}^{n-1}$ for $y \neq 0$. The Marcinkiewicz integral operator μ_Ω is given by

$$\mu_\Omega(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y)\frac{\Omega(y')}{|y|^{n-1}}dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}. \tag{1.2}$$

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In [18], E. M. Stein established the L^p boundedness $(1 < p \le 2)$ of μ_Ω provided that $\Omega \in Lip_\alpha (0 < \alpha \le 1)$. Subsequently, A. Benedek, A. Calderón, and R. Panzone proved the L^p boundedness of μ_Ω under the stronger condition that $\Omega \in \mathcal{C}^1(\mathbb{S}^{n-1})$ [10]. Since then, several authors have studied the L^p boundedness of μ_Ω under various conditions on the function Ω. A particular result that is of interest to us in this paper is the main result in [12]. In [12], Fan and Pan proved that μ_Ω is bounded on L^p for all $p \in \left(\frac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$ provided that Ω satisfies

$$\sup_{\xi' \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\xi' \cdot y'|}\right)^{1+\epsilon} d\sigma(y') < \infty \tag{1.3}$$

for some $\epsilon > 0$. For $\epsilon > 0$, we let $F(\epsilon, \mathbb{S}^{n-1})$ be the space of all integrable functions on $\mathbb{S}^{n-1}$ that satisfy the condition (1.3). The set of conditions (1.3) were introduced by Grafakos and Stevanov in [17]. Grafakos and Stevanov showed that

$$F(\epsilon, \mathbb{S}^{n-1}) \nsubseteq L(\log^+ L)(\mathbb{S}^{n-1}) \text{ and } L(\log^+ L)(\mathbb{S}^{n-1}) \nsubseteq F(\epsilon, \mathbb{S}^{n-1}).$$

Furthermore, it can be easily seen that

$$\bigcup_{q>1} L^q(\mathbb{S}^{n-1}) \subset F(\epsilon, \mathbb{S}^{n-1}), \epsilon > 0$$

For additional background information and related results on the operator μ_Ω, we advice readers to consult [2], [4]-[9], [12], and [15], among others.

Our aim in this paper is to study the L^p boundedness of a related class of Marcinkiewicz integral operators on product domains. For suitable functions Φ, Ψ : $\mathbb{R}^+ \to \mathbb{R}$ and $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(u', .) d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega(., v') d\sigma(v') = 0 \tag{1.4}$$

and

$$\Omega(tx, sy) = \Omega(x, y) \tag{1.5}$$

for any $t, s > 0$, we define the associated Marcinkiewicz integral operator on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\mathcal{M}_{\Omega,\Phi,\Psi} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F_{t',s'}^{\Phi,\Psi}(f)(x, y) \right|^2 \frac{dt'\, ds'}{2^{2(t'+s')}} \right)^{\frac{1}{2}}, \tag{1.6}$$

where

$$F_{t',s'}^{\Phi,\Psi}(f)(x, y) = \int \int_{\Lambda(t',s')} f(x - \Phi(|u|)u', y - \Psi(|v|)v') \frac{\Omega(u', v')}{|u|^{n-1} |v|^{m-1}} du\, dv \tag{1.7}$$

and $\Lambda(t',s') = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : |u| \le 2^{t'} \text{ and } |v| \le 2^{s'}\}$.
For the sake of simplicity, we denote $\mathcal{M}_{\Omega,\Phi,\Psi}$ by $\mathcal{M}_{\Omega,c}$ when $\Phi(t) = \Psi(t) = t$. In [14], Ding proved that the operator $\mathcal{M}_{\Omega,c}$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ provided that $\Omega \in L(\log^+ L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Subsequently, Chen, Fan, and Ying extended Ding's result to L^p for all $1 < p < \infty$ [11]. The condition $\Omega \in L(\log^+ L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ was very much relaxed by AL-Qassem, Al-Salman, Pan, and Chang in [2]. In fact, the authors of [2] proved that $\mathcal{M}_{\Omega,c}$ is bounded on L^p for all $1 < p < \infty$ provided that kernel satisfies the weaker condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ (see [13] for the case $p = 2$). In the same paper [2], the authors showed that condition $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ can not be replaced by any condition in the form $\Omega \in L(\log^+ L)^\alpha(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $\alpha < 1$.
The main purpose of this paper is to investigate the L^p boundedness of $\mathcal{M}_{\Omega,\Phi,\Psi}$ for mappings Φ and Ψ more general than polynomials and convex functions, provided that Ω satisfies the condition

$$(1.8) \qquad \sup_{(\xi',\eta') \in (\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} |\Omega(u',v')| \, \{G(\xi',\eta')\}^{1+\epsilon} d\sigma(u') \, d\sigma(v') < \infty,$$

for some $\epsilon > 0$, where

$$G(\xi',\eta') = \log^+(|\xi' \cdot u'|^{-1}) + \log^+(|\eta' \cdot v'|^{-1}) + \log^+(|\xi' \cdot u'|^{-1}) \, \log^+(|\eta' \cdot v'|^{-1}).$$

For $\epsilon > 0$, we let $F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ be the class of all $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ that satisfy (1.8). The class $F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ is the analogy of the class $F(\epsilon, \mathbb{S}^{n-1})$ in the one parameter setting above. It is clear that, for any $\epsilon > 0$, we have

$$\bigcup_{q>1} L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subset F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1}).$$

Moreover, it was observed in [4] that

$$F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1}) \nsubseteq L(\log^+ L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$$

and

$$L(\log^+ L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \nsubseteq F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1}).$$

Historically, in [4], Al-Salman proved the L^p boundedness of $\mathcal{M}_{\Omega,\Phi,\Psi}$ for all $p \in \left(\frac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$ provided that Ω satisfies (1.8) and the functions Φ and Ψ are either convex increasing functions or satisfy a growth condition in the form

$$(1.9) \qquad |\varphi(t)| \le C_1 t^d, \qquad |\varphi''(t)| \le C_2 t^{d-2}$$

$$(1.10) \qquad C_3 t^{d-1} \le |\varphi'(t)| \le C_4 t^{d-1}$$

where $d \neq 0$, $t \in (0,\infty)$ and C_1, C_2, C_3 and C_4 are positive constants independent of t.

In [3], Al-Salman introduced a class of functions generalizing the convexity property. To be more specific, a function $\psi : [0,\infty) \to \mathbb{R}$ is said to belong to the class $\mathscr{PC}_\lambda(d)$ $(d > 0)$ if there exist, $\lambda \in \mathbb{R}$, a polynomial P, and $\varphi \in \mathcal{C}_0([0,\infty))$ such that

$$\text{(1.11)}\qquad \begin{array}{l} \text{(i)}\, \psi(t) = P(t) + \lambda\varphi(t) \\ \text{(ii)}\, P(0) = 0 \text{ and } \varphi^{(j)}(0) = 0 \text{ for } 0 \leqslant j \leqslant d \\ \text{(iii)}\, \varphi^{(j)} \text{ is positive nondecreasing on } (0,\infty) \text{ for } 0 \leqslant j \leqslant d+1. \end{array}$$

It was shown in [3] that the class $\cup_{d\geq 0}(\mathcal{PC}_\lambda(d))$ contains properly the class of polynomials $\mathcal{P}_d$ as well as the class of convex increasing functions. The author of [3] pointed out that the function $\theta(t) = -t^2 + t^2\ln(1+t)$ is in $\mathcal{PC}_\lambda(2)$ which is neither convex nor polynomial.

In light of the aforementioned discussion, it is natural to ask the following:

Question. *Let $\mathcal{M}_{\Omega,\Phi,\Psi}$ be given by* (1.6) *and assume that $\Omega \in F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ satisfying* (1.4)-(1.5) *for some $\epsilon > 0$. Suppose that $\Phi \in \mathscr{PC}_\lambda(d)$, $\Psi \in \mathscr{PC}_\alpha(b)$ for d, $b > 0$ and λ, $\alpha \in \mathbb{R}$. Is $\mathcal{M}_{\Omega,\Phi,\Psi}$ bounded on L^p for some $1 < p < \infty$?*

In the following theorem, we give an affirmative answer to the above question:

Theorem 1.1. *Suppose that $\Omega \in F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ satisfying* (1.4)-(1.5)*. If $\Phi \in \mathscr{PC}_\lambda(d)$, $\Psi \in \mathscr{PC}_\alpha(b)$ for d, $b > 0$ and λ, $\alpha \in \mathbb{R}$. Then $\mathcal{M}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p \in \left(\dfrac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$ with L^p bounds independent of $\lambda, \alpha \in \mathbb{R}$ and the coefficients of the particular polynomials involved in the standard representation* (i) *of Φ and Ψ in* (1.11).

We remark here that Theorem 1.1 is a fundamental generalization of Theorem 1.1 in [4].
Throughout this paper, the letter C will denote of a constant that may vary at each occurrence but it is independent of the essential variables.

2. Preliminary Estimates

We start for the following result in [16]:

Lemma 2.1. ([16]) *Suppose that $P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha$ is polynomial of degree m on $\mathbb{R}^n$ and $\varepsilon < \dfrac{1}{m}$. Then there exists $A_\varepsilon > 0$ such that*

$$\int_{\mathbb{S}^{n-1}} |P(y')|^{-\varepsilon} d\sigma(y') \leq A_\varepsilon \|P\|,$$

where

$$\|P\| = \sum_{|\alpha|=m} |a_\alpha|.$$

The bound A_ε may depend on ε, m and n but it is independent of the coefficients of the polynomial.

Also, we shall need the following lemma in [1]:

Lemma 2.2.([1]) *If $\varphi \in \mathcal{C}^{d+1}[0,\infty)$ and satisfies the conditions* (i) – (ii) *in* (1.11), *then*
(i) $\varphi(\alpha r) \leq \alpha\,\varphi(r)$ $\quad$ *for* $0 \leq \alpha \leq 1$and $r > 0$
(ii) $\varphi(\alpha r) \geq \alpha\,\varphi(r)$ $\quad$ for $\alpha \geq 1$ and $r > 0$.
(iii) $\varphi^{d+1}(r) \geqslant r^{-d-1}\varphi(r)$ $\quad$ for $r > 0$.

The following well known theorem on maximal functions is significant:

Theorem 2.3. ([3]) *Suppose that $\Upsilon : \mathbb{R}^n \to \mathbb{R}^d$ is a non-constant mapping and assume that $\psi \in \mathscr{PC}_\lambda(d)$ for some $d \geqslant 0$ and $\lambda \in \mathbb{R}$. Suppose also that $\rho > 0$. If $\Omega \in L^1(\mathbb{S}^{n-1})$ is homogeneous of degree zero in $\mathbb{R}^n$, then the maximal function $\mathscr{M}_{\Psi,\Omega}$ given by*

$$\mathscr{M}_{\Psi,\Omega}(f)(x) = \sup_{j\in\mathbb{Z}} \left| \int_{\rho^j<|y|<\rho^{j+1}} f(x - \psi(|y|)\,\Upsilon(y'))\,\frac{\Omega(y')}{|y|^n}\,dy \right| \tag{2.1}$$

satisfies

$$\|\mathscr{M}_{\Psi,\Omega}(f)\|_p \leq C_p\ \|\Omega\|_1\ \|f\|_p.$$

for $1 < p < \infty$. Here, the constant C_p is independent of $\lambda, \Upsilon(y')$ and the coefficients of the particular polynomials involved in the representation (1.11) *of ψ.*

Now, we move to obtain the needed oscillatory estimates. For $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and suitable mappings $\Phi, \Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}$, we define the family of measures $\{\sigma_{\Phi,\,\Psi,\Omega,t',s'} : t',\, s' \in \mathbb{R}\}$ by
(2.2)

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} f d\sigma_{\Phi,\,\Psi,\Omega,t',s'} = \int_{\Gamma(2^{t'},2^{s'})} f(x - \Phi(|u|)\,u', y - \Psi(|v|)\,v')\frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,dudv$$

where

$$\Gamma(2^{t'}, 2^{s'}) = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : 2^{t'-1} < |u| \leq 2^{t'} \text{ and } 2^{s'-1} < |v| \leq 2^{s'}\}.$$

The corresponding maximal function is defined by

$$(\sigma_{\Phi,\,\Psi,\Omega})^* f(x,y) = \sup_{t',s'\in\mathbb{R}} ||\sigma_{\Phi,\,\Psi,\Omega,t',s'}| * (f)(x,y)|\,. \tag{2.3}$$

For simplicity, we shall let

$$\sigma_{t',s'} = \sigma_{\Phi,\,\Psi,\Omega,t',s'}.$$

Now, for $\Phi \in \mathscr{PC}_{\lambda_1}(d)$ and $\Psi \in \mathscr{PC}_{\lambda_2}(b)$ for some $b,\, d > 0$, let

$$\Phi(t) = P(t) + \lambda_1 \varphi_1(t)$$
$$\Psi(r) = Q(r) + \lambda_2 \varphi_2(r),$$

where $\lambda_1,\, \lambda_2 \in \mathbb{R}$, $P \in \mathscr{P}_d$, $Q \in \mathscr{P}_b$, $\varphi_1 \in C^{d+1}[0,\infty)$, and $\varphi_2 \in C^{b+1}[0,\infty)$. Let

$$P(t) = \sum_{k=0}^{d} c_{k,1}\, t^k \qquad \text{and} \qquad Q(t) = \sum_{k=0}^{b} c_{k,2}\, t^k. \tag{2.4}$$

For $0 \le l \le d$ and $0 \le s \le b$, let

$$P_l(t) = \sum_{k=0}^{l} c_{k,1}\, t^k \qquad \text{and} \qquad Q_s(t) = \sum_{k=0}^{s} c_{k,2}\, t^k, \tag{2.5}$$

where we use the convention that $\sum_{j\in\Theta} = 0$. Now, by (2.2), we defined the family of measure $\{\sigma^{(d+1,b+1)}_{t',s'} : t',\, s' \in \mathbb{R}\}$ via the Fourier transform by

$$\widehat{\sigma}^{(d+1,b+1)}_{t',s'}(\xi,\eta) = 2^{-(t'+s')} \int\int_{\Gamma(2^{t'},2^{s'})} e^{-i\left(\Phi(|u|)\,\xi.u' + \Psi(|v|)\,\eta.v'\right)} \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\, du\, dv. \tag{2.6}$$

For $0 \le l \le d$, $0 \le s \le b$, we defined the family of measures $\{\sigma^{(l,s)}_{t',s'} : t',\, s' \in \mathbb{R}\}$ by

$$\widehat{\sigma}^{(l,s)}_{t',s'}(\xi,\eta) = 2^{-(t'+s')} \int\int_{\Gamma(2^{t'},2^{s'})} e^{-i\left(P_l(|u|)\,\xi.u' + Q_s(|v|)\,\eta.v'\right)} \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\, du\, dv. \tag{2.7}$$

It is clear that

$$\widehat{\sigma}^{(0,0)}_{t',s'} = \widehat{\sigma}^{(0,b+1)}_{t',s'} = \widehat{\sigma}^{(d+1,0)}_{t',s'} = 0.$$

By (2.6)-(2.7), we have

$$\mathcal{M}_{\Omega,\Phi,\Psi}(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sigma^{(d+1,b+1)}_{t',s'} * (f)(x,y) \right|^2 dt'\, ds' \right)^{\frac{1}{2}}. \tag{2.8}$$

We have the following two lemmas:

Lemma 2.4. *Let $\{\sigma^{(d+1,b+1)}_{t',s'} : t',\, s' \in \mathbb{R}\}$ be the measures given in (2.6). Suppose that $\Omega \in F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ for some $\epsilon > 0$. Then*

(i) $\|\sigma^{(d+1,b+1)}_{t',s'}\| \le C$;

(ii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{t',s'}(\xi,\eta)\right| \le C \left(\log^+ |\lambda_1\, \varphi_1\, (2^{t'-1})\, \xi\,|\right)^{-1-\epsilon} \left(\log^+ |\lambda_2\, \varphi_2\, (2^{s'-1})\, \eta\,|\right)^{-1-\epsilon}$;

(iii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{t',s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{t',s'}(\xi,\eta)\right| \le C \left|\lambda_1 \varphi_1(2^{t'})\, \xi\right| \left(\log^+ |\lambda_2\, \varphi_2\, (2^{s'-1})\, \eta|\right)^{-1-\epsilon}$;

(iv) $\left|\widehat{\sigma}^{(d+1,b+1)}_{t',s'}(\xi,\eta) - \widehat{\sigma}^{(d+1,b)}_{t',s'}(\xi,\eta)\right| \le C \left|\lambda_2\, \varphi_2(2^{s'})\, \eta\right| \left(\log^+ |\lambda_1 \varphi_1\, (2^{t'-1})\, \xi\,|\right)^{-1-\epsilon}$;

(v) $\left|\widehat{\sigma}^{(d+1,b+1)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{t',s'}(\xi,\eta)+\widehat{\sigma}^{(d,b)}_{t',s'}(\xi,\eta)\right|$
$\leq C\,|\lambda_1\varphi_1(2^{t'})\,\xi|\;|\lambda_2\varphi_2(2^{s'})\,\eta|$;

(vi) $\left|\widehat{\sigma}^{(d+1,b)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{t',s'}(\xi,\eta)\right|\leq C\,|\lambda_1\varphi_1(2^{t'})\,\xi|$;

(vii) $\left|\widehat{\sigma}^{(d,b+1)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{t',s'}(\xi,\eta)\right|\leq C\,\left|\lambda_2\,\varphi_2(2^{s'})\,\eta\right|$.

Here C is independent of $t',s'\in\mathbb{R}$ and $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^m$.

Lemma 2.5. *Let $\{\sigma^{(l,s)}_{t',s'}:0\leq l\leq d,\ 0\leq s\leq b\}$ be as in (2.7). Suppose that $\Omega\in F(\epsilon,\mathbb{S}^{n-1},\mathbb{S}^{m-1})$ for some $\epsilon>0$. Then*

(i) $\|\sigma^{(l,s)}_{t',s'}\|\leq C$;

(ii) $\left|\widehat{\sigma}^{(l,s)}_{t',s'}(\xi,\eta)\right|\leq C\,\left(\log^+|c_{l,1}\,(2^{t'-1})^l\ l!\,\xi|\right)^{-1-\epsilon}\left(\log^+|\,c_{s,2}\,(2^{s'-1})^s\ s!\,\eta|\right)^{-1-\epsilon}$;

(iii) $\left|\widehat{\sigma}^{(l,s)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{t',s'}(\xi,\eta)\right|\leq C\,|c_{l,1}\,(2^{t'})^l\xi|\,\left(\log^+|\,c_{s,2}\,(2^{s'-1})^s\ s!\,\eta|\right)^{-1-\epsilon}$;

(iv) $\left|\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\,\left|c_{s,2}(2^{s'})^s\,\eta\right|\left(\log^+|c_{l,1}\,(2^{t'-1})^l\ l!\,\xi|\right)^{-1-\epsilon}$;

(v) $\left|\widehat{\sigma}^{(l,s)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{t',s'}(\xi,\eta)+\widehat{\sigma}^{(l-1,s-1)}_{t',s'}(\xi,\eta)\right|$
$\leq C\,|c_{l,1}(2^{t'})^l\,\xi|\;|c_{s,2}\,(2^{s'})^s\,\eta|$;

(vi) $\left|\widehat{\sigma}^{(l,s-1)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{t',s'}(\xi,\eta)\right|\leq C\,|c_{l,1}\,(2^{t'})^l\,\xi|$;

(vii) $\left|\widehat{\sigma}^{(l-1,s)}_{t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{t',s'}(\xi,\eta)\right|\leq C\,\left|c_{s,2}(2^{s'})^s\,\eta\right|$.

Here C is independent of $t',s'\in\mathbb{R}$ and $(\xi,\eta)\in(\mathbb{R}^n,\mathbb{R}^m)$.

Now, we shall start by presenting the proof of Lemma 2.4.

Proof of Lemma 2.4. To prove (i), we have

$$\begin{aligned}\|\sigma^{(d+1,b+1)}_{t',s'}\| &\leqslant 2^{-(t'+s')}\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(u',v')|\,dt\,dr\,d\sigma(u')\,d\sigma(v')\\ &\leqslant C\,\|\Omega\|_{L^1}\leqslant C.\end{aligned}$$

To get (ii), by polar coordinates, we have

(2.9)
$$\begin{aligned}&\left|\widehat{\sigma}^{(d+1,b+1)}_{j,k}(\xi,\eta)\right|\\ &=\left|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}2^{-(t'+s')}e^{-i\left(\Phi(t)\,\xi.u'+\Psi(r)\,\eta.v'\right)}\,\Omega(u',v')\,dt\,dr\sigma(u')\,d\sigma(v')\right|\\ &\leq 2^{-(t'+s')}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(u',v')|\;A_{\Phi,\Psi,t',s'}\;d\sigma(u')d\sigma(v')\end{aligned}$$

where

$$A_{\Phi,\Psi,t',s'}=\left|\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}e^{-i\left(\Phi(t)\,\xi.u'+\Psi(r)\,\eta.v'\right)}dt\,dr\right|. \tag{2.10}$$

By change of variables, we get

$$A_{\Phi,\Psi,t',s'} \le 2^{t'+s'-2} A_{\Phi,t'}\ A_{\Psi,s'} \tag{2.11}$$

where

$$A_{\Phi,t'} = \left| \int_1^2 e^{-i(\Phi(2^{t'-1}\,t)\,\xi.u'}\,dt \right|$$

and

$$A_{\Psi,s'} = \left| \int_1^2 e^{-i(\Psi(2^{s'-1}\,r)\,\eta.v'}\,dr \right|.$$

By Lemma 2.2, we have

$$\left|\Phi^{(d+1)}(2^{t'-1}\ t)\right| \quad \ge C\,|\lambda_1 \varphi_1\,(2^{t'-1})| \tag{2.12}$$

and

$$\left|\Psi^{(b+1)}(2^{s'-1}\ r)\right| \ge C\,|\lambda_2 \varphi_2\,(2^{s'-1})| \tag{2.13}$$

where $2^{t'-1} < t < 2^{t'}$ and $2^{s'-1} < r < 2^{s'}$. Thus, by Van der Corput lemma in [19], we obtain

$$A_{\Phi,t'} \leqslant |\lambda_1\,\varphi_1\,(2^{t'-1})\,\xi \cdot u'|^{-\frac{1}{d+1}} \tag{2.14}$$

and

$$A_{\Psi,s'} \leqslant |\lambda_2\,\varphi_2\,(2^{s'-1})\,\eta \cdot v'|^{-\frac{1}{b+1}}. \tag{2.15}$$

Thus, by combining the trivial estimates $A_{\Phi,t'} \le C$ and $A_{\Psi,s'} \le C$ with the estimates (2.14)-(2.15), we have

$$A_{\Phi,t'} \le \min\{C,\, |\lambda_1\,\varphi_1\,(2^{t'-1})\,\xi \cdot u'|^{-\frac{1}{d+1}}\}$$

and

$$A_{\Psi,s'} \le \min\{C,\, |\lambda_2\,\varphi_2\,(2^{s'-1})\,\eta \cdot v'|^{-\frac{1}{b+1}}\}.$$

Therefore,

$$\begin{aligned} A_{\Phi,t'} \quad &\le \left(\frac{C\ \log^+ |\xi' \cdot u'|^{-\frac{1}{d+1}}}{\log^+ |\lambda_1\,\varphi_1\,(2^{t'-1})\,\xi|^{\frac{1}{d+1}}} \right)^{1+\epsilon} \\ &\le C\,\left(\log^+ |\lambda_1 \varphi_1\,(2^{t'-1})\,\xi|\right)^{-1-\epsilon} \left(\log^+ \frac{1}{|\xi' \cdot u'|}\right)^{1+\epsilon}, \end{aligned} \tag{2.16}$$

where $\xi' = \dfrac{\xi}{|\xi|}$. Similarly, we get

$$A_{\Psi,s'} \le C\,\left(\log^+ |\lambda_2\,\varphi_2\,(2^{s'-1})\,\eta|\right)^{-1-\epsilon} \left(\log^+ \frac{1}{|\eta' \cdot v'|}\right)^{1+\epsilon}. \tag{2.17}$$

Thus, we arrive at the following estimate

$$\begin{aligned} & A_{\Phi,\Psi,t',s'} \\ & \leq C\, 2^{(t'+s')} \left(\log^{+} |\lambda_1 \varphi_1 (2^{t'-1}) \xi|\right)^{-1-\epsilon} \left(\log^{+} \frac{1}{|\xi' \cdot u'|}\right)^{1+\epsilon} \\ & \left(\log^{+} |\lambda_2 \varphi_2 (2^{s'-1}) \eta|\right)^{-1-\epsilon} \left(\log^{+} \frac{1}{|\eta' \cdot v'|}\right)^{1+\epsilon}. \end{aligned} \tag{2.18}$$

Finally, by (2.9), (2.18) and (1.8), we obtain

$$\left|\widehat{\sigma}^{(d+1,b+1)}_{t,'s'}(\xi,\eta)\right| \leq C \left(\log^{+} |\lambda_1 \varphi_1 (2^{t'-1}) \xi|\right)^{-1-\epsilon} \left(\log^{+} |\lambda_2 \varphi_2 (2^{s'-1}) \eta|\right)^{-1-\epsilon} \tag{2.19}$$

For the proof of (iii), we have

$$\begin{aligned} & \left|\widehat{\sigma}^{(d+1,b+1)}_{t,'s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{t,'s'}(\xi,\eta)\right| \\ & = \left| \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \int_{2^{t'-1}}^{2^{t'}} \int_{2^{s'-1}}^{2^{s'}} \Omega(u',v')\, e^{-i\Psi(r)\,\eta.v'} \left(e^{-i\Phi(t)\,\xi.u'} - e^{-iP(t)\,\xi.u'}\right) dt\, dr\, d\sigma(u')\, d\sigma(v') \right|. \end{aligned} \tag{2.20}$$

Thus, by Fubini's Theorem, we get

$$\begin{aligned} & \left|\widehat{\sigma}^{(d+1,b+1)}_{t,'s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{t,'s'}(\xi,\eta)\right| \\ & \leq 2^{-(t'+s')} \int_{\mathbb{S}^{n-1}} \int_{2^{t'-1}}^{2^{t'}} \left| \int_{\mathbb{S}^{m-1}} \int_{2^{s'-1}}^{2^{s'}} \Omega(u',v')\, e^{-i\Psi(r)\,\eta.v'}\, dr\, d\sigma(v') \right| \cdot \\ & \qquad \left| e^{-iP(t)\,\xi.u'} \left(e^{-i\,\lambda_1\varphi_1(t)\,\xi.u'} - 1\right)\right| dt\, d\sigma(u') \\ & \leq C \left|\lambda_1\varphi_1(2^{t'})\,\xi\right| \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega(u',v')| d\sigma(u') d\sigma(v') \int_1^2 \left| \int_1^2 e^{-i\Psi(2^{s'-1}r)\,\eta.v'}\, dr \right| dt \\ & \leq C \left\|\Omega\right\|_{L^1} \left|\lambda_1\varphi_1(2^{t'})\,\xi\right| \int_1^2 A_{\Psi,s'}\, dt \end{aligned} \tag{2.21}$$

where φ_1 is increasing, $2^{t'-1} < t < 2^{t'}$ and $A_{s',\Psi}$ as in (2.17). Thus,

$$\begin{aligned} & \left|\widehat{\sigma}^{(d+1,b+1)}_{t,'s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{t,'s'}(\xi,\eta)\right| \\ & \leq C \left\|\Omega\right\|_{L^1} \left|\lambda_1\varphi_1(2^{t'})\,\xi\right| \left(\log^{+} |\lambda_2 \varphi_2 (2^{s'-1}) \eta|\right)^{-1-\epsilon} \left(\log^{+} \frac{1}{|\eta' \cdot v'|}\right)^{1+\epsilon}. \end{aligned} \tag{2.22}$$

Finally, by combining (1.8) and (2.22), we establish the estimate (iii). Similarly, we can obtain the estimate (iv). We omit details.

Now, to get the estimate (v), we have

$$\begin{aligned}
&\left|\widehat{\sigma}^{(d+1,b+1)}_{t,'s'}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{t,'s'}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{t,'s'}(\xi,\eta)+\widehat{\sigma}^{(d,b)}_{t,'s'}(\xi,\eta)\right|\\
&\le 2^{-(t'+s')}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}|\Omega(u',v')|\,|e^{-i\lambda_1\varphi_1(t)\,\xi}-1|\,|e^{-i\lambda_2\varphi_2(r)\,\eta}-1|\,dt\,dr\,d\sigma(u')\,d\sigma(v')\\
&\le 2^{-(t'+s')}|\lambda_1\varphi_1(2^{t'})\,\xi|\,|\lambda_2\varphi_2(2^{s'})\,\eta|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}|\Omega(u',v')|dt\,dr\,d\sigma(u')\,d\sigma(v')\\
&\le C\,\|\Omega\|_{L^1}\,|\lambda_1\varphi_1(2^{t'})\,\xi|\,|\lambda_2\varphi_2(2^{s'})\,\eta|\\
&\le C\,|\lambda\varphi_1(2^{t'})\,\xi|\,|\lambda_2\varphi_2(2^{s'})\,\eta|.
\end{aligned}$$

For the estimate (vi), we have

$$\begin{aligned}
&\left|\widehat{\sigma}^{(d+1,b)}_{t,'s'}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{t,'s'}(\xi,\eta)\right|\\
&\le 2^{-(t'+s')}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{2^{t'-1}}^{2^{t'}}\int_{2^{s'-1}}^{2^{s'}}|\Omega(u',v')|\,\left|e^{-i\lambda_1\varphi_1(t)\,\xi\cdot u'}-1\right|dt\,dr\,d\sigma(u')\,d\sigma(v').
\end{aligned}$$

Now, since φ_1 is increasing and $2^{t'-1}<t<2^{t'}$, then by change of variables, we obtain

$$\left|\widehat{\sigma}^{(d+1,b)}_{t,'s'}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{t,'s'}(\xi,\eta)\right|\le C\,|\lambda_1\varphi_1(2^{t'})\,\xi|. \tag{2.23}$$

Similarly, we can prove (vii). We omit details. This completes the proof.

Proof of Lemma 2.5. The proof of Lemma 2.5 follows the same procedure as in the proof of Lemma 2.4. We only need to notice here that

$$\frac{d^l}{dt}P_l(2^{t'-1}\,t)=c_{l,1}\,(2^{t'-1})^l\ l! \qquad\text{and}\qquad \frac{d^s}{dr}Q_s(2^{s'-1}\,r)=c_{s,2}\,(2^{s'-1})^s\ s!.$$

We omit details.

Now, we have the following lemma on the concerned maximal functions:

Lemma 2.6. *Suppose that* $\Omega\in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *and* $\Phi,\ \Psi:\mathbb{R}^+\to\mathbb{R}$. *Let* $\mathscr{M}_{\Phi,\Psi,\Omega}$ *be the maximal function defined by*

$$\mathscr{M}_{\Phi,\Psi,\Omega}(f)(x,y)=\sup_{t',s'\in\mathbb{R}}\left|\frac{1}{2^{t'+s'}}\int\int_{\Gamma(2^{t'},2^{s'})}f(x-\Phi(|u|)u',y-\Psi(|v|)v')\,\frac{\Omega(u',v')}{|u|^{n-1}|v|^{m-1}}dudv\right|.$$

If $\Phi\in\mathscr{PC}_\lambda(d)$ *and* $\Psi\in\mathscr{PC}_\alpha(b)$ *for some* $d,b>0$, *then*

$$\|\mathscr{M}_{\Phi,\Psi,\Omega}f\|_{L^p}\le C\,\|\Omega\|_{L^1}\,\|f\|_{L^p}.$$

Proof. For $\Phi \in \mathscr{PC}_\lambda(d)$ and $\Psi \in \mathscr{PC}_\alpha(b)$, let

$$\mathscr{M}_{\Psi,\Omega}(f)(x,y) = \sup_{s' \in \mathbb{R}} \left| \frac{1}{2^{s'}} \int\limits_{2^{s'-1}<|v|<2^{s'}} f(\cdot\,, y - \Psi(|v|)v') \, \frac{\Omega(\cdot\,, v')}{|v|^{m-1}} d(v) \right|$$

and

$$\mathscr{M}_{\Phi,\Omega}(f)(x,y) = \sup_{t' \in \mathbb{R}} \left| \frac{1}{2^{t'}} \int\limits_{2^{t'-1}<|u|<2^{t'}} f(x - \Phi(|u|)u', \cdot) \, \frac{\Omega(u', \cdot)}{|u|^{n-1}} d(u) \right|$$

By using the observation $\mathscr{M}_{\Phi,\Psi,\Omega}(f) \leq (\mathscr{M}_{\Phi,\Omega} \circ \mathscr{M}_{\Psi,\Omega})(f)$ and Theorem 2.3, we get

$$\begin{aligned} \|\mathscr{M}_{\Phi,\Psi,\Omega}(f)\|_p &\leq \|\mathscr{M}_{\Phi,\Omega} \circ \mathscr{M}_{\Psi,\Omega}(f)\|_p \\ &\leq C_p \, \|\Omega\|_1 \, \|\mathscr{M}_{\Psi,\Omega}(f)\|_p \\ &\leq C \, \|\Omega\|_1 \, \|f\|_p, \end{aligned} \tag{2.24}$$

where $\circ$ denotes the composition of operators. This ends the proof.

3. Proof of Main Result

Assume that $\Omega \in F(\epsilon, \mathbb{S}^{n-1}, \mathbb{S}^{m-1})$ for some $\epsilon > 0$. Let $\sigma_{t',s'}^{(l,s)}$ be the measure defined by (2.7). Now, for $t', s' \in \mathbb{R}$, we defined the family of measure $\{\tau_{t',s'}^{(l,s)} : 0 \leq l \leq d+1, 0 \leq s \leq b+1\}$ by

(3.1)

$$\begin{aligned} &\tau_{t',s'}^{(l,s)}(\xi,\eta) \\ &= \widehat{\sigma}_{t',s'}^{(l,s)}(\xi,\eta) \textstyle\prod_{l \leqslant r \leqslant d+1} \phi((2^{t'-1})^l |c_{l,1}\,\xi|) \textstyle\prod_{s \leqslant i \leqslant b+1} \phi((2^{s'-1})^s |c_{s,2}\,\eta|) \\ &- \widehat{\sigma}_{t',s'}^{(l-1,s)}(\xi,\eta) \textstyle\prod_{l-1 \leqslant r \leqslant d+1} \phi((2^{t'-1})^l |c_{l,1}\,\xi|) \textstyle\prod_{s \leqslant i \leqslant b+1} \phi((2^{s'-1})^s |c_{s,2}\,\eta|) \\ &- \widehat{\sigma}_{t',s'}^{(l,s-1)}(\xi,\eta) \textstyle\prod_{l \leqslant r \leqslant d+1} \phi((2^{t'-1})^l |c_{l,1}\,\xi|) \textstyle\prod_{s-1 \leqslant i \leqslant b+1} \phi((2^{s'-1})^s |c_{s,2}\,\eta|) \\ &+ \widehat{\sigma}_{t',s'}^{(l-1,s-1)}(\xi,\eta) \textstyle\prod_{l-1 \leqslant r \leqslant d+1} \phi((2^{t'-1})^l |c_{l,1}\,\xi|) \textstyle\prod_{s-1 \leqslant i \leqslant b+1} \phi((2^{s'-1})^s |c_{s,2}\,\eta|). \end{aligned}$$

Here $\phi(t) \in C_0^\infty(\mathbb{R})$ such that $\phi(t) = 1$ for $|t| \leqslant \dfrac{1}{2}$ and $\phi(t) = 0$ for $|t| \geqslant 1$. Here, we set $c_{d+1,1} = \lambda_1$ and $c_{b+1,2} = \lambda_2$. By Lemma 2.4 and Lemma 2.5, we obtain that $\{\tau_{t',s'}^{(l,s)} : 0 \leq l \leq d+1, 0 \leq s \leq b+1\}$ satisfy

$$\|\tau_{t',s'}^{(l,s)}\| \leq C; \tag{3.2}$$

$$\left|\widehat{\tau}_{t',s'}^{(l,s)}(\xi,\eta)\right| \leq C \, \left(\log^+ |a_{l,t'}\, L_l(\xi)|\right)^{-1-\epsilon} \, \left(\log^+ |b_{s,s'}\, Q_s(\eta)|\right)^{-1-\epsilon}; \tag{3.3}$$

$$\left|\widehat{\tau}_{t',s'}^{(l,s)}(\xi,\eta)\right| \leq C \, |a_{l,t'}\, L_l(\xi)| \, \left(\log^+ |b_{s,s'}\, Q_s(\eta)|\right)^{-1-\epsilon}; \tag{3.4}$$

$$\left|\widehat{\tau}^{(l,s)}_{t',s'}(\xi,\eta)\right| \leq C\,|b_{s,s'}\,Q_s(\eta)|\left(\log^+|a_{l,t'}\,L_l(\xi)|\right)^{-1-\epsilon}; \tag{3.5}$$

$$\left|\widehat{\tau}^{(l,s)}_{t',s'}(\xi,\eta)\right| \leq C\;|a_{l,t'}\,L_l(\xi)|\,|b_{s,s'}\,Q_s(\eta)|; \tag{3.6}$$

and

$$\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\tau^{(l,s)}_{t',s'} = \sigma^{(d+1,b+1)}_{t',s'}, \tag{3.7}$$

where

$$L_l(\xi) = \begin{cases} \lambda_1\,\xi & ,l = d+1 \\ c_{k,1}\,\xi & ,l \neq d+1 \end{cases}\quad,\qquad Q_s(\eta) = \begin{cases} \lambda_2\,\eta & ,s = b+1 \\ c_{s,2}\,\eta & ,s \neq b+1 \end{cases},$$

$$a_{l,t'} = \begin{cases} \varphi_1(2^{t'-1}) & ,l = d+1 \\ C\,(2^{t'-1})^l & ,l \neq d+1 \end{cases}\quad,\ \text{and}\quad b_{s,s'} = \begin{cases} \varphi_2(2^{s'-1}) & ,s = b+1 \\ C\,(2^{s'-1})^s & ,s \neq b+1 \end{cases}$$

Thus, by (3.7), we have

$$\mathcal{M}_{\Omega,\Phi,\Psi}(f)(x,y) \leq C\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\tau^{(l,s)}_{t',s'}*(f)(x,y)\right)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}. \tag{3.8}$$

Let

$$\mathcal{M}^{(l,s)}_{\Omega}(f) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\tau^{(l,s)}_{t',s'}*(f)(x,y)\right)\right|^2 dt'\,ds'\right)^{\frac{1}{2}} \tag{3.9}$$

Now, we need to prove that $\mathcal{M}^{(l,s)}_{\Omega}$ is bounded on L^p for $p \in \left(\frac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$. By an elementary procedure, we choose two collections of $\mathcal{C}^\infty$ functions $\{\nu^{(l)}_k\}_{k\in\mathbb{Z}}$ and $\{\nu^{(s)}_k\}_{k\in\mathbb{Z}}$ on $(0,\infty)$ that satisfy the following properties:

$$supp(\nu^{(l)}_k) \subseteq \left[\frac{1}{a_{l,k+1}},\frac{1}{a_{l,k-1}}\right],\ \ supp(\nu^{(s)}_k) \subseteq \left[\frac{1}{b_{s,k+1}},\frac{1}{b_{s,k-1}}\right] \tag{3.10}$$

$$0 \leq \nu^{(l)}_k,\ \nu^{(s)}_k \leq 1; \tag{3.11}$$

$$\sum_{k\in\mathbb{Z}}\nu^{(l)}_k(u) = \sum_{k\in\mathbb{Z}}\nu^{(s)}_k(u) = 1; \tag{3.12}$$

$$\left|\frac{d^r\nu_k^{(l)}}{du^r}(u)\right|,\ \left|\frac{d^r\nu_k^{(s)}}{du^r}(u)\right| \le \frac{C_r}{u^r}. \tag{3.13}$$

Defined the functions $\{\upsilon_k^{(l)}) : k \in \mathbb{Z}\}$ on $\mathbb{R}^n$ and $\{\upsilon_k^{(s)}) : k \in \mathbb{Z}\}$ on $\mathbb{R}^m$ by

$$(\upsilon_k^{(l)}\hat{)}(x) = \nu_k^{(l)}(|x|^2) \text{ and } (\upsilon_k^{(s)}\hat{)}(y) = \nu_k^{(s)}(|y|^2).$$

Thus,

$$(\tau_{t',s'}^{(l,s)} * f)(x,y) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * \tau_{t',s'}^{(l,s)} * f(x,y), \tag{3.14}$$

where $\lfloor t\rfloor$ the greatest integer function less than or equal to t. Thus

$$\mathcal{M}_\Omega^{(l,s)}(f)(x,y) \le C \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} U_{j,k}^{(l,s)}(f)(x,y), \tag{3.15}$$

where

(3.16)

$$U_{j,k}^{(l,s)}(f)(x,y) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * \tau_{t',s'}^{(l,s)} * f(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}.$$

By Littlewood-Paley Theory in [18], we have

$$\|S_{j,k}(f)\|_p \le C\,\|f\|_p \tag{3.17}$$

for all $1 < p < \infty$ with constant $C > 0$, where

$$S_{j,k}(f)(x,y) = \left(\int_{-\infty}^{\infty}\left|\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * f(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}.$$

Thus, by (3.17), Lemma 2.6, (3.2), and Lemma 1 in [16], we obtain

$$\|U_{j,k}^{(l,s)}(f)\|_p \le C\ \|f\|_p, \tag{3.18}$$

for $p \in (1,\infty)$ and $C > 0$.
Next, we seek suitable L^2 - norm of $U_{j,k}^{(l,s)}(f)$. We shall adopt the same steps followed by Al-Salman in [4]. By (3.10), defined the intervals $E_k^{(l)}$ and $E_j^{(s)}$ in $\mathbb{R}$ by

$$E_k^{(l)}(\xi) = \left[\log_2(2^k\theta^{-1}(|\xi|^{-1})), \log_2(2^{k+3}\theta^{-1}(|\xi|^{-1}))\right]; \tag{3.19}$$

$$E_{(j)}^{s}(\eta) = \left[\log_2(2^j\beta^{-1}(|\xi|^{-1})), \log_2(2^{j+3}\beta^{-1}(|\xi|^{-1}))\right], \tag{3.20}$$

where

$$\theta(t) = \begin{cases} \varphi_1(t), & l = d+1 \\ t, & l \neq d+1 \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} \varphi_2(t), & s = b+1 \\ t, & s \neq b+1, \end{cases}$$

and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$.
It is clear that, from (3.19) and (3.20), the following are satisfied

$$|E_k^{(l)}(\xi)| = |E_j^{(s)}(\eta)| = 3 \tag{3.21}$$

$$\theta(2^{-k}\theta^{-1}(|\xi|^{-1})) \leq \theta(2^{t'}) \leq \theta(2^{-k+3}\theta^{-1}(|\xi|^{-1})) \tag{3.22}$$

$$\beta(2^{-j}\beta^{-1}(|\eta|^{-1})) \leq \beta(2^{s'}) \leq \beta(2^{-j+3}\beta^{-1}(|\eta|^{-1})), \tag{3.23}$$

where $(t', s') \in E_k^{(l)}(\xi) \times E_j^{(s)}(\eta)$.
Therefore, for $(t', s') \in E_k^{(l)}(\xi) \times E_j^{(s)}(\eta)$, by Lemma 2.2, (3.22), and (3.23), we have

$$\theta(2^{t'}) \leq 2^{-k+3}\,|\xi|^{-1} \qquad \text{for } k \geq 3; \tag{3.24}$$

$$\theta(2^{t'-1}) \geq 2^{-k-1}\,|\xi|^{-1} \qquad \text{for } k \leq -2; \tag{3.25}$$

$$\beta(2^{s'}) \leq 2^{-j+3}\,|\eta|^{-1} \qquad \text{for } j \geq 3; \tag{3.26}$$

$$\beta(2^{s'-1}) \geq 2^{-j-1}\,|\eta|^{-1} \qquad \text{for } j \leq -2. \tag{3.27}$$

Thus, by Plancherel's Theorem, (3.2)-(3.6) and (3.24)-(3.27), we get

$$\|U_{j,k}^{(l,s)}(f)\|_2 \leq \mathcal{B}_{j,k}\,\|f\|_2, \tag{3.28}$$

where

$$\mathcal{B}_{j,k} = \begin{cases} |kl|^{-(1+\epsilon)}\ |js|^{-(1+\epsilon)}, & \text{if } k, j \leq -2 \\ 2^{-kl-js}, & \text{if } k, j \geq 3 \\ 2^{(-js}\,|kl|^{-(1+\epsilon)}, & \text{if } k \leq -2 \text{ and } j \geq 3 \\ 2^{-kl}\,|js|^{-(1+\epsilon)}, & \text{if } k \geq 3 \text{ and } j \leq -2 \\ 1, & \text{if } k \geq -2 \text{ and } j \leq 3 \end{cases}$$

By an interpolation between (3.18) and (3.28) , we get

$$\|U_{j,k}^{(l,s)}(f)\|_p \leq C\,\mathcal{B}_{j,k}\ \|f\|_p. \tag{3.29}$$

for all $p \in \left(\frac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$. Since the series $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \mathcal{B}_{j,k}$ is converged, we get

$$\|\mathcal{M}_\Omega^{(l,s)}(f)\|_p \leq C\ \|f\|_p. \tag{3.30}$$

for all $p \in \left(\frac{2+2\epsilon}{1+2\epsilon}, 2+2\epsilon\right)$. This completes the proof.

References

[1] B. Al-Azriyah, *A class of singular integral operators*, MSc thesis, Sultan Qaboos University(2018).

[2] H. Al-Qassem, A. Al-Salman, L. C. Cheng and Y. Pan, *Marcinkiewicz integrals on product spaces*, Studia Mathematica, **167**(2005), 227–234.

[3] A. Al-Salman, *Marcinkiewicz functions with hardy space kernels*, Math. Inequal. Appl., **2(21)**(2018), 553–567.

[4] A. Al-Salman, *Rough Marcinkiewicz integrals on product spaces*, Int. Math. Forum, **2(23)**(2007), 1119–1128.

[5] A. Al-Salman, *Marcinkiewicz functions along flat surfaces with hardy space kernels*, J. Integral Equations Appl., **17(4)**(2005), 357–373.

[6] A. Al-Salman, *On Marcinkiewicz integrals along flat surfaces*, Turk J. Math., **29**(2005), 111–120.

[7] A. Al-Salman, *Marcinkiewicz integrals along subvarieties on product domains*, Int. J. Math. Math. Sci., **72**(2004), 4001–4011.

[8] A. Al-Salman and H. Al-Qassem, *Integral operators of Marcinkiewicz type*, J. Integral Equations Appl., **14(4)**(2002) 343–354.

[9] A. Al-Salman and H. Al-Qassem, *Rough Marcinkiewicz integral operators*, Int. J. Math. Math. Sci., **27(8)**(2001), 495–503.

[10] A. Benedek, A. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U. S. A., **48**(1962) 356–365.

[11] J. Chen, D. Fan and Y. Ying, *Rough Marcinkiewicz integrals with* $L(\log L)^2$ *kernels*, Adv. Math. (China), **30**(2001), 179–181.

[12] J. Chen, D. Fan and Y. Pan, *A note on a Marcinkiewicz integral operator*, Math. Nachr., **227**(2001), 33–42.

[13] Y. Choi, *Marcinkiewicz integrals with rough homogeneous kernel of degree zero in product domains*, J. Math. Appl., **261**(2001), 53–60.

[14] Y. Ding, L^2*-boundedness of Marcinkiewicz integral with rough kernel,* Hokkaido Math. J., **27(1)**(1998), 105–115.

[15] Y. Ding, D. Fan and Y. Pan, *On the* L^p *boundedness of Marcinkiewicz integrals*, Michigan Math. J., **50(1)**(2002), 17–26.

[16] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84**(1986), 541–561.

[17] L. Grafakos and A. Stefanov, L^p *bounds for singular integrals andmaximal singular integrals with rough kernels*, Indiana Univ. Math. J., **47(2)**(1998), 455–469.

[18] E. M. Stein, *On the function of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc., **88**(1958), 430–466.

[19] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton(1993).