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On $*_w$-Finiteness Conditions

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Abstract. Let D be an integral domain and let $*$ be a star-operation on D. In this article, we give new characterizations of $*_w$-Noetherian domains and $*_w$-principal ideal domains. More precisely, we show that D is a $*_w$-Noetherian domain (resp., $*_w$-principal ideal domain) if and only if every $*_w$-countable type ideal of D is of $*_w$-finite type (resp., principal).

1. Introduction

1.1 Star-operations

In this subsection, we review some terminology for star-operations. Let D be an integral domain with quotient field K and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A *star-operation* on D is a mapping $I \mapsto I_*$ from $\mathbf{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(a)_* = (a)$ and $(aI)_* = aI_*$;

(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$; and

(3) $(I_*)_* = I_*$.

The most important examples of star-operations are the d-operation, the v-operation and the w-operation. The *d-operation* is just the identity mapping, *i.e.*, $I \mapsto I_d := I$. For an element $I \in \mathbf{F}(D)$, let $I^{-1} = \{a \in K \,|\, aI \subseteq D\}$. The *v-operation* is the mapping defined by $I \mapsto I_v := (I^{-1})^{-1}$. The *w-operation* is the mapping defined by $I \mapsto I_w := \{a \in K \,|\, Ja \subseteq I$ for some nonzero finitely generated ideal J of D with $J_v = D\}$. Given any star-operation $*$ on D, we can construct a new star-operation $*_w$ induced by $*$. For all $I \in \mathbf{F}(D)$, the $*_w$*-operation* is the mapping defined by $I \mapsto I_{*_w} := \{a \in K \,|\, Ja \subseteq I$ for some $J \in \mathrm{GV}^*(D)\}$, where

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$\mathrm{GV}^*(D)$ is the set of nonzero finitely generated ideals J of D with $J_* = D$. An element of $\mathrm{GV}^*(D)$ is called a $*$-*Glaz-Vasconcelos ideal* ($*$-GV-ideal) of D. When $* = d$ (resp., $* = v$), the $*_w$-operation is precisely the same as the d-operation (resp., w-operation).

Let D be an integral domain and let $*$ be a star-operation on D. Then $*$ is said to be of *finite character* (or *finite type*) if $I_* = \bigcup\{J_* \mid J$ is a nonzero finitely generated subideal of $I\}$ for all $I \in \mathbf{F}(D)$. It was shown in [1, Theorem 2.7] that the $*_w$-operation is of finite character. An element $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I_* = I$. We say that a $*$-ideal I of D is of $*$-*finite type* (resp., $*$-*countable type*) if $I = J_*$ for some finitely generated subideal (resp., countably generated subideal) J of I.

The readers can refer to [3] for star-operations and to [1, 5] for $*_w$-operations.

1.2 $*_w$-Noetherian domains and $*_w$-principal ideal domains

In commutative algebra, finiteness conditions play important roles. Especially, Noetherian rings and principal ideal domains have been studied by many mathematicians. Due to their importance, there were several attempts to generalize finiteness conditions in order to extend well-known results and to find new algebraic structures. One of them is to use star-operations. In fact, many integral domains can be easily characterized by using star-operations. Let D be an integral domain and let $*$ be a star-operation on D. Recall that D is a $*_w$-*Noetherian domain* if D satisfies the ascending chain condition on integral $*_w$-ideals (or equivalently, every $*_w$-ideal of D is of $*_w$-finite type [10, Theorem 1.1]). In particular, if $* = d$ (resp., $* = v$), then the concept of $*_w$-Noetherian domains is exactly the same as that of Noetherian domains (resp., strong Mori domains (SM-domains)). Also, we say that D is a $*_w$-*principal ideal domain* ($*_w$-PID) if every $*_w$-ideal of D is principal. In particular, if $* = d$ (resp., $* = v$), then the notion of $*_w$-PIDs is precisely the same as that of PIDs (resp., unique factorization domains (UFDs)) (cf. [6, p. 284]).

The purpose of this article is to indicate that in order to show that an integral domain is a $*_w$-Noetherian domain or a $*_w$-PID, it is enough to check $*_w$-countable type ideals. In fact, we show that an integral domain D is a $*_w$-Noetherian domain (resp., $*_w$-PID) if and only if every $*_w$-countable type ideal of D is of $*_w$-finite type (resp., principal). We also remark that the condition '$*_w$-countable type' in the $*_w$-PID case cannot be reduced to '$*_w$-finite type'.

2. Main Results

Our first result is a characterization of $*_w$-Noetherian domains.

Theorem 2.1. *Let D be an integral domain and let $*$ be a star-operation on D. Then the following statements are equivalent.*

(1) *D is a $*_w$-Noetherian domain.*

(2) *Every $*_w$-countable type ideal of D is of $*_w$-finite type.*

Proof. (1) ⇒ (2) This implication follows from the definition of $*_w$-Noetherian domains.

(2) ⇒ (1) Suppose that every $*_w$-countable type ideal of D is of $*_w$-finite type and deny the conclusion. Let I be a $*_w$-ideal of D that is not of $*_w$-finite type and choose any element $a_1 \in I$. Then $(a_1) \subsetneq I$, because I is not of $*_w$-finite type. Let $a_2 \in I \setminus (a_1)$. Then $(a_1, a_2)_{*_w} \subsetneq I$, because I is not of $*_w$-finite type. By repeating this process, we obtain a subset $\{a_n \,|\, n \in \mathbb{N}\}$ of I such that $a_{n+1} \in I \setminus (a_1, \ldots, a_n)_{*_w}$ for all $n \geq 1$. Let $A = (\{a_n \,|\, n \in \mathbb{N}\})_{*_w}$. Then A is a $*_w$-countable type ideal of D. By the assumption, A is of $*_w$-finite type; so $A = (b_1, \ldots, b_m)_{*_w}$ for some $b_1, \ldots, b_m \in A$. For each $i = 1, \ldots, m$, there exists an element $J_i \in \mathrm{GV}^*(D)$ such that $J_i b_i \subseteq (\{a_n \,|\, n \in \mathbb{N}\})$. Let $J = J_1 \cdots J_m$. Then J is a $*$-GV-ideal of D [5, Lemma 2.3(3)] and $J(b_1, \ldots, b_m) \subseteq (\{a_n \,|\, n \in \mathbb{N}\})$. Since J and $(b_1, \ldots, b_m)$ are finitely generated, $J(b_1, \ldots, b_m) \subseteq (a_1, \ldots, a_\ell)$ for some positive integer ℓ; so $(b_1, \ldots, b_m)_{*_w} \subseteq (a_1, \ldots, a_\ell)_{*_w}$. Therefore $A = (a_1, \ldots, a_\ell)_{*_w}$. Hence $a_{\ell+1} \in (a_1, \ldots, a_\ell)_{*_w}$. However, this contradicts the choice of $a_{\ell+1}$. Thus D is a $*_w$-Noetherian domain. □

Let D be an integral domain. Recall that D is a *Noetherian domain* (resp., *strong Mori domain* (SM-domain)) if D satisfies the ascending chain condition on integral ideals (resp., integral w-ideals) (or equivalently, every ideal (resp., w-ideal) of D is finitely generated (resp., of w-finite type)). By applying $* = d$ or $* = v$ to Theorem 2.1, we obtain

Corollary 2.2. *Let D be an integral domain. Then the following assertions are equivalent.*

(1) *D is a Noetherian domain (resp., SM-domain).*

(2) *Every countably generated ideal (resp., w-countable type ideal) of D is finitely generated (resp., of w-finite type).*

We next give a characterization of $*_w$-PIDs.

Theorem 2.3. *Let D be an integral domain and let $*$ be a star-operation on D. Then the following statements are equivalent.*

(1) *D is a $*_w$-PID.*

(2) *Every $*_w$-countable type ideal of D is principal.*

Proof. (1) ⇒ (2) This implication is obvious by the definition of $*_w$-PIDs.

(2) ⇒ (1) Suppose to the contrary that D is not a $*_w$-PID and let I be a $*_w$-ideal of D which is not principal. Write $I = (\{a_\lambda \,|\, \lambda \in \Lambda\})_{*_w}$, where Λ is an uncountable set of ordinal numbers. Let $\mathcal{A} = \{\alpha \in \Lambda \,|\, (\{a_\lambda \,|\, \lambda < \alpha\})_{*_w} \subsetneq (\{a_\lambda \,|\, \lambda \leq \alpha\})_{*_w}\}$. Then $I = (\{a_\alpha \,|\, \alpha \in \mathcal{A}\})_{*_w}$. If $\mathcal{A}$ is a finite set, then I is of $*_w$-finite type; so by (2), I is principal. This is a contradiction to the choice of I. Hence $\mathcal{A}$ is an infinite set. Let $\{\alpha_n \,|\, n \in \mathbb{N}\}$ be any subset of $\mathcal{A}$, where $\alpha_n < \alpha_{n+1}$ for all $n \geq 1$. Let $C = (\{a_{\alpha_n} \,|\, n \in \mathbb{N}\})$. Then by (2), $C_{*_w} = (c)$ for some $c \in D$; so

there exists a $*$-GV-ideal J of D such that $Jc \subseteq C$. Since J is finitely generated, $Jc \subseteq (\{a_{\alpha_1}, \ldots, a_{\alpha_m}\})$ for some positive integer m; so $c \in (\{a_{\alpha_1}, \ldots, a_{\alpha_m}\})_{*_w}$. Therefore $C_{*_w} = (\{a_{\alpha_1}, \ldots, a_{\alpha_m}\})_{*_w}$. Hence $a_{\alpha_{m+1}} \in (\{a_{\alpha_1}, \ldots, a_{\alpha_m}\})_{*_w}$, which is absurd. Thus D is a $*_w$-PID. □

Let D be an integral domain and let $*$ be a star-operation on D. We say that D is a *$*_w$-Bézout domain* if every $*_w$-finite type ideal of D is principal. The next result is an immediate consequence of Theorems 2.1 and 2.3.

Corollary 2.4. ([2, Remark 3.2(4)]) *Let D be an integral domain and let $*$ be a star-operation on D. Then the following assertions are equivalent.*

(1) *D is a $*_w$-PID.*

(2) *D is both a $*_w$-Noetherian domain and a $*_w$-Bézout domain.*

Let D be an integral domain. Recall that D is a *Bézout domain* (resp., *GCD-domain*) if every finitely generated ideal (resp., w-finite type ideal) of D is principal. Also, it was shown in [9, Theorem 7.9.5] that D is a UFD if and only if every w-ideal of D is principal. By applying $* = d$ or $* = v$ to Theorem 2.3 and Corollary 2.4, we obtain

Corollary 2.5. *Let D be an integral domain. Then the following conditions are equivalent.*

(1) *D is a PID* (*resp., UFD*).

(2) *Every countably generated ideal* (*resp., w-countable type ideal*) *of D is principal.*

(3) *D is both a Noetherian domain* (*resp., SM-domain*) *and a Bézout domain* (*resp., GCD-domain*).

At this point, it is natural to ask if the condition '$*_w$-countable type' in Theorem 2.3 can be reduced to '$*_w$-finite type'. We are closing this article with the following examples which show that the condition cannot be reduced.

Example 2.6. (1) Let V be a one-dimensional nondiscrete valuation domain and let $*$ be any star-operation on V. Then $*_w = d$. (To see this, let $I \in \mathbf{F}(V)$ and let $a \in I_{*_w}$. Then we can find a $*$-GV-ideal J of V such that $Ja \subseteq I$. Since J is finitely generated, J is principal. Since $J_{*_w} = V$, $J = V$; so $a \in I$. Hence $I_{*_w} = I$.) Note that V is a nonNoetherian Bézout domain; so by Corollary 2.5, V is not a PID. Thus V is a $*_w$-Bézout domain which is not a $*_w$-PID.

(2) Let $\mathcal{E}$ be the ring of entire functions. Then $\mathcal{E}$ is a Bézout domain which is not a PID. Moreover, $\mathcal{E}$ is a GCD-domain that is not a UFD.

(3) Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Z} + X\mathbb{Q}[X]$ is a GCD-domain [7, Theorem 2.5] which is not a UFD [8, Corollary 3.6].

(4) Let F be any field and let $\mathbb{Q}_0$ be the semigroup of nonnegative rational numbers. Then the semigroup ring $F[\mathbb{Q}_0]$ is a GCD-domain which is not a UFD [4, Theorems 14.5 and 14.16].

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