KYUNGPOOK Math. J. 63(2023), 561-569
https://doi.org/10.5666/KMJ.2023.63.4.561
pISSN 1225-6951 eISSN 0454-8124

Certain Clean Decompositions for Matrices over Local Rings

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Abstract. An element $a \in R$ is strongly rad-clean provided that there exists an idempotent $e \in R$ such that $a - e \in U(R)$, $ae = ea$ and $eae \in J(eRe)$. In this article, we completely determine when a 2×2 matrix over a commutative local ring is strongly rad clean. An application to matrices over power-series is also given.

1. Introduction

An element $a \in R$ is *strongly clean* provided that it is the sum of an idempotent and a unit that commutes. A ring R is *strongly clean* provided that every element in R is strongly clean. A ring R is local if it has only one maximal right ideal. As is well known, a ring R is local if and only if for any $x \in R$, x or $1 - x$ is invertible. Strongly clean matrices over commutative local rings was extensively studied by many authors from very different view points (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]). Recently, a related cleanness of triangular matrix rings over abelian rings was studied by Diesl et al. (see [9]).

Following Diesl, we say that $a \in R$ is *strongly rad-clean* provided that there exists an idempotent $e \in R$ such that $a - e \in U(R)$, $ae = ea$ and $eae \in J(eRe)$ (see [9]). A ring R is *strongly rad-clean* provided that every element in R is strongly

* Corresponding Author.
Received December 15, 2022; revised July 21, 2023; accepted July 25, 2023.
2020 Mathematics Subject Classification: 15A09, 16E50, 16U99.
Key words and phrases: Strongly clean matrix, Strongly rad-clean matrix, Local ring, Power-series.

rad-clean. Strongly rad-clean rings form a natural subclass of strongly clean rings which have stable range one (see [4]). Let M be a right R-module, and let $\varphi \in end_R(M)$. Then we include a relevant diagram to reinforce the theme of direct sum decompositions:

$$\begin{array}{ccccc} M & = & A & \bigoplus & B \\ & & \varphi\downarrow\cong & & \downarrow\ \varphi \\ M & = & A & \bigoplus & B \end{array}$$

If such diagram holds we call this is an AB-decomposition for φ. It turns out by [2, Lemma 40] that φ is strongly π-regular if and only if there is an AB-decomposition with $\varphi|_B \in N(end(B))$ (the set of nilpotent elements).
Further, φ is strongly rad-clean if and only if there is an *AB-decomposition* with $\varphi|_B \in J(end(B))$ (the Jacobson radical of $end(B)$). Thus, strong rad-cleanness can be seen as a natural extension of strong π-regularity. In [2, Theorem 12], the authors gave a criterion to characterize when a square matrix over a commutative local ring is strongly clean. We extend this result to strongly rad-clean matrices over a commutative local ring. We completely determine when a 2×2 matrix over a commutative local ring has such clean decomposition related to its Jacobson radical. Application to the matrices over power-series is also studied.

Throughout, all rings are commutative with an identity and all modules are unitary left modules. Let M be a left R-module. We denote the endomorphism ring of M by $end(M)$ and the automorphism ring of M by $aut(M)$, respectively. The characteristic polynomial of A is the polynomial $\chi(A) = det(tI_n - A)$. We always use $J(R)$ to denote the Jacobson radical and $U(R)$ is the set of invertible elements of a ring R. $M_2(R)$ stands for the ring of all 2×2 matrices over R, and $GL_2(R)$ denotes the 2-dimensional general linear group of R.

2. Main Results

In this section, we study the structure of strongly rad-clean elements in various situations related to ordinary ring extensions which have roles in ring theory. We start with a well known characterization of strongly rad-clean element in the endomorphism ring of a module M.

Lemma 2.1. *Let $E = end(_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly rad-clean.*

(2) *There exists a direct sum decomposition $M = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in aut(P)$ and $\alpha|_Q \in J\big(end(Q)\big)$.*

Proof. See [4, Proposition 4.1.2]. □

Lemma 2.2. *Let R be a ring, let M be a left R-module. Suppose that $x, y, a, b \in end(_RM)$ such that $xa + yb = 1_M, xy = yx = 0, ay = ya$ and $xb = bx$. Then $M = ker(x) \oplus ker(y)$ as left R-modules.*

Proof. See [2, Lemma 11]. □

A commutative ring R is *projective-free* if every finitely generated projective R-module is free. Evidently, every commutative local ring is projective-free. We now derive

Lemma 2.3. *Let R be projective-free. Then $A \in M_2(R)$ is strongly rad-clean if and only if $A \in GL_2(R)$, or $A \in J\big(M_2(R)\big)$, or A is similar to $diag(\alpha, \beta)$ with $\alpha \in J(R)$ and $\beta \in U(R)$.*

Proof. $\Longrightarrow$ Write $A = E + U, E^2 = E, U \in GL_2(R), EA = AE \in J(M_2(R))$. Since R is projective-free, there exists $P \in GL_n(R)$ such that $PEP^{-1} = diag(0,0), diag(1,1)$ or $diag(1,0)$. Then (i) $PAP^{-1} = PUP^{-1}$; hence, $A \in GL_2(R)$, (ii) $(PAP^{-1})diag(1,1) = diag(1,1)(PAP^{-1}) \in J(M_2(R)$, and so $A \in J(M_2(R))$. (3) $(PAP^{-1})diag(1,0) = diag(1,0)(PAP^{-1}) \in J(M_2(R)$ and $PAP^{-1} - diag(1,0) \in GL_2(R)$. Hence, $PAP^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in J(R), b = c = 0$ and $d \in UR)$. Therefore A is similar to $diag(\alpha, \beta)$ with $\alpha \in J(R)$ and $\beta \in U(R)$.

$\Longleftarrow$ If $A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, then A is strongly rad-clean. We now assume that A is similar to $diag(\alpha, \beta)$ with $\alpha \in J(R)$ and $\beta \in U(R)$. Then A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}$ where

$$\begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} \in GL_2(R), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in J\big(M_2(R)\big)$$
$$\begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

Therefore $A \in M_2(R)$ is strongly rad-clean. $\square$

Theorem 2.4. *Let R be projective-free. Then $A \in M_2(R)$ is strongly rad-clean if and only if*

(1) *$A \in GL_2(R)\big)$, or*

(2) *$A \in J\big(M_2(R)\big)$, or*

(3) *$x^2 = tr(A)x - detA$ has roots $\alpha \in U(R), \beta \in J(R)$.*

Proof. $\Longrightarrow$ By Lemma 2.3, $A \in GL_2(R)$, or $A \in J\big(M_2(R)\big)$, or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R)$ and $\beta \in U(R)$. Then $\chi(A) = (x - \alpha)(x - \beta)$ has roots $\alpha \in U(R), \beta \in J(R)$.

$\Longleftarrow$ If (1) or (2) holds, then $A \in M_2(R)$ is strongly rad-clean. If (3) holds, we assume that $\chi(A) = (t - \alpha)(t - \beta)$. Choose $X = A - \alpha I_2$ and $Y = A - \beta I_2$. Then

$$X(\beta - \alpha)^{-1} I_2 - Y(\beta - \alpha)^{-1} I_2 = I_2,$$
$$XY = YX = 0, X(\beta - \alpha)^{-1} I_2 = (\beta - \alpha)^{-1} I_2 X,$$
$$(\beta - \alpha)^{-1} I_2 Y = Y(\beta - \alpha)^{-1} I_2.$$

By virtue of Lemma 2.2, we have $2R = ker(X) \oplus ker(Y)$. For any $x \in ker(X)$, we have $(x)AX = (x)XA = 0$, and so $(x)A \in ker(X)$. Then $ker(X)$ is A-invariant. Similarly, $ker(Y)$ is A-invariant. For any $x \in ker(X)$, we have $0 = (x)X = (x)\big(A - \alpha I_2\big)$; hence, $(x)A = (x)\alpha I_2$. By hypothesis, we have $A|_{ker(X)} \in J\big(end(ker(X))\big)$. For any $y \in ker(Y)$, we prove that

$$0 = (y)Y = (y)\big(A - \beta I_2\big).$$

This implies that $(y)A = (y)\big(\beta I_2\big)$. Obviously, $A|_{ker(Y)} \in aut\big(ker(Y)\big)$. Therefore $A \in M_2(R)$ is strongly rad-clean by Lemma 2.1. □

We have accumulated all the information necessary to prove the following.

Theorem 2.5. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly rad-clean.*

(2) *$A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R)$ and the quadratic equation $x^2 + x = -\frac{detA}{tr^2A}$ has a root in $J(R)$.*

(3) *$A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R), detA \in J(R)$ and the quadratic equation $x^2 + x = \frac{detA}{tr^2A - 4detA}$ is solvable.*

Proof. (1) ⇒ (2) Assume that $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. By virtue of Theorem 2.4, $trA \in U(R)$ and the characteristic polynomial $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$. According to Lemma 2.3, A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. Clearly, $y^2 - (\lambda + \mu)y + \lambda\mu = 0$ has a root λ in $J(R)$. Hence so does the equation

$$(\lambda + \mu)^{-1}y^2 - y = -(\lambda + \mu)^{-1}\lambda\mu.$$

Set $z = (\lambda + \mu)^{-1}y$. Then

$$(\lambda + \mu)z^2 - (\lambda + \mu)z = -(\lambda + \mu)^{-1}\lambda\mu.$$

That is, $z^2 - z = -(\lambda + \mu)^{-2}\lambda\mu$. Consequently, $z^2 - z = -\frac{detA}{tr^2A}$ has a root in $J(R)$. Let $x = -z$. Then $x^2 + x = -\frac{detA}{tr^2A}$ has a root in $J(R)$.

(2) ⇒ (3) By hypothesis, we prove that the equation $y^2 - y = -\frac{detA}{tr^2A}$ has a root $a \in J(R)$. Assume that $trA \in U(R)$. Then $\big(a(2a-1)^{-1}\big)^2 - \big(a(2a-1)^{-1}\big) = \frac{detA}{tr^2A\cdot\big(4(a^2-a)+1\big)} = \frac{detA}{tr^2A\cdot\big(-4(trA)^{-2}detA+1\big)} = \frac{detA}{tr^2A - 4detA}$. Therefore the equation $y^2 - y = \frac{detA}{tr^2A - 4detA}$ is solvable. Let $x = -y$. Then $x^2 + x = \frac{detA}{tr^2A - 4detA}$ is solvable.

(3) ⇒ (1) Suppose $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. Then $trA \in U(R), detA \in J(R)$ and the equation $x^2 + x = \frac{detA}{tr^2A - 4detA}$ has a root. Let $y = -x$. Then $y^2 - y\frac{detA}{tr^2A - 4detA}$ has a root $a \in R$. Clearly, $b := 1 - a \in R$ is a root of this

equation. As $a^2 - a \in J(R)$, we see that either $a \in J(R)$ or $1 - a \in J(R)$. Thus, $2a-1 = 1-2(1-a) \in U(R)$. It is easy to verify that $\big(a(2a-1)^{-1}trA\big)^2 - trA\cdot\big(a(2a-1)^{-1}trA\big) + detA = -\frac{tr^2A\cdot(a^2-a)}{4(a^2-a)+1} + detA = 0$. Thus the equation $y^2 - trA\cdot y + detA = 0$ has roots $a(2a-1)^{-1}trA$ and $b(2b-1)^{-1}trA$. Since $ab \in J(R)$, we see that $a+b = 1$ and either $a \in J(R)$ or $b \in J(R)$. Therefore $y^2 - trA \cdot y + detA = 0$ has a root in $U(R)$ and a root in $J(R)$. Since R is a commutative local ring, it is projective-free. By virtue of Theorem 2.4, we obtain the result. □

Corollary 2.6. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly clean.*

(2) *$I_2 - A \in GL_2(R)$ or $A \in M_2(R)$ is strongly rad-clean.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) In view of [3, Corollary 16.4.33], $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$ or $trA \in U(R), detA \in J(R)$ and the quadratic equation $x^2 - x = \frac{detA}{tr^2A-4detA}$ is solvable. Hence $x^2 + x = \frac{detA}{tr^2A-4detA}$ is solvable. According to Theorem 2.5, we complete the proof. □

Corollary 2.7. *Let R be a commutative local ring. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly rad-clean.*

(2) *$A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R), detA \in J(R)$ and $tr^2A - 4detA$ is square.*

Proof. (1) $\Rightarrow$ (2) According to Theorem 2.5, $A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R), detA \in J(R)$ and the quadratic equation $x^2 - x = \frac{detA}{tr^2A-4detA}$ is solvable. If $a \in R$ is the root of the equation, then $(2a-1)^2 = 4(a^2-a)+1 = \frac{tr^2A}{tr^2A-4detA} \in U(R)$. As in the proof of Theorem 2.5 , $2a-1 \in U(R)$. Therefore $tr^2A - 4detA = \big(trA\cdot(2a-1)^{-1}\big)^2$.

(2) $\Rightarrow$ (1) If $trA \in U(R), detA \in J(R)$ and $tr^2A - 4detA = u^2$ for some $u \in R$, then $u \in U(R)$ and the equation $x^2 + x = \frac{detA}{tr^2A-4detA}$ has a root $-\frac{1}{2}u^{-1}(trA+u)$. By virtue of Theorem 2.5, $A \in M_2(R)$ is strongly rad-clean. □

Every strongly rad-clean matrix over a ring is strongly clean. But there exist strongly clean matrices over a commutative local ring which is not strongly rad-clean as the following shows.

Example 2.8. Let $R = \mathbb{Z}_4$, and let $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in M_2(R)$. R is a commutative local ring. Then $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is a strongly clean decomposition. Thus $A \in M_2(R)$ is strongly clean. If $A \in M_2(R)$ is strongly rad-clean, there exist an idempotent $E \in M_2(R)$ and an invertible $U \in M_2(R)$ such that $A = E + U, EA = AE$ and $EAE \in J\big(M_2(R)\big)$. Hence, $AU = A(A-E) = (A-E)A = UA$, and then $E = A - U \in GL_2(R)$ as $A^4 = 0$. This implies that $E = I_2$, and so $EAE = A \notin J\big(M_2(R)\big)$, as $J(R) = 2R$. This gives a contradiction. Therefore $A \in M_2(R)$ is not strongly rad-clean.

Following Cui and Chen, an element $a \in R$ is quaspolar if there exists an idempotent $e \in comm(a)$ such that $a + e \in U(R)$ and $ae \in R^{qnil}$ (see [6]). Obviously, A is strongly J-clean $\Longrightarrow A$ is strongly rad-clean $\Longrightarrow A$ is quasipolar. But the converses are not true, as the following shows:

Example 2.9. (1) Let R be a commutative local ring and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ be in $M_2(R)$. Since $A \in GL_2(R)$, by Lemma 2.3, it is strongly rad-clean but is not strongly J-clean, as $I_2 - A \notin J(M_2(R))$.

(2) Let $R = \mathbb{Z}_{(3)}$ and $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Then $trA = 3 \in J(R)$ and $detA = 3 \in J(R)$. Hence A is quasipolar by [5, Theorem 2.6]. Note that $trA \notin U(R)$, $A \notin GL_2(R)$ and $A \notin J(M_2(R))$. Thus, A is not strongly rad-clean, in terms of Corollary 2.7.

Set $B_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B_{21}(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. We now derive

Theorem 2.10. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *Every $A \in M_2(R)$ with invertible trace is strongly rad-clean.*

(2) *For any $\lambda \in J(R), \mu \in U(R)$, the quadratic equation $x^2 = \mu x + \lambda$ is solvable.*

Proof. (1) $\Rightarrow$ (2) Let $\lambda \in J(R), \mu \in U(R)$. Choose $A = \begin{pmatrix} 0 & -\lambda \\ 1 & \mu \end{pmatrix}$. Then $A \in M_2(R)$ is strongly rad clean. Obviously, $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. In view of Theorem 2.4, we see that the quadratic equation $x^2 = \mu x + \lambda$ is solvable.

(2) $\Rightarrow$ (1) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $tr(A) \in U(R)$.

Case I. $c \in U(R)$. Then

$$diag(c,1)B_{12}(-ac^{-1})AB_{12}(ac^{-1})diag(c^{-1},1) = \begin{pmatrix} 0 & -\lambda \\ 1 & -\mu \end{pmatrix}$$

for some $\lambda, \mu \in R$. If $\lambda \in U(R)$, then $A \in GL_2(R)$, and so it is strongly rad-clean. If $\lambda \in J(R)$, then $\mu \in U(R)$. Then A is strongly rad-clean by Theorem 2.5.

Case II. $b \in U(R)$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

and the result follows from Case I.
Case III. $c, b \in J(R), a - d \in U(R)$. Then

$$B_{21}(-1)AB_{21}(1) = \begin{pmatrix} a+b & b \\ c-a+d-b & d-b \end{pmatrix}$$

where $a - d + b - c \in U(R)$; hence the result follows from Case I.
Case IV. $c, b \in J(R), a, d \in U(R)$. Then

$$B_{21}(-ca^{-1})A = \begin{pmatrix} a & b \\ 0 & d - ca^{-1}b \end{pmatrix};$$

hence, $A \in GL_2(R)$.
Case V. $c, b, a, d \in J(R)$. Then $A \in J\big(M_2(R)\big)$, and so $tr(A) \in J(R)$, a contradiction.

Therefore $A \in M_2(R)$ with invertible trace is strongly rad-clean. □

Example 2.11. Let $R = \mathbb{Z}_4$. Then R is a commutative local ring. For any $\lambda \in J(R), \mu \in U(R)$, we directly check that the quadratic equation $x^2 = \mu x + \lambda$ is solvable. Applying Theorem 2.10, every 2×2 matrix over R with invertible trace is strongly rad-clean. In this case, $M_2(R)$ is not strongly rad-clean.

Example 2.12. Let $R = \widehat{\mathbb{Z}}_2$ be the ring of 2-adic integers. Then every 2×2 matrix with invertible trace is strongly rad-clean.

Proof. Obviously, R is a commutative local ring. Let $\lambda \in J(R), \mu \in U(R)$. Then $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \in M_2(R)$ is strongly clean, by [5, Theorem 3.3]. Clearly, $det(A) = -\lambda \in J(R)$. As $R/J(R) \cong \mathbb{Z}_2$, we see that $\mu \in 1 + J(R)$, and then $det(A - I_2) = 1 - \lambda - \mu \in J(R)$. In light of [5, Lemma 3.1], the equation $x^2 = \mu x + \lambda$ is solvable. This completes the proof, by Theorem 2.10. □

We note that matrix with non-invertible trace over commutative local rings maybe not strongly rad-clean. For instance, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\widehat{\mathbb{Z}}_2)$ is not strongly rad-clean.

3. Applications

We now apply our preceding results and investigate strongly rad-clean matrices over power series over commutative local rings.

Lemma 3.1. *Let R be a commutative ring, and let $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$. Then the following hold:*

(1) *$A(x_1, \cdots, x_n) \in GL_2\big(R[[x_1, \cdots, x_n]]\big)$ if and only if $A(0, \cdots, 0) \in GL_2(R)$.*

(2) *$A(x_1, \cdots, x_n) \in J\big(M_2(R[[x_1, \cdots, x_n]])\big)$ if and only if $A(0, \cdots, 0) \in J\big(M_2(R)\big)$.*

Proof. (1) We suffice to prove for $n = 1$. If $A(x_1) \in GL_2\big(R[[x_1]]\big)$, it is easy to verify that $A(0) \in GL_2(R)$. Conversely, assume that $A(0) \in GL_2(R)$. Write

$$A(x_1) = \begin{pmatrix} \sum\limits_{i=0}^{\infty} a_i x_1^i & \sum\limits_{i=0}^{\infty} b_i x_1^i \\ \sum\limits_{i=0}^{\infty} c_i x_1^i & \sum\limits_{i=0}^{\infty} d_i x_1^i \end{pmatrix},$$

where $A(0) = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. We note that the determinant of $A(x_1)$ is $a_0 d_0 - c_0 b_0 + x_1 f(x_1)$, which is a unit plus an element of the radical of $R[[x_1]]$. Thus, $A(x_1) \in GL_2\big(R[[x_1]]\big)$, as required.

(2) It is immediate from (1). □

Theorem 3.2. *Let R be a commutative local ring, and let $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$. Then the following are equivalent:*

(1) $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$ *is strongly rad-clean.*

(2) $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]/(x_1^{m_1} \cdots x_n^{m_n})\big)$ *is strongly rad-clean.*

(3) $A(0, \cdots, 0) \in M_2(R)$ *is strongly rad-clean.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) It will suffice to prove for $n = 1$. Set $x = x_1$. Clearly, $R[[x]]$ is a commutative local ring. Since $A(0)$ is strongly clean in $M_2(R)$, it follows from Theorem 2.4 that $A(0) \in GL_2(R)$, or $A(0) \in J\big(M_2(R)\big)$, or $\chi\big(A(0)\big)$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$. If $A(0) \in GL_2(R)$ or $A(0) \in J\big(M_2(R)\big)$, in view of Lemma 3.1, $A(x) \in GL_2\big(R[[x]]\big)$ or $A(x) \in J\big(M_2(R[[x]])\big)$. Hence, $A(x) \in M_2\big(R[[x]]\big)$ is strongly rad-clean. Thus, we may assume that $\chi\big(A(0)\big) = t^2 + \mu t + \lambda$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$.

Write $\chi\big(A(x)\big) = t^2 + \mu(x)t + \lambda(x)$ where $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ and $\mu_0 = \mu, \lambda_0 = \lambda$. Let $b_0 = \alpha$. It is easy to verify that $\mu_0 = \alpha + \beta \in U(R)$. Hence, $2b_0 + \mu_0 \in U(R)$. Choose

$$b_1 = (2b_0 + \mu_0)^{-1}(-\lambda_1 - \mu_1 b_0),$$
$$b_2 = (2b_0 + \mu_0)^{-1}(-\lambda_2 - \mu_1 b_1 - \mu_2 b_0 - b_1^2),$$
$$\vdots$$

Then $y = \sum\limits_{i=0}^{\infty} b_i x^i \in R[[x]]$ is a root of $\chi\big(A(x)\big)$. In addition, $y \in J\big(R[[x]]\big)$ as $b_0 \in J(R)$. Since $y^2 + \mu(x)y + \lambda(x) = 0$, we have $\chi\big(A(x)\big) = (t-y)(t+y) + \mu(t-y) = (t-y)(t+y+\mu)$. Set $z = -y - \mu$. Then $z \in U\big(R[[x]]\big)$ as $\mu \in U\big(R[[x]]\big)$. Therefore $\chi\big(A(x)\big)$ has a root in $J\big(R[[x]]\big)$ and a root in $U\big(R[[x]]\big)$. According to Theorem 2.4, $A(x) \in M_2\big(R[[x]]\big)$ is strongly rad-clean, as asserted. □

Corollary 3.3. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *Every $A \in M_2(R)$ with invertible trace is strongly rad-clean.*

(2) *Every $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$ with invertible trace is strongly rad-clean.*

Proof. (1) $\Rightarrow$ (2) Let $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$ with invertible trace. Then $trA(0, \cdots, 0) \in U(R)$. By hypothesis, $A(0, \cdots, 0) \in M_2(R)$ is strongly rad-clean. In light of Theorem 2.4, $A(x_1, \cdots, x_n) \in M_2\big(R[[x_1, \cdots, x_n]]\big)$ is strongly rad-clean.

(2) $\Rightarrow$ (1) is obvious. $\square$

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