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Commutativity Criteria for a Factor Ring R/P Arising from P-Centralizers

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Abstract. In this paper we consider a more general class of centralizers called I-centralizers. More precisely, given a prime ideal P of an arbitrary ring R we establish a connection between certain algebraic identities involving a pair of P-left centralizers and the structure of the factor ring R/P.

1. Introduction

Throughout this paper, R will be a ring with center $Z(R)$. Let $x, y \in R$. The commutator $xy - yx$ will be denoted by $[x, y]$ and the anti-commutator $xy + yx$ will be represented by $x \circ y$. Recall that an ideal P of R is prime if for all $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and d is called the associated derivation of F. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R.
An additive mapping $T : R \to R$ is said to be a left centralizer (resp. right centralizer) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer in case T is a left and a right centralizer of R. In

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ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semi-prime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale left ring of quotients Q_r (see [5, Chapter 2]). If R has the identity element then $T : R \to R$ is a left centralizer if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. Recently there has been a great interest in the study of the relationship between the commutativity of a ring and some specific additive mappings defined on the considered ring. In this direction, several authors have studied this problem by considering left (respectively right) centralizers in prime and semi-prime rings (see for example [1, 2, 6, 7], where further references can be found).

In the following definition, we have initiated the concept of I-centralizers in rings, where I is an ideal, and extended several known results.

Definition. Let I be an ideal of a ring R and $f : R \longrightarrow R$ an additive mapping.

(1) f is called an *I-left centralizer* if $f(xy) - f(x)y \in I$ for all $x, y \in R$.

(2) f is called an *I-right centralizer* if $f(xy) - xf(y) \in I$ for all $x, y \in R$.

(3) f is called an *I-centralizer* if and only if f is both an I-left centralizer and I-right centralizer.

Example.

(1) The zero function Θ_R is an I-centralizer on R.

(2) The I_d and $-I_d$ are I-left centralizers (resp. I-right centralizers) on R, where I_d denotes the identity function.

(3) Consider the ring $R = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}$. Let I be the nonzero ideal of R defined by $I = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{Z} \right\}$. It is easy to verify that the additive mapping $T : R \to R$ defined by:

$$T \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is an I-centralizer but T is not a centralizer.

The main goal of this work is to continue on this line of investigation and study the relationship between the structure of quotient rings R/P and the behavior of P-centralizers satisfying specific algebraic identities.

In the sequel, we shall make some use of the following well-known result.

Fact 1.1. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If $aIb \subseteq P$ for all $a, b \in R$, then $a \in P$ or $b \in P$.*

Fact 1.2. *Let R be a semi-prime ring, I a nonzero ideal of R and $a \in I$ such that $aIa = 0$, then $a = 0$.*

2. Identities Involving a Pair of Left P-Centralizers

In what follows, $\overline{x}$ for x in R denotes $x + P$ in R/P.

In [4, Theorem 2.3], Aydin proved that if R is a non-commutative prime ring, F a generalized derivation of R associated with a nonzero derivation d and $a \notin Z(R)$ such that $F(x)a = aF(x)$ for all $x \in I$, then $d(x) = \lambda[x, a]$, for all $x \in I$, where I is an ideal of R.

Inspired by the above result, we here consider a more general algebraic identity involving two P-left centralizers by omitting the primeness assumption imposed on the ring R.

Theorem 2.1. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on R, satisfying the condition $\overline{T_1(x)a - aT_2(x)} \in Z(R/P)$ for all $x \in I$, where $a \in R$, then one of the following assertions holds:*

1. *$T_1(R) \subseteq P$ and $aT_2(R) \subseteq P$;*
2. *R/P is a commutative integral domain;*
3. *$[a, R] \subset P$.*

Proof. By assumption, we have

$$[T_1(x)a - aT_2(x), r] \in P \quad \text{for all} \quad r, x \in I. \tag{2.1}$$

Replacing x by xy in (2.1), we obtain

$$[T_1(x)ya, r] - [aT_2(x)y, r] \in P \quad \text{for all} \quad r, x, y \in I$$

in such a way that

$$(T_1(x)a - aT_2(x))[y, r] + \big[T_1(x)[y, a], r\big] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.2}$$

Substituting yr for y in (2.2), we get

$$\big[T_1(x)y[r, a], r\big] \in P \quad \text{for all} \quad r, x, y \in I.$$

That is

$$T_1(x)y\big[[r, a], r\big] + [T_1(x), r]y[r, a] + T_1(x)[y, r][r, a] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.3}$$

Putting $T_1(x)y$ instead of y in (2.3) and using it, one can see that

$$[T_1(x), r]T_1(x)y[r, a] \in P \quad \text{for all} \quad r, x, y \in I.$$

According to Fact 1.1, we obtain for each $r \in I$, either $[T_1(x), r]T_1(x) \in P$ or $[r, a] \in P$. Define $A = \{r \in I \,/\, [T_1(x), r]T_1(x) \in P \text{ for all } x \in I\}$ and $B = \{r \in I \,/\, [r, a] \in P\}$. Clearly, A and B are additive subgroups of I whose union is I. Hence by Brauer's trick, we have either $A = I$ or $B = I$.
In the second case, namely $[I, a] \subseteq P$. Since $RI \subseteq I$, then $[R, a] \subseteq P$.
Now consider $A = I$, in this situation

$$[T_1(x), r]T_1(x) \in P \quad \text{for all} \quad r, x \in I.$$

Substituting sr for r in the above expression, we arrive at

$$[T_1(x), s]rT_1(x) \in P \quad \text{for all} \quad r, s, x \in I. \tag{2.4}$$

Right multiplying the above equation by s and combining it with (2.4), it follows that

$$[T_1(x), s]I[T_1(x), s] \subseteq P \quad \text{for all} \quad s, x \in I.$$

Applying Fact 1.2, we conclude that $\overline{T_1(x)} \in Z(R/P)$ for all $x \in I$. Writing xt for x in the last expression, where $t \in R$, we arrive at $\bar{t} \in Z(R/P)$ or $\overline{T_1(x)} = \bar{0}$. i.e., R/P is commutative or $T_1(R) \subseteq P$ and our hypothesis reduces to

$$[r, aT_2(x)] \in P \quad \text{for all} \quad r, x \in I$$

which means that

$$a[r, T_2(x)] + [r, a]T_2(x) \in P \quad \text{for all} \quad r, x \in I. \tag{2.5}$$

Replacing x by xt in (2.5), on can see that

$$a[r, T_2(x)]t + aT_2(x)[r, t] + [r, a]T_2(x)t \in P \quad \text{for all} \quad x, t \in I. \tag{2.6}$$

Right multiplying (2.5) by t and subtracting it from (2.6), we get

$$aT_2(x)[r, t] \in P \quad \text{for all} \quad r, t, x \in I. \tag{2.7}$$

Substituting r by ru in (2.7) and employing it, we obtain

$$aT_2(x)I[u, t] \subseteq P \quad \text{for all} \quad t, u, x \in I. \tag{2.8}$$

Once again invoking Fact 1.1, it follows from equation (2.8) that $aT_2(R) \subseteq P$ or $[R, R] \subseteq P$. Finally, we have either ($T_1(R) \subseteq P$ and $aT_2(R) \subseteq P$) or $[a, R] \subseteq P$. $\square$

As an application of our Theorem, we get the following result.

Corollary 2.2. *Let R be a non-commutative prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are two left centralizers on R such that $T_1(x)a \pm aT_2(x) \in Z(R)$ for all $x \in I$, where $a \notin Z(R)$, then $T_1 = 0$ and $aT_2 = 0$.*

In [3, Theorem 2.1], it is showed that if a prime ring R admits a nonzero left centralizer T, with $T(x) \neq x$ for all x in a nonzero ideal I of R, such that $T([x,y]) = [x,y]$ for all $x, y \in I$, then R must be commutative. The author in [8] with addition of 2-torsion freeness hypothesis, extended the preceding result to a Jordan ideal.

Motivated by the preceding results we investigate a more general context which allows us to generalize the above result in two ways. First of all, we will assume that $T([x,y])$ belong to center of R/P rather than $T([x,y]) = 0$. Secondly we will investigate the behavior of the more general expression $\overline{T_1(xy) - T_2(yx)} \in Z(R/P)$ involving two P-left centralizers instead of the expression $T(xy) - T(yx) = 0$.

Theorem 2.3. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*

(1) *$\overline{T_1(xy) - T_2(yx)} \in Z(R/P)$ for all $x, y \in I$;*

(2) *($T_1(R) \subseteq P$ and $T_2(R) \subseteq P$) or R/P is a commutative integral domain.*

Proof. By given assumption, we have

$$\overline{T_1(xy) - T_2(yx)} \in Z(R/P) \quad \text{for all} \quad x, y \in I. \tag{2.9}$$

Substituting yr for y in (2.9), and by expanding this equation, we get

$$\big[T_2(y)[x,r], r\big] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.10}$$

Replacing y by $yT_2(y)$ in (2.10), we find that

$$T_2(y)\big[T_2(y)[x,r], r\big] + [T_2(y), r]T_2(y)[x,r] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.11}$$

In light of (2.10), Eq. (2.11) yields

$$[T_2(y), r]T_2(y)[x,r] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.12}$$

Writing tx for x in (2.12), one can easily to see that

$$[T_2(y), r]T_2(y)t[x,r] \in P \quad \text{for all} \quad r, t, x, y \in I.$$

According to Fact 1.1, we obtain either R/P is an integral domain or $[T_2(y), r]T_2(y) \in P$ for all $r, y \in I$. Arguing as above, the last relation assures that $\overline{T_2(y)} \in Z(R/P)$ for all $y \in I$ and our hypothesis becomes

$$T_1(x)[y,x] + [T_1(x), x]y \in P \quad \text{for all} \quad x, y \in I. \tag{2.13}$$

Putting yu instead of y in (2.13), we get

$$T_1(x)y[u,x] \in P \quad \text{for all} \quad u,x,y \in I.$$

By the primeness of P, we conclude that $T_1(R) \subseteq P$ or R/P is an integral domain. Now if $T_1(R) \subseteq P$, then equation (2.9) yields $\overline{T_2(y)x} \in Z(R/P)$ for all $x,y \in I$. Commuting this expression with r, we find that $T_2(y)I[x,r] \subseteq P$. Once again applying Fact 1.1, it follows that $T_2(R) \subseteq P$ or R/P is a commutative integral domain. □

As an application of Theorem 2.3, the following corollary gives a generalization of some results in [3, 8].

Corollary 2.4. *Let R be a prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are nonzero two left centralizers on R, then the following assertions are equivalent:*

(1) $T_1(xy) \pm T_2(yx) \in Z(R)$ *for all* $x,y \in I$*;*

(2) *R is a commutative integral domain.*

Corollary 2.5. *Let R be a prime ring and I a nonzero ideal of R. Suppose that T is a nonzero left centralizer on R, then the following assertions are equivalent:*

(1) $T([x,y]) \in Z(R)$ *for all* $x,y \in I$*;*

(2) $T(x \circ y) \in Z(R)$ *for all* $x,y \in I$*;*

(3) *R is a commutative integral domain.*

In [3, Theorems 3.1 and 3.3], it is proved that a prime ring R must be a commutative integral domain if it admits a non trivial left centralizer T such that $T(xy) - xy \in Z(R)$ or $T(xy) - yx \in Z(R)$ for all x,y in a nonzero ideal I of R. This result can be obtained as an immediate application of Corollary 2.5.

Corollary 2.6. *Let R be a prime ring and I a nonzero ideal of R. Suppose that T is a non trivial left centralizer on R, then the following assertions are equivalent:*

(1) $T(xy) \pm xy \in Z(R)$ *for all* $x,y \in I$*;*

(2) $T(xy) \pm yx \in Z(R)$ *for all* $x,y \in I$*;*

(3) *R is a commutative integral domain.*

The following theorem exhibits a connection between the commutativity of R/P and range inclusion results of a pair of P-left centralizers.

Theorem 2.7. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*

(1) $\overline{T_1(x)T_2(x)} \in Z(R/P)$ *for all* $x \in I$*;*

(2) $T_1(R) \subseteq P$ *or* $T_2(R) \subseteq P$ *or* R/P *is a commutative integral domain.*

Proof. For non-trivial implications. Assume that

$$\overline{T_1(x)T_2(x)} \in Z(R/P) \quad \text{for all} \quad x \in I. \tag{2.14}$$

A Linearization of (2.14) gives

$$\overline{T_1(x)T_2(y) + T_1(y)T_2(x)} \in Z(R/P) \quad \text{for all} \quad x, y \in I.$$

This means that

$$[T_1(x), r]T_2(y) + T_1(x)[T_2(y), r] + T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.15}$$

Substituting $yT_2(x)$ for y in (2.15) and combining it from the above expression, we get

$$\big(T_1(x)T_2(y) + T_1(y)T_2(x)\big)[T_2(x), r] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.16}$$

Putting tr instead of r in (2.16), we obtain

$$\big(T_1(x)T_2(y) + T_1(y)T_2(x)\big)t[T_2(x), r] \in P \quad \text{for all} \quad r, t, x, y \in I.$$

In view of the primeness of P, we find that either $T_1(x)T_2(y) + T_1(y)T_2(x) \in P$ for all $x, y \in I$ or $[T_2(x), r] \in P$ for all $r, x \in I$.
In the latter case, taking $x = xs$, it is obviously to see that

$$T_2(x)[s, r] \in P \quad \text{for all} \quad r, s, x \in I. \tag{2.17}$$

Writing xu for x and using Fact 1.1, we arrive at $T_2(R) \subseteq P$ or R/P is commutative. Now consider the first case, i.e., $T_1(x)T_2(y) + T_1(y)T_2(x) \in P$ for all $x, y \in I$. Replacing y by yw in this equation, it follows that $T_1(y)(T_2(x)w - wT_2(x)) \in P$ for all $w, x, y \in I$. Thereby obtaining,

$$T_1(y)z(T_2(x)w - wT_2(x)) \in P \quad \text{for all} \quad w, x, y, z \in I.$$

Therefore, either $T_1(R) \subseteq P$ or $T_2(x)w - wT_2(x) \in P$ for all $w, x \in I$. In the last case, putting $x = xy$, we easily get $T_2(x)[w, y] \in P$ for all $w, x, y \in I$ proving that $T_2(R) \subseteq P$ or R/P is an integral domain. □

Corollary 2.8. *Let R be a prime ring and I a nonzero ideal of R. If T_1 and T_2 are two nonzero left centralizers on R such that $T_1(x)T_2(x) \in Z(R)$ for all $x \in I$, then R is a commutative integral domain.*

Theorem 2.9. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*

(1) $\overline{[T_1(x), T_2(y)]} \in Z(R/P)$ *for all* $x, y \in I$*;*
(2) $\overline{T_1(x) \circ T_2(y)} \in Z(R/P)$ *for all* $x, y \in I$*;*
(3) $T_1(R) \subseteq P$ *or* $T_2(R) \subseteq P$ *or* R/P *is a commutative integral domain.*

Proof. Wee only need to prove (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).
(1) $\Longrightarrow$ (3) For all $x, y \in I$, we suppose that

$$\overline{[T_1(x), T_2(y)]} \in Z(R/P). \tag{2.18}$$

This may be rewritten as

$$\big[[T_1(x), T_2(y)], r\big] \in P \quad \text{for all} \quad r, x, y \in I. \tag{2.19}$$

Analogously, replacing yt for y, where $t \in R$ in (2.19), and by appropriate expansion, get

$$[T_1(x), T_2(y)][t, r] + T_2(y)\big[[T_1(x), t], r\big] + [T_2(y), r][T_1(x), t] \in P. \tag{2.20}$$

Letting $t = T_1(x)$ in (2.20), one can see that

$$[T_1(x), T_2(y)][T_1(x), r] \in P \quad \text{for all} \quad r, x, y \in I.$$

Keeping in mind that $\overline{[T_1(x), T_2(y)]} \in Z(R/P)$, we get

$$[T_1(x), T_2(y)]I[T_1(x), r] \subseteq P \quad \text{for all} \quad r, x, y \in I. \tag{2.21}$$

In light of the primeness of P, we find that either $[T_1(x), T_2(y)] \in P$ or $[T_1(x), r] \in P$ for all $x \in I$. Consequently, I is a union of two additive subgroups I_1 and I_2, where

$$I_1 = \{x \in I \,/\, [T_1(x), T_2(y)] \in P \text{ for all } y \in I\} \quad \text{and} \quad I_2 = \{x \in I \,/\, [T_1(x), I] \subseteq P\}.$$

According to Brauer's trick, we are forced to conclude that either $I = I_1$ or $I = I_2$. If $I = I_1$, i.e. $[T_1(x), T_2(y)] \in P$ for all $x, y \in I$, then replacing y by ys, one obtains

$$T_2(y)[T_1(x), s] \in P \quad \text{for all} \quad s, x, y \in I. \tag{2.22}$$

Substituting yu for y in (2.22), we obviously get

$$T_2(y)u[T_1(x), s] \in P \quad \text{for all} \quad s, u, x, y \in I.$$

So again an appeal to Fact 1.1, gives either $T_2(R) \subseteq P$ or $[T_1(x), s] \in P$ for all $x, s \in I$.
Now if $I = I_2$, that is $[T_1(x), r] \in P$ for all $x, r \in I$, then putting xw instead of x, we obtain

$$T_1(x)[z, r] \in P \quad \text{for all} \quad x, y, z \in I. \tag{2.23}$$

Writing xw for x in (2.23), we get

$$T_1(x)w[z, r] \in P \quad \text{for all} \quad r, w, x, z \in I.$$

Accordingly, it follows that $T_1(R) \subseteq P$ or R/P is a commutative integral domain.
(2) $\Longrightarrow$ (3) Can be proved by using the same steps as we did before. □

Corollary 2.10. *Let R be a prime ring and I a nonzero ideal of R. If T_1 and T_2 are two nonzero left centralizers on R, then the following assertions are equivalent:*

(1) $[T_1(x), T_2(y)] \in Z(R)$ *for all* $x, y \in I$;

(2) $T_1(x) \circ T_2(y) \in Z(R)$ *for all* $x, y \in I$;

(3) *R is a commutative integral domain.*

Using similar arguments as above with necessary variation, we can prove the following theorem.

Theorem 2.11. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on I of R, then the following assertions are equivalent:*

(1) $\overline{T_1(x)T_2(y) - [x, y]} \in Z(R/P)$ *for all* $x, y \in I$;

(2) $\overline{T_1(x)T_2(y) - x \circ y} \in Z(R/P)$ *for all* $x, y \in I$;

(3) *R/P is a commutative integral domain.*

Let R be a prime ring. Letting $P = (0)$ in the previous theorem, we deduce that, if $T_1(x)T_2(y) - [x, y] \in Z(R)$ or $T_1(x)T_2(y) - x \circ y \in Z(R)$ for all $x, y \in I$, then R is commutative. The following corollary shows that the same conclusion remains satisfied for semi-prime rings.

Corollary 2.12. *Let R be a semi-prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are two left centralizers on R, then the following assertions are equivalent:*

(1) $T_1(x)T_2(y) \pm [x, y] \in Z(R)$ *for all* $x, y \in I$;

(2) $T_1(x)T_2(y) \pm x \circ y \in Z(R)$ *for all* $x, y \in I$;

(3) *R is commutative.*

Proof. We have only to prove (1) $\Longrightarrow$ (3), while the implication (2) $\Longrightarrow$ (3) can be proved similarly. The ring R is semi-prime, then there exists a family $\mathcal{P}$ of prime ideals such that $\bigcap_{P \in \mathcal{P}} P = (0)$. Then we may suppose existence of a two left centralizers T_1 and T_2 satisfying $T_1(x)T_2(y) \pm [x, y] \in Z(R)$ for all $x, y \in I$. Thereby obtaining, $[T_1(x)T_2(y) \pm [x, y], r] = 0 \in \bigcap_{P \in \mathcal{P}} P$ for all $r, x, y \in I$, therefore, Theorem 2.11 yields that for all $P \in \mathcal{P}$, R/P is commutative which, because of $\bigcap_{P \in \mathcal{P}} P = (0)$, assures that R is commutative. $\square$

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