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Generalized k-Balancing and k-Lucas Balancing Numbers and Associated Polynomials

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Abstract. In this paper, we define the generalized k-balancing numbers $\{B_n^{(k)}\}$ and k-Lucas balancing numbers $\{C_n^{(k)}\}$ and associated polynomials, where n is of the form $sk+r$, $0 \leq r < k$. We give several formulas for these new sequences in terms of classic balancing and Lucas balancing numbers and study their properties. Moreover, we give a Binet style formula, Cassini's identity, and binomial sums of these sequences.

1. Introduction

Special number sequences such as the Fibonacci, Lucas, Horadam, Jacobsthal, and balancing numbers sequences are widely studied in number theory. Finding generalizations of such number sequences, establishing new identities, and finding applications of these sequences in other branches of mathematics have become very a popular research goal; see, for example, [6, 13]. Mikkawy and Sogabe [2] introduced a new family of k-Fibonacci numbers $F_n^{(k)}$ where n is of kind $sk + r$, $0 \leq r < k$. Among other properties, then gave a relation to the classic Fibonacci numbers. Later, Özkan et al. [8] further studied this sequence and introduced a new family of k-Lucas numbers. Kumari et al. [7] extended the study to Mersenne numbers and investigated some new families of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials.

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Motivated by these works on new families of sequences, in this paper, we study a new family of k-balancing and k-Lucas balancing numbers and associated polynomials, where the concept of balancing numbers and balancers was originally introduced in 1999 by Behera and Panda [1].

A natural number n is said to be a balancing number [1] with balancer r if it satisfies the Diophantine equation

$$1+2+3+...+(n-1)=(n+1)+(n+2)+...+(n+r).$$

The balancing numbers B_n and Lucas balancing numbers C_n are defined as

$$B_{n+2} = 6B_{n+1}-B_n, n\geq 0 \text{ with } B_0=0, B_1=1, \tag{1.1}$$

$$\text{and} \quad C_{n+2} = 6C_{n+1}-C_n, n\geq 0 \text{ with } C_0=1, C_1=3. \tag{1.2}$$

The first few terms of balancing and Lucas-balancing sequence are

n	**0**	**1**	2	3	4	5	6	7	8	...
B_n	**0**	**1**	6	35	204	1189	6930	40391	235416	...
C_n	**1**	**3**	17	99	577	3363	19601	114243	665857	...

Closed form formulas play an important roll in establishing many algebraic identities. The closed form formulas for balancing and Lucas-balancing numbers [11], are given as

$$B_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}=\frac{\lambda_1^n-\lambda_2^n}{4\sqrt{2}} \text{ and } C_n=\frac{\lambda_1^n+\lambda_2^n}{2}, \tag{1.3}$$

where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$ are the roots of the characteristic equation $x^2-6x+1=0$.

We will use the following useful relations for the balancing and Lucas-balancing numbers.

Lemma 1.1. ([12]) *For all integers m and n, we have*

1. $2B_mC_m=B_{2m}$.
2. $B_{m+n}+B_{m-n}=2B_mC_n$.
3. $B_{m+n}-B_{m-n}=2C_mB_n$.
4. $B_{m-n}B_{m+n}=B_m^2-B_n^2$.
5. $C_{n-m}C_{n+m}-C_n^2=\frac{1}{2}(C_{2m}-1)$.
6. $C_{2n}=2C_n^2-1$.
7. $C_n^2=8B_n^2+1$.

Fibonacci numbers have many generalizations, in which both the initial values and/or the recurrence relation are modified. The k-Fibonacci numbers, tribonacci numbers, Horadam numbers, generalized Fibonacci and Leonardo numbers, higher order Fibonacci numbers, are some examples of generalization of Fibonacci numbers. Likewise, k-balancing numbers $\{B_{k,n}\}$ and k-Lucas balancing numbers $\{C_{k,n}\}$, both generalisations of balancing numbers, were introduced and studied by Özkoc and Tekcan [9] and Ray [11]. These sequences are given by the following recurrences:

$$B_{k,n+2} = 6B_{k,n+1} - B_{k,n}, n \geq 0 \text{ with } B_{k,0} = 0, B_{k,1} = 1,$$
$$\text{and} \quad C_{k,n+2} = 6C_{k,n+1} - C_{k,n}, n \geq 0 \text{ with } C_{k,0} = 1, C_{k,1} = 3k.$$

Later in [10], Ray extended the k-balancing numbers $B_{k,n}$ to the sequence of balancing polynomials $\{B_n(x)\}$ by replacing k with a real variable x and presented numerous properties of balancing polynomials. Frontczak [3] also studied the balancing polynomials by relating them to Chebyshev polynomials.

In this paper, we give a new generalization of balancing and Lucas-balancing numbers à la [2], which we call the generalized k-balancing and k-Lucas balancing numbers. They are defined in Section 2. Then in Section 3, we give associated polynomials having a connection with balancing polynomials.

2. Generalized k-Balancing Numbers

Our main defintion is as follows.

Definition 2.1. Let $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ then $\exists!\ s, r \in \mathbb{N} \cup \{0\}$ such that $n = sk + r$, $0 \leq r < k$. The generalized k-balancing and k-Lucas balancing numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined as

$$B_n^{(k)} = \frac{1}{(4\sqrt{2})^k}(\lambda_1^s - \lambda_2^s)^{k-r}(\lambda_1^{s+1} - \lambda_2^{s+1})^r,$$
$$C_n^{(k)} = \frac{1}{2^k}(\lambda_1^s + \lambda_2^s)^{k-r}(\lambda_1^{s+1} + \lambda_2^{s+1})^r, \tag{2.1}$$

where λ_1 and λ_2 are the roots of the characteristic equation corresponding to balancing sequence (1.1).

From Definition 2.1 and Eqn. (1.3), one gets the following relation between the generalized k-balancing and k-Lucas balancing numbers and the balancing and Lucas balancing numbers.

$$B_n^{(k)} = B_s^{k-r}B_{s+1}^r \quad \text{and} \quad C_n^{(k)} = C_s^{k-r}C_{s+1}^r, \quad \text{where } n = sk + r. \tag{2.2}$$

If $k = 1$ then $r = 0$ and hence $n = s$. Therefore, $B_n^{(1)}$ and $C_n^{(1)}$ are the classic balancing and Lucas balancing numbers i.e. $B_n^{(1)} = B_n$ and $C_n^{(1)} = C_n$.

In the case that $k = 2$ or 3 we note some identities showing the relations between generalized k-balancing numbers and balancing numbers:

1. $B_{2s}^{(2)} = B_s^2$.
2. $B_{2s+1}^{(2)} = B_s B_{s+1}$.
3. $B_{3s}^{(3)} = B_s^3$.
4. $B_{3s+1}^{(3)} = B_s^2 B_{s+1}$.
5. $B_{3s+2}^{(3)} = B_s B_{s+1}^2$.

1. $C_{2s}^{(2)} = C_s^2 = (C_{2s} + 1)/2$.
2. $C_{2s+1}^{(2)} = C_s C_{s+1}$.
3. $C_{3s}^{(3)} = C_s^3$.
4. $C_{3s+1}^{(3)} = C_s^2 C_{s+1}$.
5. $C_{3s+2}^{(3)} = C_s C_{s+1}^2$.

One can also check that $B_{2s+1}^{(2)} = 6B_{2s}^{(2)} - B_{2s-1}^{(2)}$ and $B_{3s+1}^{(3)} = 6B_{3s}^{(3)} - B_{3s-1}^{(3)}$. The same recurrences hold for generalized k-Lucas balancing numbers.

For $k = 1, 2, 3, 4, 5$, a list of first few numbers of the generalized k-balancing and k-Lucas balancing numbers are displayed in Tables 1 and 2.

$n \downarrow$	$B_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0	$B_0^{(k)}$	0	0	0	0	0
1	$B_1^{(k)}$	1	0	0	0	0
2	$B_2^{(k)}$	6	1	0	0	0
3	$B_3^{(k)}$	35	6	1	0	0
4	$B_4^{(k)}$	204	36	6	1	0
5	$B_5^{(k)}$	1189	210	36	6	1
6	$B_6^{(k)}$	6930	1225	216	36	6
7	$B_7^{(k)}$	40391	7140	1260	216	36

Table 1: Generalized k-balancing numbers

.

$n \downarrow$	$C_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0	$C_0^{(k)}$	1	1	1	1	1
1	$C_1^{(k)}$	3	3	3	3	3
2	$C_2^{(k)}$	17	9	9	9	9
3	$C_3^{(k)}$	99	51	27	27	27
4	$C_4^{(k)}$	577	289	153	81	81
5	$C_5^{(k)}$	3363	1683	867	459	243
6	$C_6^{(k)}$	19601	9801	4913	2601	1377
7	$C_7^{(k)}$	114243	57123	28611	14739	7803

Table 2: Generalized k-Lucas balancing numbers

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Theorem 2.2. *For $k, s \in \mathbb{N}$, we have*

$$B_{sk}^{(k)} = B_s^k \quad \text{and} \quad C_{sk}^{(k)} = C_s^k.$$

Proof. If $r = 0$ then $n = sk$ and hence from Eqn. (2.2) the above results are proved. □

Lemma 2.3. *We have*

$$B_{sk-1}^{(k)} = B_{s-1}B_s^{k-1} \qquad \text{and} \qquad C_{sk-1}^{(k)} = C_{s-1}C_s^{k-1}.$$

Proof. Since,

$$B_{sk-1}^{(k)} = B_{sk-k+k-1}^{(k)} = B_{(s-1)k+(k-1)}^{(k)} = B_{s-1}B_s^{k-1}.$$

Similarly, the second result holds. □

Thus, we conclude that

$$B_{sk+r}^{(k)} = B_s^{k-r}B_{s+1}r.$$

A similar identity holds for $C_{sk+r}^{(k)}$.

Theorem 2.4. *For $a \geq 1$ and $n \in \mathbb{N}$ such that $n = sk + r$, $0 \leq r < k$, we have*

$$(W_n^{(k)})^a = (W_{as}^{(a)})^{k-r}(W_{a(s+1)}^{(a)})^r, \quad \text{where } W_i = B_i \text{ or } W_i = C_i.$$

Proof. From Eqn. (2.2) for $W_i = B_i$ or $W_i = C_i$, we can write

$$(W_n^{(k)})^a = (W_s^{k-r}W_{s+1}^r)^a = (W_s^a)^{k-r}(W_{s+1}^a)^r = (W_{as}^{(a)})^{k-r}(W_{a(s+1)}^{(a)})^r.$$

Thus using Theorem 2.2 in the above equation, the result holds. □

In particular for $a = 2$ in Theorem 2.4, we have

$$(B_n^{(k)})^2 = (B_{2s}^{(2)})^{k-r}(B_{2(s+1)}^{(2)})^r \quad \text{and} \quad (C_n^{(k)})^2 = (C_{2s}^{(2)})^{k-r}(C_{2(s+1)}^{(2)})^r.$$

Theorem 2.5. *For $k, s \in \mathbb{N}$ such that $n = sk + 1$, the following relations are verified:*

$$B_n^{(k)} = 6B_{sk}^{(k)} - B_{sk-1}^{(k)} \qquad \text{and} \qquad C_n^{(k)} = 6C_{sk}^{(k)} - C_{sk-1}^{(k)}.$$

Proof. For the first identity, from Theorem 2.2 and Lemma 2.3, we write

$$\begin{aligned} 6B_{sk}^{(k)} - B_{sk-1}^{(k)} &= 6B_s^k - B_{s-1}B_s^{k-1} \\ &= B_s^{k-1}(6B_s - B_{s-1}) \\ &= B_s^{k-1}B_{s+1} = B_{sk+1}^{(k)}. \end{aligned}$$

A similar argument holds for the second identity. □

Theorem 2.6. *Let $s, k \in \mathbb{N}$, then for fixed k, s, the following results hold:*

$$\sum_{a=0}^{k-1} \binom{k-1}{a} B_{sk+a}^{(k)} = B_s (B_s + B_{s+1})^{k-1}, \tag{2.3}$$

$$\sum_{a=0}^{k-1} (-1)^a \binom{k-1}{a} B_{sk+a}^{(k)} = (-1)^{k-1} B_s (B_{s+1} - B_s)^{k-1}, \tag{2.4}$$

$$\sum_{a=0}^{k-1} B_{sk+a}^{(k)} = \frac{B_s (B_{k(s+1)}^{(k)} - B_{sk}^{(k)})}{B_{s+1} - B_s}, \tag{2.5}$$

$$\sum_{a=0}^{k-1} a B_{sk+a}^{(k)} = \frac{B_{s(k+2)+1}^{(k+2)} - k B_{s(k+2)+k}^{(k+2)} + (k-1) B_{s(k+2)+k+1}^{(k+2)}}{(B_s - B_{s+1})^2}. \tag{2.6}$$

Proof. For the first identity (2.3), using relation (2.2), we write

$$\begin{aligned}
\sum_{a=0}^{k-1} \binom{k-1}{a} B_{sk+a}^{(k)} &= \sum_{a=0}^{k-1} \binom{k-1}{a} B_s^{k-a} B_{s+1}^a \\
&= B_s \sum_{a=0}^{k-1} \binom{k-1}{a} B_{s+1}^a B_s^{k-1-a} \\
&= B_s (B_s + B_{s+1})^{k-1} \quad \text{(using the Binomial theorem)}.
\end{aligned}$$

Similarly, for the second identity (2.4), we have

$$\begin{aligned}
\sum_{a=0}^{k-1} (-1)^a \binom{k-1}{a} B_{sk+a}^{(k)} &= (-1)^{k-1} \sum_{a=0}^{k-1} (-1)^{k-1-a} \binom{k-1}{a} B_s^{k-a} B_{s+1}^a \\
&= (-1)^{k-1} B_s \sum_{a=0}^{k-1} \binom{k-1}{a} B_{s+1}^a (-B_s)^{k-1-a} \\
&= (-1)^{k-1} B_s (B_{s+1} - B_s)^{k-1} \quad \text{(using the Binomial theorem)}.
\end{aligned}$$

For the third identity (2.5), since from (2.2) we write $B_{sk+a}^{(k)} = B_s^{k-a} B_{s+1}^a = B_s^k (B_{s+1}/B_s)^a$. Thus

$$\begin{aligned}
\sum_{a=0}^{k-1} B_{sk+a}^{(k)} &= \sum_{a=0}^{k-1} B_s^k \Big(\frac{B_{s+1}}{B_s}\Big)^a = B_s^k \sum_{a=0}^{k-1} \Big(\frac{B_{s+1}}{B_s}\Big)^a \\
&= B_s^k \frac{(B_{s+1}/B_s)^k - 1}{B_{s+1}/B_s - 1} \\
&= B_s \Big(\frac{B_{s+1}^k - B_s^k}{B_{s+1} - B_s}\Big) \\
&= \frac{B_s}{B_{s+1} - B_s} (B_{k(s+1)}^{(k)} - B_{sk}^{(k)}).
\end{aligned}$$

And, for the last identity (2.6), note that $\sum_{a=1}^{k} ax^{a-1} = (1-kx^{k-1}+(k-1)x^k)/(1-x)^2$. Thus,

$$\begin{aligned}\sum_{a=0}^{k-1} aB_{sk+a}^{(k)} &= B_s^{k-1}B_{s+1}\sum_{a=0}^{k-1} a\Big(\frac{B_{s+1}}{B_s}\Big)^{a-1}\\ &= B_s^{k-1}B_{s+1}\Big(\frac{1-k(B_{s+1}/B_s)^{k-1}+(k-1)(B_{s+1}/B_s)^k}{(1-B_{s+1}/B_s)^2}\Big)\\ &= \frac{B_s^{k-1}B_{s+1}-kB_{s+1}^k+(k-1)B_{s+1}^{k+1}/B_s}{(1-B_{s+1}/B_s)^2}\\ &= \frac{B_s^{k+1}B_{s+1}-kB_s^2B_{s+1}^k+(k-1)B_sB_{s+1}^{k+1}}{(B_s-B_{s+1})^2}\\ &= \frac{B_{s(k+2)+1}^{(k+2)}-kB_{s(k+2)+k}^{(k+2)}+(k-1)B_{s(k+2)+k+1}^{(k+2)}}{(B_s-B_{s+1})^2} \quad \text{(using Eqn. (2.2))}.\end{aligned}$$

□

Note that Theorem 2.6 is also valid for the generalized k-Lucas balancing numbers $\{C_s^k\}$.

Theorem 2.7. *For $k, s \in \mathbb{N}$, we have*

$$B_{s+1}^k - B_s^k = B_{sk+k}^{(k)} - B_{sk}^{(k)} \qquad \textit{and} \qquad C_{s+1}^k - C_s^k = C_{sk+k}^{(k)} - C_{sk}^{(k)}.$$

Proof. Results follow from Eqn. (2.2). □

Theorem 2.8 (Cassini's identity)**.** *For $s, k \geq 2$, we have*

$$B_{sk+a}^{(k)}B_{sk+a-2}^{(k)} - (B_{sk+a-1}^{(k)})^2 = \begin{cases} -B_s^{2k-2} & a = 1,\\ 0 & a \neq 1,\end{cases}$$

$$\textit{and} \quad C_{sk+a}^{(k)}C_{sk+a-2}^{(k)} - (C_{sk+a-1}^{(k)})^2 = \begin{cases} 8C_s^{2k-2} & a = 1,\\ 0 & a \neq 1.\end{cases}$$

Proof. For $a \neq 1$, from Eqn. (2.2) we write

$$\begin{aligned} B_{sk+a}^{(k)}B_{sk+a-2}^{(k)} - (B_{sk+a-1}^{(k)})^2 &= (B_s^{k-a}B_{s+1}^a)(B_s^{k-a+2}B_{s+1}^{a-2}) - (B_s^{k-a+1}B_{s+1}^{a-1})^2\\ &= B_s^{2k-2a+2}[B_{s+1}^aB_{s+1}^{a-2} - (B_{s+1})^{2a-2}]\\ &= 0\end{aligned}$$

and if $a = 1$ then with Eqn. (2.2) and Lemma 2.3

$$\begin{aligned} B_{sk+1}^{(k)}B_{sk-1}^{(k)} - (B_{sk}^{(k)})^2 &= (B_s^{k-1}B_{s+1})(B_{s-1}B_s^{k-1}) - (B_s^k)^2\\ &= B_s^{2k-2}[B_{s+1}B_{s-1} - (B_s)^2]\\ &= -B_s^{2k-2} \qquad \text{(simplified using Eqn. (1.3))}.\end{aligned}$$

A similar argument holds for the second identity. □

Theorem 2.9. *For integers s, s_1, s_2 and $k \geq 1$, we have*

1. $B_{2sk}^{(k)} = 2^k B_{sk}^{(k)} C_{sk}^{(k)}$.
2. $(B_{s_1+s_2} + B_{s_1-s_2})^k = 2^k B_{s_1 k}^{(k)} C_{s_2 k}^{(k)}$.
3. $(B_{s_1+s_2} - B_{s_1-s_2})^k = 2^k C_{s_1 k}^{(k)} B_{s_2 k}^{(k)}$.
4. $B_{2s_1}^{(2)} - B_{2s_2}^{(2)} = B_{s_1+s_2} B_{s_1-s_2}$.
5. $C_{2sk}^{(k)} = (2C_s^2 - 1)^k = (2B_s^2 + 1)^k$.
6. $(C_{s_1+s_2} + C_{s_1-s_2})^k = 2^k C_{s_1 k}^{(k)} C_{s_2 k}^{(k)}$.
7. $(C_{s_1+s_2} - C_{s_1-s_2})^k = 16^k B_{s_1 k}^{(k)} B_{s_2 k}^{(k)}$.
8. $C_{2s_1}^{(2)} - C_{2s_2}^{(2)} = 8 B_{s_1+s_2} B_{s_1-s_2}$.

Proof. From Theorem 2.2 and 1 of Lemma 1.1, note that for the first identity, we have

$$B_{2sk}^{(k)} = B_{2s}^k = (2B_s C_s)^k = 2^k B_s^k C_s^k = 2^k B_{sk}^{(k)} C_{sk}^{(k)}.$$

For the second identity, from 2 of Lemma 1.1, we have

$$(B_{s_1+s_2} + B_{s_1-s_2})^k = (2B_{s_1} C_{s_2})^k = 2^k B_{s_1}^k C_{s_2}^k = 2^k B_{s_1 k}^{(k)} C_{s_2 k}^{(k)}.$$

For the third identity, from 3 of Lemma 1.1, we have

$$(B_{s_1+s_2} - B_{s_1-s_2})^k = (2C_{s_1} B_{s_2})^k = 2^k C_{s_1}^k B_{s_2}^k = 2^k C_{s_1 k}^{(k)} B_{s_2 k}^{(k)}.$$

For the fourth identity, from Theorem 2.2 and 4 of Lemma 1.1, we write

$$B_{2s_1}^{(2)} - B_{2s_2}^{(2)} = B_{s_1}^2 - B_{s_2}^2 = B_{s_1+s_2} B_{s_1-s_2}.$$

The argument for identities $5-8$ are similar to that of $1-4$ using Theorem 2.2 and Lemma 1.1. □

3. Generalized k-Balancing and k-Lucas Balancing Polynomials

For $n \geq 0$, the balancing and Lucas balancing polynomials $B_n(x)$ and $C_n(x)$ satisfy the recurrence relation

$$W_{n+2}(x) = 6xW_{n+1}(x) - W_n(x) \tag{3.1}$$

but with the initial values as $B_0(x) = 0,\ B_1(x) = 1$ and $C_0(x) = 1,\ C_1(x) = 3x$, respectively. The Binet type formulas for these polynomials are, respectively,

$$B_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\sqrt{9x^2-1}} \quad \text{and} \quad C_n(x) = \frac{\lambda_1^n(x) + \lambda_2^n(x)}{2}, \tag{3.2}$$

where $\lambda_1 = (3x + \sqrt{9x^2-1})/2$ and $\lambda_2 = (3x - \sqrt{9x^2-1})/2$ are roots of the characteristic equation $\lambda^2 - 6x\lambda + 1 = 0$.

Now, we define the generalized k-balancing and k-Lucas balancing polynomials in a similar fashion to the preceding section.

Definition 3.1. Let $k \in \mathbb{N}$ and for $n \geq 0$, $\exists!\ s, r \in \mathbb{N} \cup \{0\}$ such that $n = sk + r$, $0 \leq r < k$. Then the generalized k-balancing and k-Lucas balancing polynomials $B_n^{(k)}(x)$ and $C_n^{(k)}(x)$ are defined as

$$\begin{aligned} B_n^{(k)}(x) &= \left(\frac{\lambda_1^s(x) - \lambda_2^s(x)}{\sqrt{9x^2-1}}\right)^{k-r} \left(\frac{\lambda_1^{s+1}(x) - \lambda_2^{s+1}(x)}{\sqrt{9x^2-1}}\right)^{r}, \\ \text{and} \quad C_n^{(k)}(x) &= \left(\frac{\lambda_1^s(x) + \lambda_2^s(x)}{2}\right)^{k-r} \left(\frac{\lambda_1^{s+1}(x) + \lambda_2^{s+1}(x)}{2}\right)^{r}. \end{aligned}$$

From Binet's formula (3.2) and Definition 3.1, we deduce the following relations between newly introduced sequences and existing one

$$W_{sk+r}^{(k)}(x) = W_s^{k-r}(x) W_{s+1}^{r}(x), \quad \text{where } W_i(x) = B_i(x) \text{ or } C_i(x). \tag{3.3}$$

For the case $k = 1$, we get $r = 0$. Hence, from Eqn. (3.3,) we have $W_s^{(1)}(x) = W_s(x)$.

For instance at $k = 2, 3$ in (3.3), we have noted some identities showing relations between newly introduced polynomials sequences and classic balancing/Lucas balancing polynomials:

1. $W_{2s}^{(2)}(x) = W_s^2(x)$.
2. $W_{2s+1}^{(2)}(x) = W_s(x) W_{s+1}(x)$.
3. $W_{3s}^{(3)}(x) = W_s^3(x)$.
4. $W_{3s+1}^{(3)}(x) = W_s^2(x) W_{s+1}(x)$.
5. $W_{3s+2}^{(3)}(x) = W_s(x) W_{s+1}^2(x)$.
6. $W_{4s}^{(4)}(x) = W_s^4(x)$.
7. $W_{4s+1}^{(4)}(x) = W_s^3(x) W_{s+1}(x)$.
8. $W_{4s+2}^{(4)}(x) = W_s^2(x) W_{s+1}^2(x)$.
9. $W_{4s+3}^{(4)}(x) = W_s(x) W_{s+1}^3(x)$.

Also the recurrence relations $W_{2s+1}^{(2)}(x) = 6W_{2s}^{(2)}(x) - W_{2s-1}^{(2)}(x)$ for $k = 2$ and $W_{3s+1}^{(3)}(x) = 6W_{3s}^{(3)}(x) - W_{3s-1}^{(3)}(x)$ for $k = 3$ are verified.

Theorem 3.2. *For $k, s \in \mathbb{N}$, we have*

$$W_{sk}^{(k)}(x) = W_s^k(x), \quad \textit{where} \quad W_i(x) = B_i(x) \textit{ or } C_i(x).$$

Proof. If $r = 0$ then $sk + r = sk$ and hence from Eqn. (3.3) the result follows immediately. □

By a similar argument to Lemma 2.3, we have $W_{sk-1}^{(k)}(x) = W_{s-1}(x) W_s^{k-1}(x)$ which will be used in the next theorem.

Theorem 3.3. *For $k, s \in \mathbb{N}$ such that $n = sk + 1$, the following recurrence relation is satisfied.*

$$W_{sk+1}^{(k)}(x) = 6xW_{sk}^{(k)}(x) - W_{sk-1}^{(k)}(x).$$

Proof. From Eqn. (3.3) and Eqn. (3.1), we have

$$\begin{aligned} 6xW_{sk}^{(k)}(x) - W_{sk-1}^{(k)}(x) &= 6xW_s^k(x) - W_{s-1}(x)W_s^{k-1}(x) \\ &= W_s^{k-1}(x)(6xW_s(x) - W_{s-1}(x)) \\ &= W_s^{k-1}(x)W_{s+1}(x) \\ &= W_{sk+1}^{(k)}(x). \end{aligned}$$

□

Theorem 3.4. *We have*

$$W_{sk-1}^{(k)}(x) = W_{s-1}(x)W_s^{k-1}(x), \quad \textit{where} \quad W_i(x) = B_i(x) \textit{ or } C_i(x).$$

Proof. Since,

$$W_{sk-1}^{(k)}(x) = W_{sk-k+k-1}^{(k)}(x) = W_{(s-1)k+(k-1)}^{(k)}(x) = W_{s-1}(x)W_s^{k-1}(x).$$

□

An analogue argument to the preceding section proves the following theorems, so we omit the proofs.

Theorem 3.5. *For $k, s \in \mathbb{N}$ we have,*

$$W_{s+1}^k(x) - W_s^k(x) = W_{sk+k}^{(k)}(x) - W_{sk}^{(k)}(x).$$

Theorem 3.6 (Cassini's identity)**.** *Let $k, s \geq 2$, we have*

$$W_{nk+a}^{(k)}(x)W_{nk+a-2}^{(k)}(x) - (W_{nk+a-1}^{(k)})^2(x) = \begin{cases} -B_n^{2k-2}(x), & : \textit{if } a = 1 \textit{ and } W_n(x) = B_n(x), \\ 8C_n^{2k-2}(x), & : \textit{if } a = 1 \textit{ and } W_n(x) = C_n(x), \\ 0, & : a \neq 1. \end{cases}$$

Theorem 3.7. *For integers s, s_1, s_2 and $k \geq 1$, the following relations are valid:*

1. $B_{2sk}^{(k)}(x) = B_{sk}^{(k)}(x)[B_{s+1}(x) - B_{s-1}(x)]^k$.
2. $B_{2s_1}^{(2)}(x) - B_{2s_2}^{(2)}(x) = B_{s_1-s_2}(x)B_{s_1+s_2}(x)$.
3. $[B_{s+1}(x) - B_{s-1}(x)]^k = 2^k C_{sk}^{(k)}(x)$.
4. $[3xB_s(x) - B_{s-1}(x)]^k = 2^k C_{sk}^{(k)}(x)$.
5. $[B_{s+1}(x) - 3xB_s(x)]^k = C_{sk}^{(k)}(x)$.
6. $[B_{s+1}^2(x) - B_{s-1}^2(x)]^k = 6^k x^k B_{2sk}^{(k)}(x)$.
7. $[C_{s+1}^2(x) - C_{s-1}^2(x)]^k = 6^k x^k (9x^2 - 1)^k B_{2sk}^{(k)}(x)$.

8. $C_{2s_1}^{(2)}(x) + C_{2s_2}^{(2)}(x) = C_{s_1-s_2}(x)C_{s_1+s_2}(x) + 1.$

9. $C_{2s}^{(2)}(x) - (9x^2 - 1)B_{2s}^{(2)}(x) = 1.$

10. $[3xC_{s-1}(x) + (9x^2 - 1)B_{s-1}(x)]^k = C_{sk}^{(k)}(x).$

11. $B_{2sk}^{(k)}(x) = 2^k B_{sk}^{(k)}(x)C_{sk}^{(k)}(x).$

12. $C_{2sk}^{(k)}(x) = [2C_s^2(x) - 1]^k.$

Proof. The arguments for these identities are analog to the proof of Theorem 2.9 and can be easily verified using Proposition 2.3 of [3] and Section 3 of [10] along with Theorem 3.2. □

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