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Some Results on Simple-Direct-Injective Modules

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Abstract. A module M is called a simple-direct-injective module if, whenever A and B are simple submodules of M with $A \cong B$ and B is a direct summand of M, then A is a direct summand of M. Some new characterizations of these modules are proved. The structure of simple-direct-injective modules over a commutative Dedekind domain is fully determined. Also, some relevant counterexamples are indicated to show that a left simple-direct-injective ring need not be right simple-direct-injective.

1. Preliminaries and Introduction

Throughout, all rings R are associative with identity and all modules are unitary right R-modules. For a module M, we denote by $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$ and $E(M)$ the Jacobson radical, the socle and the injective hull of M, respectively. We write $N \subseteq M$ if N is a subset of M, and $N \leq M$ if N is a submodule of M. The notation $N \leq_d M$ means that N is a direct summand of M. For two modules X, Y over a ring R, the set of R-homomorphisms from X to Y is denoted by $\mathrm{Hom}_R(X, Y)$. For a ring R, we denote by $J(R)$ the Jacobson radical of R. Let M be a module over a ring R. Recall that M is called a $C2$-*module* if every submodule of M which is isomorphic to a direct summand is itself a direct summand of M. In addition, the module M is said to be a $C3$-*module* if the sum of any two direct summands of M with zero intersection is again a direct summand of M. It was shown in [4, Proposition 2.1] that the "simple" versions of $C2$ and $C3$-modules coincide. Camillo et al. [4] called these modules *simple-direct-injective* modules. Then M is a simple-direct-injective

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module if, whenever A and B are simple submodules of M with $A \cong B$ and B a direct summand of M, then A is a direct summand of M. Equivalently, for any simple direct summands A and B of M with $A \cap B = 0$, $A \oplus B$ is a direct summand of M. A number of examples of simple-direct-injective modules appeared in some research papers. Among others, note that every indecomposable right R-module is simple-direct-injective by [4, Example 2.3(1)]. Moreover, according to [4, Example 2.3(2)], every cyclic R-module is simple-direct-injective if R is a commutative ring. It is clear that every R-module M which has no simple direct summands is simple-direct-injective. For example, we can take a module M with $M = \mathrm{Rad}(M)$. Also, if a module $M = M_1 \oplus M_2$ is a direct sum of two nonsimple indecomposable R-modules M_1 and M_2, then M has no simple direct summands (see [8, Example 2.6(1)]). Next, we present some known characterizations of various rings in terms of simple-direct-injective modules.

1. Given a ring R, the right R-module R_R is simple-direct-injective if and only if every projective right R-module is simple-direct-injective ([4, Corollary 2.15]).
2. A ring R is an artinian serial ring with $J(R)^2 = 0$ if and only if every simple-direct-injective right R-module is a $C3$-module (a quasi-injective module) ([4, Theorem 3.4]).
3. A ring R is semisimple if and only if every simple-direct-injective right R-module is injective ([4, Corollary 3.5]).
4. A ring R is a right V-ring if and only if every 2-generated (finitely cogenerated) right R-module is simple-direct-injective ([4, Proposition 4.1]).
5. A von Neumann regular ring R is a right V-ring if and only if every cyclic right R-module is simple-direct-injective ([4, Theorem 4.4]).

Recently, Büyükaşık et al. [3] gave a complete characterization of simple-direct-injective abelian groups. They proved in [3, Theorem 2] that an abelian group M is simple-direct-injective if and only if for each prime number p, the p-primary component $T_p(M)$ is semisimple or $\mathrm{Soc}(T_p(M)) \subseteq \mathrm{Rad}(T_p(M))$.

The motivation of this paper comes from these three works: [3], [4] and [8]. Our goals are to extend the preceding characterization of simple-direct-injective abelian groups to modules over commutative Dedekind domains, investigate some properties of simple-direct-injective modules and rings and construct some useful examples.

In Section 2, we present some easy examples. In Section 3, we obtain several equivalent conditions for modules to being simple-direct-injective and give some properties of this type of modules. Among others, we prove that a module M is simple-direct-injective if and only if for every pair of idempotents $e, f \in S = \mathrm{End}_R(M)$ such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$, there exist orthogonal idempotents $g, h \in S$ such that $eS = gS$ and $fS = hS$. It is also shown that a module M is simple-direct-injective if and only if for every submodule K of M such that $K = K_1 \oplus K_2$, K_1 and K_2 are simple (perspective) direct summands of M, every homomorphism $\varphi : K \to M$ can be extended to an endomorphism $\theta : M \to M$. In addition, we prove that for an R-module $M = \oplus_{i\in I} M_i$ which is a direct sum of submodules M_i ($i \in I$) such that for every direct summand X of M, $X = \oplus_{i\in I}(X \cap M_i)$ (for example, if $\mathrm{Hom}_R(M_i, M_j) = 0$, for all distinct $i, j \in I$), then M is simple-direct-injective if and only if M_i is simple-direct-injective for all $i \in I$. This result is useful to investigate simple-direct-injective modules over commutative Dedekind domains. In Section 4, we fully characterize simple-direct-injective modules over commutative Dedekind domains. In particular, we prove an extension of [3, Theorem 2] to modules over commutative Dedekind domains. In Section 5, we investigate simple-direct-injective rings. A ring R is called left (right) simple-direct-injective if the left (right) R-module ${}_RR$ (R_R) is simple-direct-injective. We show that being a simple-direct-injective ring is not left-right symmetric. Moreover, we prove that the class of left (right) simple-direct-injective rings is closed under direct products. Some necessary conditions for the endomorphism ring of a module to be right simple-direct-injective are investigated.

2. Some Examples

In this section, we provide some other examples of simple-direct-injective modules. It is well known that a simple submodule of a module M is either small in M or a direct summand of M. This clearly implies that any module M with $\mathrm{Soc}(M) \cap \mathrm{Rad}(M) = 0$ is simple-direct-injective since all simple submodules of M are direct summand. Next, we present an important class of modules satisfying this condition.

Example 2.1. Let R be any ring and let M be a regular right R-module (i.e. every cyclic (finitely generated) submodule of M is a direct summand of M (see [15, Remark 6.1])). Therefore M is a simple-direct-injective module since every simple submodule of M is a direct summand of M. In particular, any projective module over a von Neumann regular ring is simple-direct-injective by [15, Proposition 6.7(4)].

A module M is called *dual Rickart* if for every endomorphism f of M, $f(M)$ is a direct summand of M (see [9] and [11]). It is easily seen that every dual Rickart module is simple-direct-injective, but the converse is not true, in general. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is simple-direct-injective but it is not dual Rickart. Now using [9, Theorem 3.2 and Corollary 3.3], we obtain the following two examples of simple-direct-injective modules. For the undefined notions here we refer to [7].

Example 2.2. Let R be a prime right Goldie ring such that R is not right primitive and let an R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is simple-direct-injective.

Example 2.3. Let R be a prime PI-ring which is not artinian and let an R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is simple-direct-injective.

It is clear that any module M with $\mathrm{Soc}(M) \subseteq \mathrm{Rad}(M)$ (for example, $\mathrm{Soc}(M) = 0$ or $\mathrm{Rad}(M) = M$) is simple-direct-injective since every simple submodule of M is small in M.

Example 2.4. Let R be a local right artinian ring which is not semisimple (for example, we can take $R = \mathbb{Z}/p^n\mathbb{Z}$ for some prime number p and some integer $n \geq 2$). Clearly the right R-module R_R has no simple direct summands. Therefore R_R is a simple-direct-injective R-module. Note that $\mathrm{Soc}(R_R) \subseteq \mathrm{Rad}(R_R)$, $\mathrm{Soc}(R_R) \neq 0$ and $\mathrm{Rad}(R_R) \neq R_R$.

It is well known that the notion of simple-direct-injective modules is closed under direct summands. To construct counterexamples showing that the simple-direct-injective property is not inherited by submodules and direct sums, we need the following lemma.

Lemma 2.5. *Let M be a module having a simple submodule S such that S is not a direct summand of M (e.g., M is indecomposable with* $\mathrm{Soc}(M) \neq 0$ *and* $\mathrm{Soc}(M) \neq M$*). Then $M \oplus \mathrm{Soc}(M)$ is not simple-direct-injective.*

Proof. Assume that $M \oplus \mathrm{Soc}(M)$ is simple-direct-injective. Then $M \oplus S$ is simple-direct-injective by [3, Lemma 6]. Now consider the inclusion map $i : S \to M$. By [4, Proposition 2.1], $\mathrm{Im}(i) = S$ is a direct summand of M. This contradicts our assumption. Therefore $M \oplus \mathrm{Soc}(M)$ can not be simple-direct-injective. □

Example 2.6. Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of indecomposable nonsimple submodules M_i $(i \in I)$ such that $\mathrm{Soc}(M_{i_0}) \neq 0$ for some $i_0 \in I$. Then the module $N = M \oplus M$ is simple-direct-injective since N has no simple direct summands. However, its submodule $M \oplus \mathrm{Soc}(M)$ is not simple-direct-injective by Lemma 2.5. Moreover, note that both M and $\mathrm{Soc}(M)$ are simple-direct-injective. As an explicit example, we can take for M any direct sum of indecomposable nonsimple $\mathbb{Z}$-modules (e.g., $\mathbb{Z}$, $\mathbb{Q}$) such that at least one of them must be isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ or to $\mathbb{Z}(p^\infty)$ for some prime number p and some integer $n \geq 2$.

3. Some Properties of Simple-Direct-Injective Modules

In this section we provide some new equivalent formulations of being a simple-direct-injective module and establish some properties of this type of modules. In the proof of [4, Proposition 2.1], one can easily see that the implication (1) ⇒ (2) is true even if the condition "B is simple" is deleted from the statement (2). We thus get the following proposition. Its proof is given for completeness.

Proposition 3.1. *The following conditions are equivalent for a module M:*

(i) *M is simple-direct-injective;*

(ii) *For any direct summands A and B of M such that A is simple and $A \cap B = 0$, $A \oplus B \leq_d M$.*

Proof. (i) $\Rightarrow$ (ii) Let A and B be direct summands of M such that A is simple and $A \cap B = 0$. Then $M = B \oplus B'$ for some submodule B' of M. Let $\pi : B \oplus B' \to B'$ be the projection map with $\text{Ker}\pi = B$. Since $A \cap B = 0$, $\pi(A) \cong A$. Since M is simple-direct-injective, $\pi(A) \leq_d M$. Hence $\pi(A) \leq_d B'$. Therefore there exists a submodule X of B' such that $B' = X \oplus \pi(A)$. It follows that $M = B \oplus B' = \pi(A) \oplus B \oplus X$. Note that $\pi(A) \oplus B = A \oplus B$. Thus $M = A \oplus B \oplus X$ and hence $A \oplus B \leq_d M$.

(ii) $\Rightarrow$ (i) This follows from [4, Proposition 2.1]. □

Recall that two direct summands A and B of a module M are called *perspective* (see [5]) if $M = A \oplus X = B \oplus X$ for some submodule X of M. Following [1], two idempotents e and f of a ring R are called *perspective* if eR and fR are perspective direct summands of the right R-module R_R (i.e. there exists a right ideal C of R such that $R = eR \oplus C = fR \oplus C$). It was shown in [1, Proposition 2.13] that a module M is simple-direct-injective if and only if for any simple perspective direct summands A, B of M with $A \cap B = 0$, $A \oplus B$ is a direct summand of M. Using the notion of perspectivity of idempotents, we obtain the following characterization of simple-direct-injective modules. The proof of this result is similar to that of [12, Lemma 4.5] (see also [1, Lemma 3.1]).

Theorem 3.2. *The following are equivalent for a module M and $S = \text{End}_R(M)$:*

(i) *M is simple-direct-injective;*

(ii) *For every pair of idempotents $e, f \in S$ such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$, there exist orthogonal idempotents $g, h \in S$ such that $eS = gS$ and $fS = hS$;*

(iii) *For every pair of perspective idempotents $e, f \in S$ such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$, there exist orthogonal idempotents $g, h \in S$ such that $e(M) = g(M)$ and $f(M) = h(M)$;*

(iv) *For every pair of perspective idempotents $e, f \in S$ such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$, there exists an idempotent g of S such that $e(M) = g(M)$ and $f(M) \subseteq (1 - g)(M)$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that M is simple-direct-injective. Let $e^2 = e, f^2 = f \in S$ with $e(M) \cap f(M) = 0$ such that $e(M)$ and $f(M)$ are simple modules. Then $M = e(M) \oplus f(M) \oplus N$ for some submodule N of M (see Proposition 3.1). Let $g : M \to e(M)$ be the projection map with $\text{Ker}g = f(M) \oplus N$ and $h : M \to f(M)$ be the projection map with $\text{Ker}h = e(M) \oplus N$. Clearly, $g(M) = e(M)$ and $h(M) = f(M)$. Then by [12, Lemma 1.1], $gS = eS$ and $hS = fS$. It is not hard to see that h and g are orthogonal.

(ii) $\Rightarrow$ (iii) Clear by [12, Lemma 1.1].

(iii) $\Rightarrow$ (iv) Let $e, f \in S$ be perspective idempotents such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$. By hypothesis, there exist orthogonal idempotents

$g, h \in S$ such that $e(M) = g(M)$ and $f(M) = h(M)$. Note that $f(M) = h(M) \subseteq \text{Ker} g = (1 - g)(M)$, as desired.

(iv) $\Rightarrow$ (i) Let A and B be simple perspective direct summands of M with $A \cap B = 0$. Then $A = e(M)$ and $B = f(M)$ for some idempotents e and f of the ring S. By (iv), there exists an idempotent g of S such that $A = g(M)$ and $B \subseteq (1-g)(M)$. Hence B is a direct summand of $(1-g)(M)$. Since $M = g(M) \oplus (1-g)M$, it follows that $A \oplus B$ is a direct summand of M. Now using [1, Proposition 2.13], we conclude that M is a simple-direct-injective module. This completes the proof. □

The following lemma is needed to prove another characterization of simple-direct-injective modules.

Lemma 3.3. *Let A and B be direct summands of a module M such that $A \cap B = 0$. Then the following are equivalent:*

(i) *$C = A \oplus B$ is a direct summand of M;*

(ii) *Every homomorphism $\varphi : C \to M$ can be extended to an endomorphism $\theta : M \to M$.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (i) Note that $M = B \oplus U$ for some submodule U of M. Let $\pi : C \to B$ be the projection of C on B along A and let $\mu : B \to M$ denote the inclusion map. By (ii), the homomorphism $\mu\pi : C \to M$ can be extended to an endomorphism $\theta : M \to M$. Since $\theta(A) = \mu\pi(A) = 0$, we have $A \subseteq \text{Ker}\theta$. Moreover, it is clear that $\theta(y) = y$ for all $y \in B$ and hence $\text{Ker}\theta \cap B = 0$. Now take an element $m \in M$. As $M = B \oplus U$, $\theta(m) = b + u$ for some elements $b \in B$ and $u \in U$. Therefore $\theta(m) = \theta(b) + u$. Thus $u = \theta(m - b)$ and so $m - b \in \theta^{-1}(U)$. This yields $m = (m - b) + b \in \theta^{-1}(U) + B$. It follows that $M = \theta^{-1}(U) + B$.

Now to show that $B \cap \theta^{-1}(U) = 0$, take $a \in B \cap \theta^{-1}(U)$. Then $\theta(a) = a \in U \cap B = 0$. Therefore $M = \theta^{-1}(U) \oplus B$.

Let V be a submodule of M such that $M = A \oplus V$. Then $\theta^{-1}(U) = \theta^{-1}(U) \cap (A \oplus V) = A \oplus (\theta^{-1}(U) \cap V)$ because $A \subseteq \text{Ker}\theta \subseteq \theta^{-1}(U)$. Hence $M = \theta^{-1}(U) \oplus B = A \oplus (\theta^{-1}(U) \cap V) \oplus B = C \oplus (\theta^{-1}(U) \cap V)$. This completes the proof. □

Proposition 3.4. *The following are equivalent for a module M:*

(i) *M is simple-direct-injective;*

(ii) *For every submodule K of M such that $K = K_1 \oplus K_2$ and K_1 and K_2 are simple perspective direct summands of M, every homomorphism $\varphi : K \to M$ can be extended to an endomorphism $\theta : M \to M$.*

Proof. This follows by combining the preceding lemma with [1, Proposition 2.13]. □

Proposition 3.5. *The following are equivalent for a module M:*

(i) *M is simple-direct-injective;*

(ii) *For every endomorphisms α, β of M such that* $\mathrm{Ker}\beta = \mathrm{Ker}\alpha$ *and* $\mathrm{Ker}\alpha$ *is a direct summand of M which is a maximal submodule of M, there exists an endomorphism γ of M such that $\gamma\alpha = \beta$.*

Proof. (i) $\Rightarrow$ (ii) Let α and β be endomorphisms of M such that $\mathrm{Ker}\beta = \mathrm{Ker}\alpha$ and $\mathrm{Ker}\alpha$ is a direct summand of M which is a maximal submodule of M. Then $M = \mathrm{Ker}\alpha \oplus L$ for some simple submodule L of M. Clearly $\alpha_{|_L} : L \to M$ is a monomorphism. Since M is simple-direct-injective, it follows that $\alpha(L)$ is a direct summand of M. So there exists a homomorphism $\eta : M \to L$ such that $\eta\alpha_{|_L} = 1_L$. Now since $\mathrm{Ker}\alpha = \mathrm{Ker}\beta$, the homomorphism $\overline{\beta} : M/\mathrm{Ker}\alpha \to M$ given by $m + \mathrm{Ker}\alpha \mapsto \beta(m)$ is well defined. In addition, consider the isomorphism $\overline{\alpha} : L \to M/\mathrm{Ker}\alpha$ defined by $l \mapsto l + \mathrm{Ker}\alpha$ for all $l \in L$. Set $\gamma = \overline{\beta}\overline{\alpha}\eta$. Note that γ is an endomorphism of M. To show that $\gamma\alpha = \beta$, take $m \in M$. Then $m = x + l$ for some $x \in \mathrm{Ker}\alpha$ and $l \in L$. Therefore,

$$\begin{aligned}\gamma\alpha(m) &= \overline{\beta}\overline{\alpha}\eta\alpha(m) = \overline{\beta}\overline{\alpha}\eta\alpha(x+l) = \overline{\beta}\overline{\alpha}\eta\alpha(l) = \overline{\beta}\overline{\alpha}(l) = \overline{\beta}(l + \mathrm{Ker}\alpha)\\ &= \overline{\beta}(x + l + \mathrm{Ker}\alpha) = \overline{\beta}(m + \mathrm{Ker}\alpha) = \beta(m).\end{aligned}$$

This implies that γ is the desired endomorphism of M.

(ii) $\Rightarrow$ (i) Let K be a simple direct summand of M. Hence $M = K \oplus K'$ for some submodule K' of M. Assume that K is isomorphic to a simple submodule S of M. So there exists an isomorphism $\mu : K \to S$. Consider the endomorphisms φ and θ of M defined respectively by $k + k' \mapsto \mu(k)$ and $k + k' \mapsto k$ for any $k \in K$ and $k' \in K'$. Then $\varphi\theta$ is an endomorphism of M such that $\mathrm{Ker}\varphi\theta = \mathrm{Ker}\theta = K'$ is a maximal submodule of M. By hypothesis, there exists an endomorphism $\gamma : M \to M$ such that $\gamma\varphi\theta = \theta$. Hence $(\gamma\varphi - 1_M)\theta = 0$. Since θ is an epimorphism, we have $\gamma\varphi = 1_M$. Thus $\mathrm{Im}\varphi = S$ is a direct summand of M (see [2, Lemma 5.1]). Therefore M is a simple-direct-injective module. $\square$

Recall that any module M is called *pseudo-N-injective*, if every monomorphism $f : K \to M$, where $K \leq N$, can be extended to a homomorphism from N to M. The next proposition characterizes modules whose submodules are simple-direct-injective in terms of the pseudo-injectivity.

Proposition 3.6. *The following conditions are equivalent for a module M:*

(i) *Every submodule of M is simple-direct-injective;*

(ii) *For any submodules A and B of M with B simple and $A \cap B = 0$, B is pseudo-A-injective.*

Proof. (i) $\Rightarrow$ (ii) Let A and B be submodules of M such that B is simple and $A \cap B = 0$. Let us show that B is pseudo-A-injective. Let $0 \neq f : X \to B$ be a monomorphism with $X \leq A$. Since B is simple, f is an isomorphism. Now $X \cong B \leq_d A \oplus B$ and B and X are submodules of $A \oplus B$. By hypothesis, $A \oplus B$ is simple-direct-injective and B is simple. Therefore $X \leq_d A \oplus B$, and hence $X \leq_d A$. Thus f can be extended to A.

(ii) $\Rightarrow$ (i) Let L be a submodule of M. We will show that L is simple-direct-injective. For, let $L = A \oplus B$ with B simple and let $f : B \to A$ be a nonzero homomorphism. By hypothesis, B is pseudo-A-injective. Since $B \cong f(B)$, $f(B)$ is pseudo-A-injective. This implies that the identity homomorphism $1_{f(B)} : f(B) \to f(B)$ can be extended to a homomorphism $g : A \to f(B)$. Now $g \circ i = 1_{f(B)}$, where $i : f(B) \to A$ is the inclusion map. Therefore $i(f(B)) = f(B) \leq_d A$. From [4, Proposition 2.1 (1) $\Leftrightarrow$ (3)], it follows that L is simple-direct-injective. □

Next, we present some examples of modules satisfying the conditions in the hypothesis of Proposition 3.6.

Example 3.7. (i) Let M be a module such that $\mathrm{Soc}(M) = 0$. Then every submodule of M is simple-direct-injective.

(ii) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty)$, where p is a prime number. It is well known that any nonzero proper submodule of M is isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$ for some integer $k \geq 1$. It follows that every submodule of M is simple-direct-injective.

(iii) Let R be a commutative principal ideal ring. By [4, Example 2.3(2)], every cyclic R-module is simple-direct-injective. Thus, every submodule of a cyclic R-module is simple-direct-injective.

In the following proposition, we provide sufficient conditions under which a direct sum of simple-direct-injective modules is simple-direct-injective. This result should be contrasted with Lemma 2.5 and Example 2.6.

Proposition 3.8. *Let R be any ring and let an R-module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Suppose that one of the following conditions is fulfilled:*

(i) *For every simple direct summand X of M, $X \subseteq M_i$ for some $i \in I$; or*

(ii) $\mathrm{Hom}_R(M_i, M_j) = 0$ *for all distinct $i, j \in I$; or*

(iii) *For every direct summand X of M, $X = \oplus_{i \in I}(X \cap M_i)$.*

Then M is simple-direct-injective if and only if M_i is simple-direct-injective for all $i \in I$.

Proof. If M is simple-direct-injective, then M_i is simple-direct-injective for all $i \in I$ by [3, Lemma 6]. Conversely, assume that each M_i ($i \in I$) is simple-direct-injective.

(i) Let A and B be simple direct summands of M with $A \cap B = 0$. By hypothesis, there exist j and k in I such that $A \subseteq M_j$ and $B \subseteq M_k$. Since A and B are direct summands of M, $A \leq_d M_j$ and $B \leq_d M_k$. Assume that $j \neq k$. Then clearly $A \oplus B \leq_d M_j \oplus M_k$, and hence $A \oplus B \leq_d M$. Now assume that $j = k$. Since M_j is simple-direct-injective, it follows that $A \oplus B$ is a direct summand of M_j and hence it is a direct summand of M. Therefore M is simple-direct-injective by [4, Proposition 2.1].

(ii) Let S be a simple direct summand of M. It is well known that S has the exchange property (see [16, Proposition 1]). Then $M = S \oplus (\oplus_{i\in I} M_i')$ for some submodules $M_i' \leq M_i$ ($i \in I$). It follows that $M_{i_0}' \neq M_{i_0}$ for some $i_0 \in I$. Moreover, since $\oplus_{i\in I} M_i'$ is a maximal submodule of M, we have $M_i' = M_i$ for every $i \neq i_0$. This implies that $M = (S \oplus M_{i_0}') \oplus (\oplus_{i\neq i_0} M_i)$. Thus $S \oplus M_{i_0}' \cong M_{i_0}$. But M_{i_0} is fully invariant in M as $\mathrm{Hom}_R(M_{i_0}, M_j) = 0$ for every $j \neq i_0$. Then $S \oplus M_{i_0}' \subseteq M_{i_0}$ and hence $S \subseteq M_{i_0}$. The result now follows from (i).

(iii) This follows from the fact that the condition (iii) implies (i). □

The following example shows that the conditions (i), (ii) and (iii) in the hypothesis of Proposition 3.8 are not superfluous.

Example 3.9. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}(p^\infty)$, where p is a prime number. By Lemma 2.5, M is not a simple-direct-injective module. Let $0 \neq a \in \mathbb{Z}(p^\infty)$ with $pa = 0$ and set $S = (\bar{1}, a)\mathbb{Z}$. It is clear that $S \oplus \mathbb{Z}(p^\infty) = M$. On the other hand, $S \nsubseteq \mathbb{Z}/p\mathbb{Z} \oplus 0$ and $S \nsubseteq 0 \oplus \mathbb{Z}(p^\infty)$.

In the next example, we present another simple-direct-injective module.

Example 3.10. Let $M = M_1 \oplus M_2$ be an R-module such that $\mathrm{Rad}(M_1) = M_1$ and $\mathrm{Rad}(M_2) = 0$. It is clear that M_1 has no simple direct summands. In addition, all simple submodules of M_2 are direct summand. Therefore M_1 and M_2 are simple-direct-injective. Assume that either $\mathrm{Soc}(M_1) = 0$ or $\mathrm{Soc}(M_2) = 0$. Then for any simple submodule S of M, we have either $S \subseteq M_1$ or $S \subseteq M_2$. Using Proposition 3.8, we conclude that M is simple-direct-injective.

Next, we will be concerned with factor modules of simple-direct-injective modules. We begin with a result which characterizes the class of commutative rings over which the class of simple-direct-injective modules is closed under factor modules.

Proposition 3.11. *Let R be a right simple-direct-injective ring (for instance, R is commutative). Then the following conditions are equivalent:*

(i) *All factor modules of simple-direct-injective R-modules are simple-direct-injective;*

(ii) *R is a right V-ring.*

Proof. (i) ⇒ (ii) By [4, Theorem 2.14], every free right R-module is simple-direct-injective. Now the condition (i) implies that every right R-module is simple-direct-injective. Thus R is a right V-ring by [4, Proposition 4.1].

(ii) ⇒ (i) This follows from [4, Proposition 4.1]. □

From the preceding proposition, it follows that any commutative ring R which is not von Neumann regular has a simple-direct-injective R-module M such that M/N is not simple-direct-injective for some submodule N of M. Next, we provide an explicit example.

Example 3.12. Let R be a commutative ring having a maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^k \neq \mathfrak{m}^{k+1}$ for some integer $k \geq 1$ (for example, we can take a discrete valuation ring with maximal ideal $\mathfrak{m}$). Consider the R-module $M = R/\mathfrak{m} \oplus R/\mathfrak{m}^{k+1}$. It is easily seen that $0 \neq \mathfrak{m}^k/\mathfrak{m}^{k+1} \subseteq \mathrm{Soc}(R/\mathfrak{m}^{k+1})$. So $R/\mathfrak{m}^{k+1}$ contains a simple submodule S which is isomorphic to $R/\mathfrak{m} \oplus 0$ but $0 \oplus S$ is not a direct summand of M. Therefore M is not simple-direct-injective. On the other hand, it is clear that $M \cong R^{(2)}/(\mathfrak{m} \oplus \mathfrak{m}^{k+1})$. Moreover, since R is commutative, R is simple-direct-injective and so $R^{(2)}$ is a simple-direct-injective R-module by [4, Theorem 2.14].

4. Simple-Direct-Injective Modules over Dedekind Domains

This small section is devoted to the study of simple-direct-injective modules over commutative Dedekind domains.

Let M be a module over a commutative domain R. We denote by $T(M)$ the set of all elements x of M for which $Ann_R(x) \neq 0$. It is well known that $T(M)$ is a submodule of M which is called the *torsion submodule* of M. The module M is said to be a *torsion module* if $T(M) = M$. If $T(M) = 0$, the module M is said to be *torsion-free*. We begin by some examples of simple-direct-injective modules over a commutative domain.

Remark 4.1. Let R be a commutative domain with quotient field Q such that $R \neq Q$.

- (i) Since R_R is an indecomposable R-module, it follows that R_R is a simple-direct-injective R-module (see [4, Example 2.3(1)]). Hence, every projective R-module is simple-direct-injective by [4, Corollary 2.15].
- (ii) All torsion-free R-modules are simple-direct-injective since they have no simple submodules.
- (iii) If R is a Dedekind domain, then for any index set I, the R-module $M = (Q/R)^{(I)}$ is simple-direct-injective since M is injective.

Let R be a commutative Dedekind domain with quotient field Q and let M be an R-module. Let $\mathfrak{p}$ be a nonzero prime ideal of R. The set $T_{\mathfrak{p}}(M) = \{x \in M \mid x\mathfrak{p}^k = 0$ for some non-negative integer $k\}$ is a submodule of M which is called the $\mathfrak{p}$-*primary component* of M. It is well known that every torsion R-module is a direct sum of its $\mathfrak{p}$-primary components. The set of all nonzero prime ideals of R is denoted by $\mathbb{P}$.

In [3, Theorem 2], the authors characterized simple-direct-injective abelian groups. The next theorem is an extension of this characterization.

Theorem 4.2. *Let R be a commutative Dedekind domain. Let $\mathbb{P}$ be the set of all nonzero prime ideals of R. Then the following are equivalent for an R-module M:*

- (i) *M is simple-direct-injective;*
- (ii) *$T(M)$ is simple-direct-injective;*

(iii) *For every* $\mathfrak{p} \in \mathbb{P}$, *the* $\mathfrak{p}$*-primary component* $T_{\mathfrak{p}}(M)$ *is simple-direct-injective;*

(iv) *For every* $\mathfrak{p} \in \mathbb{P}$, $T_{\mathfrak{p}}(M)$ *is semisimple or* $\mathrm{Soc}(T_{\mathfrak{p}}(M)) \subseteq \mathrm{Rad}(T_{\mathfrak{p}}(M))$*;*

(v) *For every* $\mathfrak{p} \in \mathbb{P}$, $T_{\mathfrak{p}}(M)$ *is semisimple or* $T_{\mathfrak{p}}(M)$ *has no simple direct summands.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from [3, Corollary 5].

(ii) $\Rightarrow$ (iii) This follows by using [3, Lemma 6] since each $T_{\mathfrak{p}}(M)$ ($\mathfrak{p} \in \mathbb{P}$) is a direct summand of $T(M)$.

(iii) $\Rightarrow$ (ii) By Proposition 3.8.

(iv) $\Leftrightarrow$ (v) This follows from the fact that any simple submodule S of a module M is either small in M or a direct summand of M.

(iii) $\Rightarrow$ (v) Let $\mathfrak{p} \in \mathbb{P}$. Assume that $T_{\mathfrak{p}}(M)$ has a simple direct summand S but $T_{\mathfrak{p}}(M)$ is not semisimple. Then $S \cong R/\mathfrak{p}$ and $T_{\mathfrak{p}}(M) = S \oplus E$ for some submodule E of $T_{\mathfrak{p}}(M)$. It is clear that E is not semisimple. Therefore there exists $x \in E$ such that $xR \cong R/\mathfrak{p}^n$ for some integer $n \geq 2$. Note that xR is an indecomposable R-module which is not simple. It is easily seen that xR contains a simple submodule S' which is isomorphic to $R/\mathfrak{p}$. Since $T_{\mathfrak{p}}(M)$ is simple-direct-injective and $S \cong S'$, it follows that S' is a direct summand of $T_{\mathfrak{p}}(M)$ and hence S' is a direct summand of xR, a contradiction.

(v) $\Rightarrow$ (iii) This is immediate. □

Let R be a commutative domain with field of fractions Q. Recall that an R-submodule F of Q is called a *fractional ideal* of R if $rF \subseteq R$ for some nonzero element r of R. As an application of Theorem 4.2, the next corollary characterizes finitely generated simple-direct-injective modules over commutative Dedekind domains.

Corollary 4.3. *Let R be a commutative Dedekind domain and let M be a finitely generated R-module. Then M is simple-direct-injective if and only if*

$$M \cong (R/\mathfrak{q}_1)^{(m_1)} \oplus \cdots \oplus (R/\mathfrak{q}_s)^{(m_s)} \oplus (R/{\mathfrak{p}_1}^{n_1}) \oplus \cdots \oplus (R/{\mathfrak{p}_t}^{n_t}) \oplus I_1 \oplus \cdots \oplus I_k,$$

where k, s and t are non-negative integers, $\mathfrak{q}_i$ $(1 \leq i \leq s)$ and $\mathfrak{p}_i$ $(1 \leq i \leq t)$ are nonzero prime ideals of R such that $\mathfrak{q}_i \neq \mathfrak{p}_j$ for all $(i, j) \in \{1, \ldots, s\} \times \{1, \ldots, t\}$, m_i $(1 \leq i \leq s)$ and n_i $(1 \leq i \leq t)$ are positive integers, and I_j $(1 \leq j \leq k)$ are nonzero fractional ideals of R.

Proof. The necessity follows from [14, Theorem 6.16], Example 3.12 and [3, Lemma 6]. Conversely, suppose that M satisfies the stated conditions. Then clearly,

$$T(M) \cong (R/\mathfrak{q}_1)^{(m_1)} \oplus \cdots \oplus (R/\mathfrak{q}_s)^{(m_s)} \oplus (R/{\mathfrak{p}_1}^{n_1}) \oplus \cdots \oplus (R/{\mathfrak{p}_t}^{n_t}).$$

It is easily seen that for each nonzero prime ideal $\mathfrak{p}$ of R, we have either $\mathrm{Soc}(T_{\mathfrak{p}}(M)) = T_{\mathfrak{p}}(M)$ or $\mathrm{Soc}(T_{\mathfrak{p}}(M)) \subseteq \mathrm{Rad}(T_{\mathfrak{p}}(M))$. Thus M is simple-direct-injective by Theorem 4.2. □

Example 4.4. Let R be a commutative Dedekind domain with quotient field Q. Let $\mathbb{P}$ denote the set of all nonzero prime ideals of R.

(i) Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that M_1 is torsion-free and M_2 is semisimple. Then $T(M) = M_2$. From Theorem 4.2, we infer that M is a simple-direct-injective module.

(ii) Consider the R-module $M = \prod_{\mathfrak{p}\in\mathbb{P}} R/\mathfrak{p}$. Clearly, $T(M) = \oplus_{\mathfrak{p}\in\mathbb{P}} R/\mathfrak{p}$ is simple-direct-injective. Thus M is also simple-direct-injective by Theorem 4.2.

5. Simple-Direct-Injective Rings

A ring R is called *left* (*right*) *simple-direct-injective* if the left (right) R-module ${}_RR$ (R_R) is simple-direct-injective. We begin by providing examples of left (and right) simple-direct-injective rings.

Example 5.1. (i) It is clear that semisimple rings, local rings and von Neumann regular rings are left and right simple-direct-injective.

(ii) It is clear that a left (right) V-ring is a left (right) simple-direct-injective ring.

(iii) Recall that a ring R is said to be *left* (*right*) *Kasch* if every simple left (right) R-module can be embedded in ${}_RR$ (R_R). By [13, Proposition 1.46], every left (right) Kasch ring is right (left) simple-direct-injective.

Next, we exhibit some examples to illustrate that the property of being a simple-direct-injective ring is not left-right symmetric.

Example 5.2. (i) Here we are using the ring R given in [10, Examples 8.29(6)]; that is, R is the ring of matrices of the form

$$\gamma = \begin{bmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$$

over a division ring D. It is shown in [10, Examples 8.29(6)] that $\mathrm{Soc}({}_RR) = \mathrm{Rad}(R)$. Thus the left R-module ${}_RR$ has no simple direct summands. Therefore R is left simple-direct-injective. On the other hand, let us show that R is not a right simple-direct-injective ring. Consider the elements $r_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $r_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ of R. It is easily seen that r_1 is an idempotent of R and r_1R is a simple direct summand of R_R. In addition, r_2R is a simple submodule of R_R

which is isomorphic to r_1R. Note that r_2R is not a direct summand of R_R since r_2R contains no nonzero idempotents of R. This proves that the ring R is not right simple-direct-injective.

(ii) Consider the ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Then the set of all idempotents of R is $\left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{0} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \overline{1} & \overline{1} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 1 \end{bmatrix} \right\}$. Thus the direct summands of the right R-module R_R are: $\begin{bmatrix} \overline{0} & \overline{0} \\ 0 & 0 \end{bmatrix} R$, $\begin{bmatrix} \overline{0} & \overline{0} \\ 0 & 1 \end{bmatrix} R = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$, $\begin{bmatrix} \overline{0} & \overline{1} \\ 0 & 1 \end{bmatrix} R = \left\{ \begin{bmatrix} \overline{0} & \overline{n} \\ 0 & n \end{bmatrix} \mid n \in \mathbb{Z} \right\}$, $\begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 0 \end{bmatrix} R = \begin{bmatrix} \overline{1} & \overline{1} \\ 0 & 0 \end{bmatrix} R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 1 \end{bmatrix} R = R$. It is obvious that none of them is simple. Therefore R is a right simple-direct-injective ring. On the other hand, note that $R \begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{bmatrix}$ is a simple direct summand of the left R-module ${}_RR$. Moreover, the submodules $\begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ of ${}_RR$ are isomorphic. But $\begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ is not a direct summand of ${}_RR$ since it contains no nonzero idempotents of R. Therefore R is not a left simple-direct-injective ring.

Next, we present an example of a ring R which is neither left simple-direct-injective nor right simple-direct-injective. This example shows also that a direct sum of two simple-direct-injective modules need not be simple-direct-injective, in general (see also Example 2.6).

Example 5.3. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. Then the set of all idempotents of R is

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mid b \in F \right\}.$$

Note that $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} \cong \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is a simple right R-module. In addition, $\begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ is a direct summand of the right R-module R_R but $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is not. This implies that the ring R is not right simple-direct-injective. Similarly, $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \cong R \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ is a simple left R-module. Moreover, $\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ is a direct summand of the left R-module ${}_RR$ but $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is not. Therefore R is not a left simple-direct-injective ring.

On the other hand, note that $R_R = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} \oplus \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. Moreover, $U = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ is a simple right R-module and $V = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is an injective right R-module as V is a direct summand of $E(R_R) = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. In particular, U and V are simple-direct-injective right R-modules.

Let x be an element of a ring R. We denote by $Ann_R(x)$ the right annihilator of x in R, that is, $Ann_R(x) = \{r \in R \mid xr = 0\}$. Next, we show that the class of left (right) simple-direct-injective rings is closed under direct products.

Proposition 5.4. *Let $\{R_i \mid i \in I\}$ be a family of rings and $R = \prod_{i \in I} R_i$. Then R is a right simple-direct-injective ring if and only if each R_i $(i \in I)$ is a right simple-direct-injective ring.*

Proof. Suppose that R is a right simple-direct-injective ring and fix $j \in I$. Then R_j regarded as a right R-module is simple-direct-injective since it is a direct summand of R_R (see [3, Lemma 6]). Moreover, it is easily seen that the submodules of R_j are the same whether it is regarded as a right R-module or as a right R_j-module. Thus R_j regarded as a right R_j-module is also a simple-direct-injective module. Conversely, assume that each R_i is a right simple-direct-injective ring. Let S and E be simple submodules of the right R-module R_R with $S \cong E \leq_d R_R$. Clearly, $E = eR$ for some nonzero idempotent e of R. Let $f : S \to E$ be an R-isomorphism and let $s = (s_i)_{i \in I} \in R$ such that $s \in S$ and $f(s) = e$. Then $s \neq 0$ and $sR = S$. Since $R/Ann_R(s) \cong S$, $Ann_R(s) = \prod_{i \in I} Ann_{R_i}(s_i)$ is a maximal right ideal of R. It is easily seen that the right R-modules $R/Ann_R(s)$ and $\prod_{i \in I} R_i/Ann_{R_i}(s_i)$ are isomorphic. So there exists $j \in I$ such that $Ann_{R_j}(s_j)$ is a maximal right ideal of R_j and $Ann_{R_i}(s_i) = R_i$ for all $i \neq j$. Hence $s_i = 0$ for all $i \neq j$ and $s_j R_j$ is a simple submodule of the right R_j-module $M_j = {R_j}_{R_j}$. Now let $(e_i)_{i \in I} \in R$ such that $e = (e_i)_{i \in I}$. It is clear that $Ann_R(e) = Ann_R(s)$ and hence $e_i = 0$ for all $i \neq j$ and $s_j R_j \cong e_j R_j$ (as right R_j-modules). Since R_j is right simple-direct-injective and e_j is an idempotent of R_j, it follows that $s_j R_j$ is a direct summand of M_j. Therefore sR is a direct summand of R_R. Consequently, R is right simple-direct-injective. □

As an application of the preceding proposition, we obtain the following examples of left (and right) simple-direct-injective rings.

Example 5.5. (i) Every direct product of right indecomposable rings (e.g., local rings) is a right simple-direct-injective ring.

(ii) Every direct product of copies of the ring R given in Example 5.2(i) is a left simple-direct-injective ring which is not a right simple-direct-injective ring.

In the next proposition, we provide some necessary conditions for the endomorphism ring of a module to be right simple-direct-injective. Its proof is similar in spirit to that of [12, Proposition 4.6]. We first prove the following lemma.

Lemma 5.6. *Let R be a ring and let M be an R-module with $S = End_R(M)$. Let e and f be idempotents in S. Then:*

(i) *If $e(M)$ is a simple R-module, then eS is a minimal right ideal of S.*

(ii) *If* $e(M) \cap f(M) = 0$*, then* $eS \cap fS = 0$.

Proof. (i) Suppose that $e(M)$ is a simple R-module. Let $0 \neq s = es \in eS$. Then $0 \neq s(M) \subseteq e(M)$. As $e(M)$ is simple, we have $s(M) = e(M)$ is a direct summand of M. Hence $sS = eS$ by [12, Lemma 1.1]. This implies that eS is a simple S-module.

(ii) Assume that $e(M) \cap f(M) = 0$. Suppose on the contrary that $eS \cap fS \neq 0$. Let $0 \neq s \in eS \cap fS$. Then $es = s$ and $fs = s$ and hence $s(M) \subseteq e(M) \cap f(M)$. But $s(M) \neq 0$. So $e(M) \cap f(M) \neq 0$, a contradiction. It follows that $eS \cap fS = 0$. □

Proposition 5.7. *Let* R *be a ring and let* M *be an* R*-module with* $S = End_R(M)$*. If the ring* S *is right simple-direct-injective, then* M *and* S *satisfy the following two conditions:*

(a) M *is a simple-direct-injective* R*-module, and*

(b) *for every pair of idempotents* $e, f \in S$ *with* $eS \cap fS = 0$*, we have* $e(M) \cap f(M) = 0$.

The converse holds when the following condition is satisfied:

(c) *For any idempotent* $e \in S$ *such that* eS *is a minimal right ideal of* S*,* $e(M)$ *is a simple* R*-module.*

Proof. Assume that S is a right simple-direct-injective ring. Let e and f be idempotents in S such that $e(M)$ and $f(M)$ are simple and $e(M) \cap f(M) = 0$. Then eS and fS are minimal right ideals of S with $eS \cap fS = 0$ by Lemma 5.6. Applying Theorem 3.2, there exist orthogonal idempotents $g, h \in S$ such that $eS = gS$ and $fS = hS$. By using again Theorem 3.2, we deduce that M is a simple-direct-injective R-module. This proves (a).

Now to prove (b), let e and f be idempotents in S such that $eS \cap fS = 0$. Since the right S-module S_S is simple-direct-injective, there exist orthogonal idempotents $g, h \in S$ such that $eS = gS$ and $fS = hS$ by Theorem 3.2. Thus $e(M) = g(M)$ and $f(M) = h(M)$ by [12, Lemma 1.1]. Hence $e(M) \cap f(M) = g(M) \cap h(M) = 0$ as g and h are orthogonal.

The converse follows by Theorem 3.2. □

The following example shows that the condition (c) in Proposition 5.7 is not necessary for the endomorphism ring of a module to be right simple-direct-injective.

Example 5.8. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. Then the right R-module $M = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is an indecomposable module which is not simple. Moreover, we have $S = End_R(M) \cong F$ (as rings). Clearly, S is a right simple-direct-injective ring. On the other hand, the right ideal $1_M S$ is minimal, but $1_M(M) = M$ is not a simple R-module.

Let R be a ring and M an R-R-bimodule. Then the trivial extension $R \propto M$ is a ring whose underlying group is $R \times M$ with the multiplication defined by $(r,m)(s,n) = (rs, rn + ms)$, where $r, s \in R$ and $m, n \in M$.

Proposition 5.9. *Let R be a ring and let M be an R-R-bimodule such that $eM(1-e) = 0$ for any idempotent $e \in R$. If R is a right simple-direct-injective ring, then so is the ring $T = R \propto M$.*

Proof. Suppose that R is right simple-direct-injective and let A be a simple direct summand of the right T-module T_T. From [6, Proof of Proposition 4.5], it follows that $A = (eR, eM)$ for some nonzero idempotent e of R. But $(0, eM)$ is a proper T-submodule of A. Then $(0, eM) = 0$ and hence $A = (eR, 0)$. Let B be another simple direct summand of T_T such that $A \cap B = 0$. As above $B = (fR, 0)$ for some nonzero idempotent f of R with $fM = 0$. Thus $A \oplus B = (eR, 0) \oplus (fR, 0) = (eR + fR, 0)$. Moreover, it is easily checked that eR and fR are minimal right ideals of R with $eR \cap fR = 0$. Since R is a right simple-direct-injective ring, $eR + fR$ is a direct summand of the right R-module R_R. Therefore there exists an idempotent g of R such that $(eR + fR) \oplus gR = R$. This implies that $(A \oplus B) \oplus (gR, M) = T$. It follows that T is a right simple-direct-injective ring. □

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