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Dating Sun's Locations at Equinoxes Inscribed on *Cheonsang Yeolcha Bunyajido*

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Abstract

The inscription of *Cheonsang Yeolcha Bunyajido* (天象列次分野之圖) has the sun's locations at the equinoxes, which must have been copied from the astronomical treatises in Chinese historical annals, *Songshu* (宋書) and *Jinshu* (晉書). According to the treatises, an astronomer *Wang Fan* (王蕃, 228–266 CE) referred those values from a calendrical system called *Qianxiangli* (乾象曆, 223 CE), from which it is confirmed that it adopted the sun's location at the winter solstice of the $(21\frac{1}{4})^{\rm th}$ *du* of the 8th lunar lodge *Dou* (斗) as the reference direction for equatorial lodge angles. This indicates that the sun's locations at equinoxes and solstices in the calendrical system are the same as those in *Jingchuli* (景初曆, 237 CE). Hence, we propose that the sun's location at the autumnal equinox in *Cheonsang Yeolcha Bunyajido* should be corrected from '*wu du shao ruo*' (五度少弱), meaning the $(5\frac{1}{6})^{\rm th}$ *du*, to '*wu du ruo*' (五度弱), meaning the $(4\frac{11}{12})^{\rm th}$ *du*, of the first lunar lodge *Jiao* (角), as seen in *Jingchuli*. We reconstruct the polar coordinate system used in circular star charts, assuming that the mean motion rule was applied and its reference direction was the sun's location at the winter solstice. Considering the precession, we determined the observational epoch of the sun's location at the winter solstice to be $t_{\rm o} = -18.3 \pm 43.0$ adopting the observational error of the so-called archaic determinatives (古度). It is noteworthy that the sun's locations at equinoxes inscribed in *Cheonsang Yeolcha Bunyajido* originated from *Houhan Sifenli* (後漢 四分曆) of the Latter *Han* dynasty (85 CE), while the coordinate origin in the star chart is related to *Taichuli* (太初曆) of the Former *Han* dynasty (104 BCE).

Keywords: history and philosophy of astronomy — astrometry — reference systems — atlases

1. Introduction

In ancient Chinese astronomy, there were two kinds of star charts: rectangular chart called *Hengtu* (橫圖) and circular chart called *Yuantu* (圓圖). The circular chart is also called *Kaitu* (蓋圖), to which the Korean star chart engraved in January 1396 CE called *Cheonsang Yeolcha Bunyajido* (天象列次分野之圖, hereafter, the C-Map in abbreviation) belongs. The projection of the circular chart is described in the Astronomical Treatises (天文志, hereafter, *Tianwenzhi*) in History of the *Sui* dynasty (581–619 CE) (隋書, hereafter, *Suishu*). It is a sort of polar coordinates, where each point on a plane is given by a radius from a north celestial pole and an angle from a given reference direction. Specifically, the radius is proportional to the polar distance, and the angular component is represented by the lodge angle corresponding to the right ascension.

Since the C-Map adopts an equatorial coordinate system, we can determine its observational epoch by considering the effects of precession. Two kinds of information on the C-Map have been analyzed to estimate the observational epoch. One is the positions of equinoxes on the star chart itself, and the other is the sun's locations at equinoxes written in the inscription of the C-Map. The inscription says,

> The ecliptic and the equator: The ecliptic is the path the sun travels along. The equator is the exact middle of the north and south celestial poles. One-half of the ecliptic lies outside the equator, and the other half lies inside. In the east, the ecliptic crosses the equator at *wu du shao ruo* (五度少弱 or the $(5\frac{1}{6})^{\rm th}$ *du*) of *Jiao*[1][1]; in the west, it crosses the equator at *shi si du shao qiang* (十四度少強 or the $(14\frac{1}{3})^{\rm th}$ *du*) of *Kui*[15] (奎).

Determination of the epoch has long been investigated, beginning from the article of Rufus (1913), who estimated the epoch to be the first century from the position of the autumnal equinox on the chart, which, he thought, was drawn more accurately. The sun's locations at the equinoxes in the inscrip-

[1] *Jiao*[1] represents the first lunar lodge called *Jiao* (角) among the 28 lunar lodges. The number in parenthesis is the entry number of the lunar lodge.

tion have been analyzed to determine the reference epoch by researchers (Lee 1986; Park 1995). Their calculations showed that the sun's locations were observed around 50 BCE. However, since the ecliptic was not drawn on the star chart faithfully and reliably, it has been a challenging problem to determine the epoch from the positions of equinoxes on the star chart. In this paper, we will see that the epochs obtained by previous researchers from the sun's locations on the C-Map inscription may also be unreliable.

The sentences quoted above have been regarded as being copied from the *Tianwenzhi* of the History of the *Jin* dynasty (266–420 CE) (晉書, hereafter, *Jinshu*), which was completed in 648 CE during the early *Tang* (唐) dynasty (618–907 CE). The same sentences are found in the *Tianwenzhi* of the History of the *Liu Song* (劉宋) dynasty (420–479 CE) (宋書, hereafter, *Songshu*). *Songshu* was written in 492–493 CE by *Shen Yue* (沈約, 441–513 CE) of the Southern dynasties (南朝). Hence, these facts are prone to give us a misconception that the star chart of the C-Map originated from a period as early as the *Jin* dynasty or as late as the *Tang* dynasty.

It seems that little attention has been paid to the contents of these original texts. Neither their origin nor typographical errors has not been verified. These records should also be cross-checked with other the Calendrical Treatises (律曆志, hereafter, *Lulizhi*) and *Tianwenzhi* as well. This paper will inspect the original texts to find their relationship with the calendar systems developed during the Latter *Han* dynasty (後漢). More importantly, although the sun's locations at equinoxes were thought to have been represented with reference to each determinative star in previous studies, we should calculate them with reference to the sun's location at the winter solstice to properly convert the locations of the equinoxes into the right ascensions.

To demonstrate these, in Section 2, we will clarify the coordinate system used in the star chart of the C-Map. In section 3, we clarify the reference point and reference direction in the polar coordinate and demonstrate that the direction from the north celestial pole to the center of the *Xingji* (星紀) or *Chou* (丑) sector should be aligned with the sun's location at the winter solstice and plays a role of the reference direction. Then, we will estimate the actual value of the sun's location at the winter solstice by correcting the sun's locations at the equinoxes engraved on the inscription of the C-Map in Section 4. Then, we will determine the observational epoch of the sun's location at the winter solstice in Section 5. We will summarize the results and argue which historical occasions may be related to the measurements of the sun's position in Section 6.

2. Coordinate System of the C-Map Star Chart

The sun's locations at equinoxes are written in *Tianwenzhi*'s of *Songshu* and *Jinshu*. Two records are similar, but those in *Songshu* are relatively more readable.

> At the end of the Latter *Han* dynasty, *Liu Ji* (陸積, 188–219 CE), a person of the *Wu* (吳) dynasty, was good at astronomy. He was the first person to infer what the theory of *Hun Tian* (渾天) means. *Wang Fan* (王蕃, 228–266 CE), a person from the province of *Lu Jiang* (廬江), became an emperor's attendant (中常侍, *zhong chang shi*) in the court of the *Wu* dynasty. He was good at mathematics, and he conveyed *Qianxiangli* (乾象曆) developed by *Lui Hong* (劉洪, 130–196 CE). When *Wang Fan* manufactured an armillary sphere (渾儀) based on the method in *Qianxiangli*, he established the relevant angles: "··· The ecliptic is the path the sun travels along. One-half of the ecliptic lies outside the equator, and the other half lies inside. In the east, the ecliptic crosses the equator at *wu du shao ruo* (五度少弱 or the $(5\frac{1}{6})^{\text{th}}$ *du*) of *Jiao* (角); in the west, it crosses the equator at *shi si du shao qiang* (十四度少強 or the $(14\frac{1}{3})^{\text{th}}$ *du*) of *Kui* (奎). When the ecliptic lies outside the equator, it becomes farthest away from the equator by 24 *du* when the sun lies at the 21$^{\text{th}}$ *du* of *Dou* (斗). When the ecliptic lies inside the equator, it also becomes farthest away from the equator by 24 *du* when the sun lies at the 25$^{\text{th}}$ *du* of *Jing* (井)." (*Tianwen 1*, *zhi* 13, *Songshu*)[2]

In this section, the system of fractions and the coordinate system used in the above sentences will be explained. The system of fractions is explained in ancient calendrical methods such as *Houhan Sifenli*[3] (後漢 四分曆), *Jingchuli*[4] (景初曆), and *Qingxiangli*[5] (乾象曆). According to the explanations, *shao* (少) is one-quarter, *ban* (半) is two-quarters, and *tai* (太) is three-quarters, and in addition $+\frac{1}{12}$ is *qiang* (強) and $-\frac{1}{12}$ is *ruo* (弱). Here, *shao* means small, *ban* means half, *tai* means large, *qiang* means strong, and *ruo* means weak. Hence, *wu du shao ruo* or 5 *du* and *shao ruo* means $5(\frac{1}{4}-\frac{1}{12}) = 5\frac{1}{6}$ *du*, and *shi si du shao qiang* or 14 *du* and *shao qiang* means $14(\frac{1}{4}+\frac{1}{12}) = 14\frac{1}{3}$ *du*.

Here, *du* (度) is a Chinese angular unit for angles. One circuit of the solar motion in the sky is equivalent to $365\frac{1}{4}$ days in ancient calendar systems. Assuming that the sun follows the mean motion, one day can be regarded as equal to one *du*. In addition, if we do not know the existence of precession, the angular span corresponding to one cycle of the year or the tropical year is equal to $365\frac{1}{4}$ *du*. This value also corresponds to *zhou tian* (周天) that was translated as 'Circuits of Heaven' by Cullen (2017). However, *zhou tian* is the sidereal year expressed in days, so it had been often revised after Chinese astronomers discovered the precession.

The numbers in the lodge angles, such as $5\frac{1}{6}$ *du* of *Jiao*,

[2] 漢末吳人陸績善天文, 始推渾天意. 王蕃者, 廬江人, 吳時爲中常侍, 善數術, 傳劉洪乾象曆. 依乾象法而制渾儀, 立論考度曰: ··· 黃道, 日之所行也. 半在赤道外, 半在赤道内, 與赤道東交於角五少弱, 西交於奎十四少強. 其出赤道外極遠者, 去赤道二十四度, 斗二十一度是也. 其入赤道内極遠者, 亦二十四度, 井二十五度是也. (宋書1, 志13, 天文1)

[3] 'The table of solar terms' in *Lifa*, *Lulizhi xia*, *zhi* 3, *Hou Hanshu*.

[4] 'To find the double-hour of occurrence' in *Lulizhi xia*,*zhi* 8, *Jinshu*.

[5] 'To obtain the polar distances of the moon' in *Qianxiangli*, *Lulizhi zhong*, *Zhi* 7, *Jinshu*. Cullen (2017) translated the title of this calendar as the Uranic Manifestation System.

$14\frac{1}{3}$ *du* of *Kui*, 21 *du* of *Dou*, and 25 *du* of *Jing* in the quoted sentences, were defined not by cardinal numbers but by ordinal numbers. Moreover, the first *du* in the span of a lunar lodge was usually called '*chu du*' (初度), which is zero *du* in cardinal numbers. Since the lodge angles were measured with reference to the determinative stars, '*chu du*' (初度) is equal to a range $[-\frac{1}{2}, +\frac{1}{2})$ *du*. So, 21 *du* of *Dou* must be interpreted as the 21st *du* in the span of the lunar lodge *Dou*, which is equivalently 20 *du* away from the determinative star of *Dou* in the right ascension and corresponding to a number range $[20-\frac{1}{2}, 20+\frac{1}{2})$ *du* = $[19\frac{1}{2}, 20\frac{1}{2})$ *du*.

We also need to understand a celestial coordinate system used by ancient Chinese astronomers, similar to the equatorial coordinate system in modern astronomy. In the coordinate system, lodge angles (入宿度) correspond to right ascensions, and polar distances (去極度) correspond to declinations. The polar distance of an object is the angular distance from the north celestial pole to the object. The lodge angle of an object is defined as the eastward angular difference in right ascension from its closest-to-the-west reference star to the object. The reference stars are called the determinative stars (宿距星). For example, the determinative star of the first lunar lodge *Jiao*[1] is α Vir, and the determinative star of the 15th lunar lodge *Kui*[15] is ζ And.

The lodge angles between neighbouring determinative stars along the equator are called lodge spans (赤道宿度). The lodge spans were crucial quantities to define the coordinate system, and so they were usually reexamined with new observational measurements on the occasions of calendar reforms. There were five times of significant observations for the lodge spans (Qu 2008). In particular, the spans measured in 104 BCE had been referred to as 'archaic angular spans' (古度 *gu du*) and used for a long time until they were updated by *Yi Xing* (一行, 683–727 CE, *Zhang Sui* 張遂 as a secular name) in the 8th century. It is noteworthy that these archaic spans are engraved on the inscription of the C-Map.

In practical observations, the lodge angles of celestial objects were measured with reference to determinative stars (Ahn 2021). Hence, if a lodge angle of an object is given as the $(\Delta_X + 1)^{\text{th}}$ *du* of the lunar lodge L, then we can convert the lodge angle into its right ascension, α_X, by

$$\alpha_X = \alpha_L(t_\text{o}) + \Delta_X, \tag{1}$$

where $\alpha_L(t_\text{o})$ is the right ascension of the determinative star of the lunar lodge L at an observational epoch t_o. Here we see that the uncertainty in α_X depends on the uncertainty in $\alpha_L(t_\text{o})$.

On the other hand, the lodge angles of the sun's locations at the 24 solar terms (二十四氣 or *er shi si qi*), including equinoxes and solstices, were defined differently. Most of all, their intervals were calculated rather than observed ones. Only the reference direction or the sun's location at the winter solstice was determined with observations. The sun's locations at the other solar terms were also given as lodge angles with reference to each determinative star. Still, those were just representations in the given coordinate system, in which the lodge spans are just a sort of graduations. Thus, when we have the right ascension of the sun, denoted by $\alpha_\text{WS}(t_\text{o})$, at the winter solstice at an observational epoch t_o, we have to obtain the sun's right ascension, denoted by $\alpha_X(t)$, at a specific solar term X at an observational epoch t. The angular interval between them is calculated in a calendrical method to be L_X so that $L_X = \alpha_X(t) - \alpha_\text{WS}(t_\text{o})$. If the lodge angle of the sun's location at the winter solstice is given as the $(\Delta_\text{WS} + 1)^{\text{th}}$ *du* of the j^{th} lunar lodge D whose lodge span is given as L_j, the lodge angle of the sun's location at a certain seasonal grant X, denoted by the $(\Delta_X + 1)^{\text{th}}$ *du* of a k^{th} lunar lodge L, should satisfy an equation

$$L_X = (L_j - \Delta_\text{WS}) + \sum_{i=j+1}^{k-1} L_i + \Delta_X, \tag{2}$$

where L_i's are the lodge spans for the lunar lodges intervening the two seasonal grants. Thus, the right ascension of the sun's location at the solar term should be

$$\alpha_X(t) = \alpha_\text{WS}(t_\text{o}) + (L_j - \Delta_\text{WS}) + \sum_{i=j+1}^{k-1} L_i + \Delta_X. \tag{3}$$

Here we see that the uncertainty in $\alpha_X(t)$ depends on the uncertainty in $\alpha_\text{WS}(t_\text{o})$ and L_i's. The two converting methods, expressed by Equation (1) and Equation (3), will result in different values and uncertainties. Therefore, we must select a proper method for converting lodge angles to right ascensions for each case. For the cases of the sun's locations at equinoxes and solstices, the latter method or Equation (3) will be proper.

As a prerequisite, we explain the concepts of *Dou fen* (斗分) and *Xu fen* (虛分). The summation of the lodge spans is equal to one *zhou tian* or one sidereal year. According to the description in Page 219 of Qu (2008),

> Before the *Tang* period, in no-precession calendar systems, the fraction day in a tropical year was added to the end of the lunar lodge of *Dou*[8]. Such a fraction was called *Dou fen*. After *Zu Chongzhi* (祖冲之, 429–500 CE), who developed his calendrical system *Damingli* (大明曆), first adopted the calculation for the precession into his calendrical method, the fractional day of the sidereal year that is greater than 365.25 *du* was put to the end of the lunar lodge of *Xu*[11]. This fraction day was called *Liu Xu Zhi Cha* (六虛之差). Hence, such a fractional part of the sidereal year was always called *Xu fen*.

It is noteworthy that the lodge spans listed on the inscription of the C-Map has *Dou fen*, which means the lodge span of *Dou*[8] is $26\frac{1}{4}$ *du*.

3. Reference Point and Reference Direction

The projection of a circular star chart forms a polar coordinate system. The lodge angle and the radius of any point in this polar coordinate can be the azimuthal angle and the polar distance, respectively. This polar coordinate should set the origin and the reference direction. The origin is obviously the north celestial pole. The reference direction is clarified in the New History of the *Tang* dynasty (新唐書, hereafter, *Xin Tangshu*), which describes the drawing method of circular star charts (Ahn 2015). According to *Xin Tangshu*,

> Calculate the sun's location at the winter solstice, and regard it as the center of either *chen* (辰) or *ci* (次). That is established as the reference direction of lodge angles. (*Tianwen* 1, *zhi* 21, *Xin Tangshu*)[6]

Here *ci*'s (次) are twelve equal sectors divided along the equator based on the motion of Jupiter called *sui* (歲), which completes an entire circuit in almost twelve years. The years were named after the *ci* in which the *sui* was located. *chen*(辰)'s are also twelve equal sectors on the sky divided along the equator based on the motion of Jupiter's imaginary counterrotating object called *taisui* (太歲). They are defined on the inscription on the top part of the C-Map as conveyed from *Tianwenzhi* of *Jinshu*. Although the term *Yeolcha* (列次, *Lieci* in Chinese *pinyin* notation) in the title of the C-Map means "arranged in the order of *ci*'s", we only see 12 *chen*'s on the star chart. On circular star charts, *chen*'s are defined counterclockwise and *ci*'s are defined clockwise. (See Pages 42–44 of Pan (1989) for detailed explanations on these concepts.) Since one *zhou tian* is $365\frac{1}{4}$ *du*, one sector has a span of $30\frac{7}{16}$ *du* or $30\frac{14}{32}$ *du*.

Why does the sun's location at winter solstice become the reference direction for azimuthal angles? In ancient Chinese astronomy, the winter solstice had been measured with a gnomon, and the difference between two successive winter solstices is equivalent to the length of a tropical year. Hence, the winter solstice might have been a reference point for time and angle measurements.

Then, which *ci* (次) or equivalently *chen* (辰) is aligned with the sun's location at the winter solstice? We can find statements related to this question in the comments of *Zu Chongzhi* (祖沖之, 429–500 CE). According to the *Yuanjia lifa* (元嘉曆法) part in *Lulizhi* of *Songshu*,

> In the sixth year of the *Daming* reign period (462 CE), ⋯ *Zu Chongzhi* presented a report to the throne: "⋯, And there are some ideas related to establishing methods. Firstly, we define *Zi* (子) as the beginning *chen* (辰). It is located at due north. Its *yao*[7] (爻) corresponds to the first nine. It is the lead of *Dou*'s energy (斗氣, *dou qi*). The lunar lodge of *Xu* (虛) corresponds to the north. It is the center of all lunar lodges. The fundamental energy (元氣, *yuan qi*) stems from here. Thus, the lunar lodge of *Xu* should be located in this *ci* (次). The old scholar *Yu Xi* (虞喜, c.270–345 CE) argued this concept. In the present calendar, the sun's location at *shang yuan*[8] (上元) is set to be the 1st *du* of the lunar lodge *Xu*. ⋯." (*Lilu xia*, *zhi* 3, *Songshu*)[9]

The quoted sentences state that the *Zi* (子) direction or the first *chen* (辰) aligns itself to the due north. The top of the star chart of the C-Map aligns with the *chen* of *Zi*, and its bottom aligns with the *chen* of *Wu* (午). Conventionally, the *Zi* direction has been regarded as the due north, whereas the *Wu* direction has been considered the due south. Hence, the meridian line is called the *Zi-Wu* line.

Zu Chongzhi also stated that the lunar lodge *Xu*[11] belongs to the north. In ancient Chinese astronomy, the lunar lodge *Xu*[11] is located at the center of the seven north lodges. According to *Lulizhi* in *Hanshu* (漢書), the lunar lodges of *Xu*[11] and *Wei*[12] belong to the equatorial sector of *Xuanxiao* (玄枵). *Xu* (虛) means 'emptiness' and *Xuanxiao* means 'mystical dark emptiness.' These were thought to be the characteristics of the winter season and the north direction from which the vitality of life stems.

Zu Chongzhi also mentioned that the sun was located at 'one *du* of *Xu*[11]' at *shang yuan* (上元). Here *shang yuan* is defined as the ideal epoch of a calendar system when several astronomical phenomena occurred simultaneously[10]. Qu (2008) is referred to understand this concept. This statement is interpreted that *Xu*[11] should be placed in the *Zi* direction because the sun was located in the direction of *Xu*[11] at the time of *shang yuan*.

In the sentences preceding the above ones, *Zu Chongzhi* listed the sun's locations at past times, including one estimated from the culminating stars of the emperor *Yao* (堯). (See Appendix 1 for the translated text.) He analyzed those locations to derive the drift rate of winter solstice due to the precession that had been discovered by an astronomer named *Yu Xi* (虞喜, c.270–345 CE). Ancient astronomers believed that the sun was located in the lunar lodge of *Xu*[11] at the winter solstice in the time of the emperor *Yao* (堯). The history of Chinese calendars is thought to have begun in the era of *Yao*. The observational records of culminating stars during the era are the earliest available data for ancient Chinese astronomers to estimate the precession rate (Qu 2005). Consequently, these data were highly valued as important information for calculating calendars. After *Zu Chongzhi* introduced the precession calculations in his calendrical system, astronomers of the *Tang*, *Song*, and *Yuan* dynasties[11] required in their calendar systems that the sun should be located in the span of the lunar lodge

[6] 乃步冬至日所在, 以正辰次之中, 以立宿距. (天文1 志21 新唐書)

[7] The basic unit of *Zhou yi* (周易, Book of Changes) is the hexagram (*gua*, 卦), a figure composed of six stacked horizontal lines. Each broken or unbroken line is called *yao* (爻).

[8] This term is the High Origin translated by Cullen (2017).

[9] 大明六年,⋯ 祖沖之上表曰: "⋯ 又設法者, 其一, 以子爲辰首, 位在正北, 爻應初九, 斗氣之端, 虛爲北方, 列宿之中, 元氣肇初, 宜在此次. 前儒虞喜, 備論其義. 今曆上元日度, 發自虛一. (律曆下, 志3, 宋書)

[10] At the year of the High Origin (*shang yuan*), the Jupiter/year (歲, *sui*) was at *jiazi* (甲子). On the day of *jiazi*, when it was the new moon day, and it was the winter solstice exactly at midnight, the sun, the moon, and the five planets gathered together in the direction of the 1st *du* of *Xu*[11]. The acceleration and deceleration of *yin* and *yang* also began from here. ([*Daming*] *Lifa*, *Lulizhi*, *Songshu*)

[11] *Yu Xi* (c.270–345 CE) discovered the existence of precession. *Zu Chongzhi* demonstrated the reality of the precession with observations and inserted it in his calendrical system. After long debates on its reality, the precession was settled into Chinese calendrical systems. The concept of *shang yuan* was abandoned in the new calendar system called *Shoushili* (1280 CE), and it was no longer used.

Xu[11] at the winter solstice during the era of *Yao* (堯) (Qu 2005).

In contrast to *Zu Chongzhi*'s idea, we see on the star atlas of the C-Map that the center of the *Zi* sector is aligned with the determinative line of the lunar lodge *Wei*[12], instead of *Xu*[11]. We also see that the center of the *Chou* (丑) sector is aligned with the determinative line of the lunar lodge *Niu*[9]. The star chart depicts the ecliptic circle as having the maximum polar distance around the *Chou* sector, which means that the winter solstice is located in that sector. Therefore, we propose that the sun lies in the center of *Chou* sector at the winter solstice.

Then, why the center of the *Zi* (子) sector is aligned with the determinative star of *Wei*[12] instead of *Xu*[11]? The *Chou* (丑) sector in the star chart corresponds to the equatorial sector of *Xingji* (星紀), and the *Zi* sector corresponds to the equatorial sector of *Xuanxiao* (玄枵). According to the inscription on the top-right part of the C-Map, both sectors have angular spans of 30 *du*. According to the archaic lodge spans given in the catalogue on the inscription of the C-Map, the lodge spans of *Niu*[9], *Nu*[10], and *Xu*[11] are 8 *du*, 12 *du*, and 10 *du*, respectively. Hence, counting the angles from the center of *Xingji* or 'the 1st (or zero) *du* of *Niu*,' the center of the *Zi* sector should be 'the 1st (or zero) *du* of *Wei*[12].' Therefore, it is just a coincidence that the center of the *Zi* sector lies on the determinative line of *Wei*[12], which means that the center has a lodge angle of 'the 1st (or zero) *du* of *Wei*[12]' in the star chart of the C-Map.

Zu Chongzhi insisted that the *Zi* direction must be in the span of *Xu*[11], and he also emphasized that the sun's position at the winter solstice at *shang yuan* was in the span of *Xu*[11]. After his calendrical system *Damingli*, the fractional part of *zhou tian* has been usually attached in the lodge span of *Xu*[11], which is called *Xu fen*. However, in the star chart of the C-Map, the *Zi* direction is aligned with the determinative line of *Wei*. Moreover, the catalogue of 28 lunar lodges in the C-Map has *Dou fen*. Hence, the C-Map must have been designed by reflecting the ideas before *Zu Chongzhi*.

In conclusion, in the star chart of *Cheonsang Yeolcha Bunyajido* (天象列次分野之圖), the *Zi* sector corresponds to the equatorial sector of *Xianxiao* and the *Chou* sector corresponds to the equatorial sector of *Xingji*. The sun's location at the winter solstice is aligned with the center of the *Xingji* sector. Notably, the literal meaning of *Xingji* is the celestial norm. The center of the *Xingji* sector in the star chart of C-Map is coincided with 'the 1st (or zero) *du* of *Niu*[9].' Notably, two calendar systems extant in Chinese history fulfil this condition. One is *Santongli* (三統曆), which inherited the legacy of *Taichuli* (太初曆) promulgated in 104 BCE during the Former *Han* dynasty. The other is *Shenlongli* (神龍曆), which was used during the period of the empress *Wu Zetian* (則天武后) from 705 to 707 CE. According to Table 3-41 on Page 164 of Qu (2005), the lodge angle was also adopted as the sun's location at *shang yuan*.

4. Sun's Location at the Winter Solstice

4.1. *Taichuli* and *Santongli*

The early Former *Han* (前漢) dynasty succeeded the calendrical system called *Zhuanxuli* (顓頊曆) that had been used in the *Qin* (秦) dynasty. The calendrical system was reformed in 104 BCE and replaced by *Taichuli* (太初曆 or Grand Inception system), which was developed by *Deng Ping* (鄧平, ?-?), *Tang Du* (唐都, ?-?), *Luoxia Hong* (落下閎, 156 BCE–87 BCE) *et al.* based on the observations. The calendrical system is not extant, but its revised version called *Santongli* (三統曆 or Triple Concordance system) is extant in *Lulizhi* of *Hanshu*. *Santongli* was the work of *Liu Xin* (劉歆, c.50 BCE–23 CE) in 20 BCE and began to be used in 7 BCE. According to the table for the 12 equatorial sectors in *Santongli* in *Lulizhi* of *Hanshu*,

> *Xingji* (星紀, celestial norm): It begins with the 12th *du* of *Dou* (斗), (which corresponds to) the solar term of *daxue* (大雪, the great snow). The center of the sector lies on the 1st *du* of *Qian Niu* (牽牛初度), (which corresponds to) the solar term of *dongzhi* (冬至, the winter solstice). [The *Xia* (夏) dynasty regarded the sector as the 11th month, the *Shang* (商) dynasty regarded the sector as the 12th month, and the *Zhou* (周) dynasty regarded the sector as the 1st month.] The sector ends with the 7th *du* of *Wunu*[12] (婺女). (*sui shu*, *Lulizhi1 xia*, *Hanshu*)[13]

Here, the sun's location at the winter solstice was given as *Qian Niu chu du* (牽牛初度), which means the 1st (or zero) *du* of the lunar lodge *Qian Niu*[9][14]. The same value can also be seen in the arguments on calendars given by *Jia Kui* (賈逵, 174–228 CE) in *Lulizhi* of the History of the Latter *Han* dynasty (後漢書, hereafter, *Hou Hanshu*). The sun's locations at the solstices and equinoxes in *Santongli* are shown in Table 1. It is apparent in the table that the intervals between neighbouring solstices and equinoxes are not uniform.

Santongli inherited the legacy of *Taichuli*. *Taichuli* defines the month having winter solstice as the 11th lunar month, and the beginning of the year lies in the first month. This definition was a revival of the calendrical system of the *Xia* (夏) dynasty, as can be seen in the referred sentences. *Santongli* adopted the mean motion rule or the *ping qi* rule (平氣法)[15], so we can obtain the lodge angle for the next nodal solar term (節氣 or *Jieqi*) called *xiaohan* (小寒, the little cold), the next medial solar term (中氣, *zhongqi*) called *dahan* (大寒, the great cold), and so forth by successively adding $15\frac{7}{32}$ *du*.[16]

[12] This is the archaic name for the 10th lunar lodge of *Nu*[10].

[13] 星紀, 初斗十二度, 大雪. 中牽牛初, 冬至. [於夏爲十一月, 商爲十二月, 周爲正月.] 終於婺女七度. (歲術, 律曆志1下, 漢書)

[14] This is the archaic name for the 9th lunar lodge of *Niu*[9].

[15] In ancient Chinese astronomy, the 24 solar terms were defined by a fixed interval of time from the winter solstice, called the *ping qi* rule (Qu 2008). Here, Kepler's laws, coordinate transformations, and the precession of equinoxes are neglected, assuming that the sun's apparent motion follows the mean motion along the equator. On the other hand, after the 17th century, the 24 solar terms were defined by a fixed span of angle, which is called the *ding qi* rule (定氣法).

[16] Here, the interval $15\frac{7}{32}$ *du* is derived as follows. Before the precession

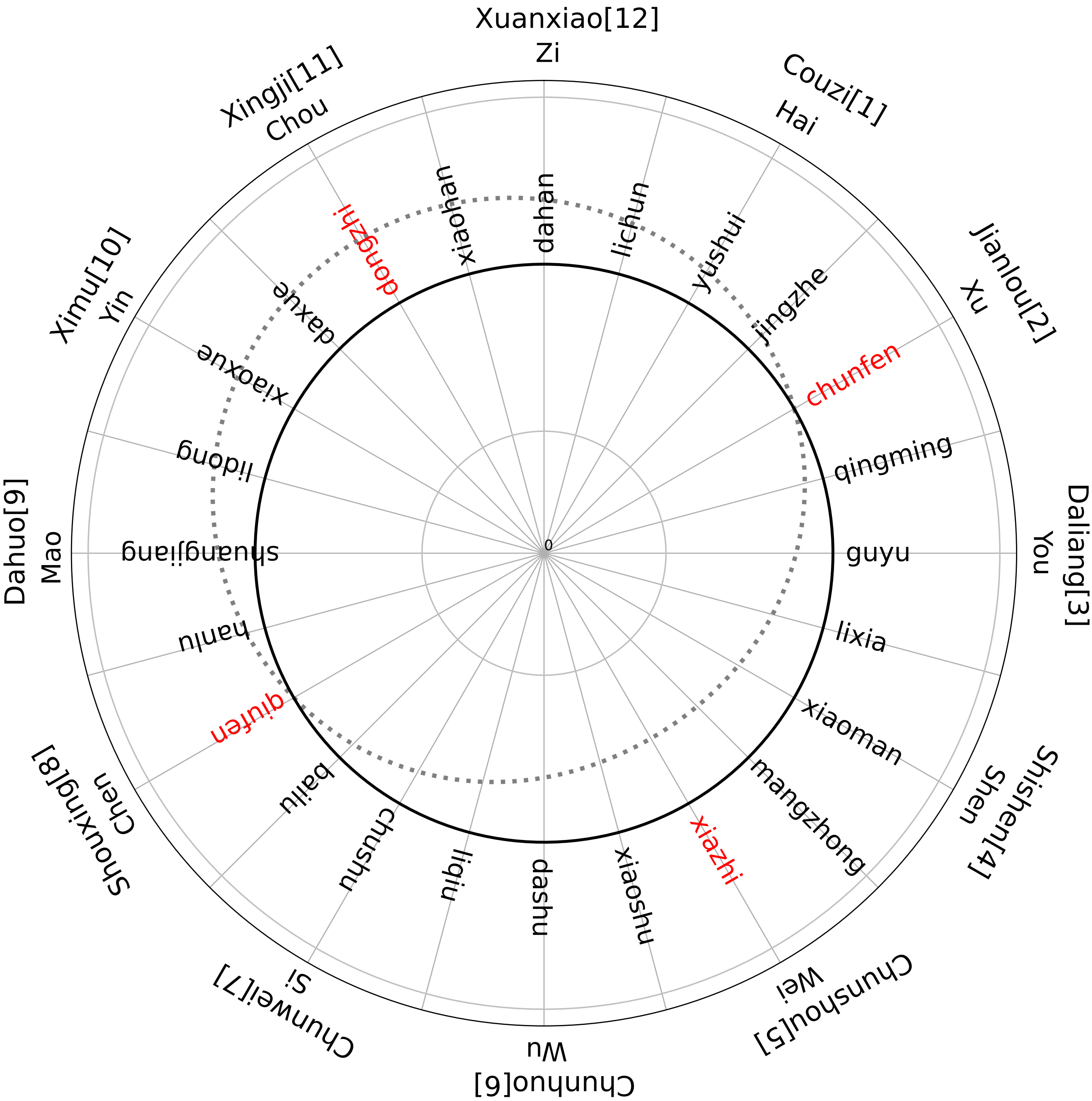

Figure 1. Schematic diagram for the 12 equatorial *ci* (次) sectors (names of months in parentheses), the 12 *chen* (辰) sectors, and the 24 solar terms in a circular star-chart. The solid circle in the middle is the equator, and the dotted circle eccentric to the equator is the ecliptic. The top of the circular chart should be aligned with the north direction or the center of the *Zi* (子) sector, which corresponds to the equatorial sector of *Xuanxiao* (玄枵) and whose representative lunar lodge is the 11th lunar lodge *Xu* (虛). The sun's location at the winter solstice (冬至, *dongzhi*) must be aligned with the center of the beginning equatorial sector called *Xingji* (星紀). The month having winter solstice is defined as the 11^{th} month since *Taichuli* (太初曆).

In *Santongli*, one lunar month is given as $29\frac{43}{81}$ days and the 11^{th} month is defined by the month of the winter solstice. At the winter solstice, the sun is located at the center of the *Xingji* sector. Since one month is roughly close to the lodge span for one *ci* sector or two *qi*'s, the 12^{th} month approximately coincides with the span of *Xuanxiao*. Likewise, the center of the *Couzi* (娵訾) sector should be aligned with the solar term of *yushui* (雨水 or the rain waters), and the beginning of the 1^{st} month corresponds to the solar term of *lichun* (立

was not discovered and not adopted in calendars, the mean motion rule was adopted for dividing the solar terms. Since one *zhou tian* was $365\frac{1}{4}$ *du* or $360\frac{21}{4}$ *du*, the mean angular interval between neighbouring solar terms should be $15\frac{7}{32}$ *du*. Its fractional part of $\frac{7}{32}$ can be exactly represented using the 32-equipartition system.

春 or the establishment of spring). Successively, a diagram demonstrating the 12 equatorial sectors (次, *ci*'s), 24 solar terms, and 12 lunar months is constructed in Figure 1.

4.2. *Houhan Sifenli*

In 85 CE during the Latter *Han* dynasty (後漢), *Li Fan* (李梵) and *Bian Xin* (編訢) edited *Houhan Sifenli* (後漢 四分曆 or *Han* Quarter Remainder system), which is extant in *Lulizhi* of *Hou Hanshu*. *Houhan Sifenli* contains a table of the sun's locations, the lengths of day and night, and the shadow lengths of a gnomon at solar terms. According to the table, the sun was located at the $(21\frac{1}{4})^{\text{th}}$ *du* of *Dou*[8] at the winter solstice. Moreover, the angular intervals between neighbouring solstices and equinoxes are uniform, as seen in Table 1. This uniformity originates from adopting the 32-equipartition system as the system of fractions.

After determining the sun's location at the winter solstice, the sun's locations at other solar terms, including solstices and equinoxes, can be calculated. A quarter of *zhou tian* or $91\frac{10}{32} = 91\frac{5}{16}$ *du* can be represented without remainders with the 32-equipartition system. A detailed discussion on this issue can be seen on Pages 81–84 of Pan (1989).

4.3. *Jingchuli*

Jingchuli (景初曆) was promulgated in 237 CE. One 24$^{\text{th}}$ of one *zhou tian* has a fractional part of $\frac{7}{32}$, and a quarter of one *zhou tian* has a fractional part of $\frac{5}{16}$. Since *Jingchuli* adopts the 12-equipartition system, neither $\frac{7}{32}$ nor $\frac{5}{16}$ can be decomposed into a fractional number $\frac{x}{4} + \frac{y}{12}$, where $x = 0, 1, 2, 3$ and $y = 0, \pm 1$. Hence, any fraction in the 12-equipartition system must be chosen as closest to the exact value. Considering that $\frac{1}{4} < \frac{5}{16} < \frac{2}{4}$, a solution of $x = 1, y = +1$ is found to minimize $|(\frac{x}{4} + \frac{y}{12}) - \frac{5}{16}|$. In this case, an angular interval of $91\frac{1}{3}$ *du* is applied three times and an interval of $91\frac{1}{4}$ *du* is applied once to calculate the solstices and equinoxes. This is how the sun's locations at solstices and equinoxes were calculated in *Jingchuli*, as can be confirmed in Table 1.

4.4. *Qianxiangli* and the C-Map Inscription

There are records on the sun's locations at solstices and equinoxes in *Tianwenzhi*'s of *Songshu* and *Jinshu*. The sun's locations at the solstices do not have fractions, so they must have been rounded down. The sun's location at the vernal equinox is the same as that in *Jingchuli*, while the location at the autumnal equinox differs from that in *Jingchuli*. The sun's location at the autumnal equinox is given as *wu du shao ruo* (五度少弱) or the $(5\frac{1}{6})^{\text{th}}$ *du* of *Jiao*[1] in *Songshu* and *Jinshu*, while it is given as the *wu du ruo* (五度弱) or the $(4\frac{11}{12})^{\text{th}}$ *du* of *Jiao*[1] in *Jingchuli*. It is notable that only the locations at the equinoxes are engraved on the inscription of the C-Map.

For the case of values in *Songshu* and *Jinshu*, the angular span from the vernal equinox through the summer solstice and to the autumnal equinox is $182\frac{5}{6}$ *du*. The angular span from the autumnal equinox through the winter solstice to the vernal equinox is $182\frac{5}{12}$ *du*. They are different from each other, which violates the mean motion rule.

The bisect of $\frac{5}{6}$ can be exactly expressed in the 12-equipartition system. Hence, the sun's location at the summer solstice can be represented as the $(25\frac{3}{4})^{\text{th}}$ *du* or the (25 and *tai* 太)$^{\text{th}}$ *du* of *Jing*[22]. Being rounded down, it becomes the 25$^{\text{th}}$ *du* of *Jing*[22], as is recorded in *Songshu* and *Jinshu*.

On the other hand, the bisect of $\frac{5}{12}$ *du* cannot be expressed in terms of the 12-equipartition system. In other words, $\frac{5}{24}$ cannot be decomposed into fractional numbers $\frac{x}{4} + \frac{y}{12}$, where $x = 0, 1, 2, 3$ and $y = 0, \pm 1$. Thus, considering a fact that $0 < \frac{5}{24} < \frac{1}{4}$, a solution $x = 1, y = -1$ is found to minimize $|(\frac{x}{4} + \frac{y}{12}) - \frac{5}{24}|$. Then, $182\frac{5}{12}$ *du* is divided into either $91\frac{1}{6}$ *du* or $91\frac{1}{4}$ *du*. Therefore, the sun's location at the winter solstice can be obtained by adding either $91\frac{1}{6}$ *du* or $91\frac{1}{4}$ *du* to the sun's position at the autumnal equinox of the $(5\frac{1}{6})^{\text{th}}$ *du* of *Jiao*[1]. Since the sum of lodge spans for the seven lunar lodges belonging to the east Azure Dragon (東方青龍) is 75 *du* (*Jiao* 12 + *Kang* 9 + *Di* 15 + *Fang* 5 + *Xin* 5 + *Wei* 18 + *Ji* 11 = 75 *du*), the sun's locations at the winter solstice is obtained to be $x = (21\frac{1}{3})^{\text{th}}$ *du* or $x = (21\frac{5}{12})^{\text{th}}$ *du* by solving equations $[75 - 5\frac{1}{6}] + [Dou\ x] = 91\frac{1}{6}$ or $[75 - 5\frac{1}{6}] + [Dou\ x] = 91\frac{1}{4}$. In other words, the sun's location at the winter solstice should be either the (21 and *shao qiang* 少強)$^{\text{th}}$ *du* or the (21 and *ban ruo* 半弱)$^{\text{th}}$ *du* of *Dou*[8]. Being rounded down, they will be commonly the 21$^{\text{th}}$ *du* of *Dou*[8], which agrees with the records in *Songshu* and *Jinshu*. These results in the C-Map cases are shown in Table 1 as C-Map(1) and C-Map(2).

However, although one *zhou tian* is quadrisected as equally as possible, the angular intervals between solstices and equinoxes in the inscription of the C-Map are not uniform, which violates the mean motion rule. This disagreement stems from the choice of the 12-equipartition system. However, although the *Jingchuli* system adopted the 12-equipartition system, this problem does not matter as much. Here, it must be an interesting fact that there is only one Chinese letter difference between the sun's location at the autumnal equinox in *Songshu*/*Jinshu* and that in *Jingchuli*; that is, *wu du ruo* (五度弱) vs *wu du shao ruo* (五度少弱). An extra Chinese letter *shao* (少) might have been erroneously inserted in the sun's location value at the winter solstice written in *Songshu* and *Jinshu*.

According to the sentences at the beginning part of Section 2, *Wang Fan* (王蕃) conveyed *Qianxiangli* (乾象曆), and he developed an armillary sphere based on the information in it. *Qianxiangli* was developed by *Liu Hong* (劉洪) in the Latter *Han* dynasty and promulgated in 223 CE in the *Wu* (吳) dynasty of the Three Kingdoms period in Chinese history. The details of *Qianxiangli* are extant in *Lulizhi* of *Jinshu*. Interestingly, it is found in the section of 'Predicting the sun's location (推日度)'[17] that the sun's location at the winter solstice was defined in *Qianxiangli* as '5 *du* before the lunar lodge of *Niu*'. According to the archaic lodge spans, the lodge span of *Dou*[8]

[17]"To predict the *du* of the sun: Multiply accumulated days (積日) by the Era Factor (紀法 or 589). Remove what fills the Circuits of Heaven (周天 or 215,130). From what remains cast out Era Factor, and the [number of these] obtained is *du*. Count off the *du* starting from '5 *du* before *Niu*' (牛前五度), and cast out the successive lodges. What does not fill a lodge is where the sun is located at midnight at the Celestial Standard Conjunction (天正朔)." (The translation is given on Page 255 of Cullen (2017).)

Table 1. The sun's locations at equinoxes and solstices and their angular spans for several calendrical systems used before the Northern and Southern dynasties. In *Jingchuli* (景初曆), the sun's location at the winter solstice is defined as the $(21\frac{1}{4})^{\text{th}}$ *du* of the 8^{th} lunar lodge *Dou* (斗). Here all lodge angles are ordinal numers. *Songshu* (宋書) and *Jinshu* (晉書) recorded the sun's location at equinoxes and solstices, which were adopted from *Qianxiangli* (乾象曆) by *Wang Fan* (王蕃, 228–266 CE) to make an armillary sphere. *Qianxiangli* has the same value for the sun's location at the winter solstice. The sun's locations at solstices in *Songshu* and *Jinshu* were rounded down. Since the mean motion rule was used in these calendars, the spans must be the same as much as possible. Hence, the sun's location at the autumnal equinox given in *Songshu/Jinshu* needs to be corrected to *wu du ruo* (五度弱) of the 1^{st} lunar lodge *Jiao* (角). The C-Map inscriptions inherited only two lodge angles for equinoxes from *Songshu* and *Jinshu*, so their lodge angles for solstices are estimated assuming the mean motion rule and shown in parentheses.

Calendar	w.solstice	span	v.equinox	span	s.solstice	span	a.equinox	span
Santongli	*Niu* 1	$\leftarrow 91 \rightarrow$	*Lou* 4	$\leftarrow 91 \rightarrow$	*Jing* 31	$\leftarrow 91 \rightarrow$	*Jiao* 10	$\leftarrow 92\frac{1}{4} \rightarrow$
Sifenli	*Dou* $21\frac{8}{32}$	$\leftarrow 91\frac{5}{16} \rightarrow$	*Kui* $14\frac{10}{32}$	$\leftarrow 91\frac{5}{16} \rightarrow$	*Jing* $25\frac{20}{32}$	$\leftarrow 91\frac{5}{16} \rightarrow$	*Jiao* $4\frac{30}{32}$	$\leftarrow 91\frac{5}{16} \rightarrow$
Jingchuli	*Dou* $21\frac{1}{4}$	$\leftarrow 91\frac{1}{3} \rightarrow$	*Kui* $14\frac{1}{3}$	$\leftarrow 91\frac{1}{4} \rightarrow$	*Jing* $25\frac{7}{12}$	$\leftarrow 91\frac{1}{3} \rightarrow$	*Jiao* $4\frac{11}{12}$	$\leftarrow 91\frac{1}{3} \rightarrow$
Song/Jin	*Dou* 21	$\leftarrow 91\frac{7}{12} \rightarrow$	*Kui* $14\frac{1}{3}$	$\leftarrow 90\frac{2}{3} \rightarrow$	*Jing* 25	$\leftarrow 92\frac{1}{6} \rightarrow$	*Jiao* $5\frac{1}{6}$	$\leftarrow 90\frac{5}{6} \rightarrow$
C-Map(1)	[*Dou* $21\frac{1}{3}$]	$\leftarrow 91\frac{1}{4} \rightarrow$	*Kui* $14\frac{1}{3}$	$\leftarrow 91\frac{5}{12} \rightarrow$	[*Jing* $25\frac{3}{4}$]	$\leftarrow 91\frac{5}{12} \rightarrow$	*Jiao* $5\frac{1}{6}$	$\leftarrow 91\frac{1}{6} \rightarrow$
C-Map(2)	[*Dou* $21\frac{5}{12}$]	$\leftarrow 91\frac{1}{6} \rightarrow$	*Kui* $14\frac{1}{3}$	$\leftarrow 91\frac{5}{12} \rightarrow$	[*Jing* $25\frac{3}{4}$]	$\leftarrow 91\frac{5}{12} \rightarrow$	*Jiao* $5\frac{1}{6}$	$\leftarrow 91\frac{1}{4} \rightarrow$

is $26\frac{1}{4}$ *du*, so the sun's location at the winter solstice should be the $(21\frac{1}{4})^{\text{th}}$ *du* of *Dou*[8] in *Qianxiangli*.

Since it is said that *Wang Fan* made an armillary sphere based on *Qianxiangli*, he probably adopted the lodge angle of the $(21\frac{1}{4})^{\text{th}}$ of *Dou*[8] as the sun's location at the winter solstice. Hence, the sun's locations at the two solstices in *Songshu* and *Jinshu* must be the rounded-down values of those in *Jingchuli*. Nevertheless, the sun's location at the vernal equinox in *Songshu* and *Jinshu* is the same as that in *Qianxiangli*. Therefore, adopting the mean motion rule, the locations at the autumnal equinox and the summer solstice in *Songshu* and *Jinshu* must have had the same values as those in *Jingchuli*. However, only one character difference exists between the sun's locations at the autumnal equinox. Thus, it is highly probable that one character of *shao* (少) was inserted incorrectly in the sun's location at the autumnal equinox in *Songshu* and *Jinshu*. As a result of the above reasoning, *Wang Fan*'s value of the sun's location at the winter solstice recorded in *Songshu* and *Jinshu* agrees with those in *Qianxiangli*, *Jingchuli*, and *Houhan Sifenli*. Therefore, all of them can be regarded as the *Houhan Sifenli* series.

Conclusively, the sun's location at the winter solstice in *Qianxiangli*, *Songshu/Jinshu*, *Jingchuli*, *Houhan Sifenli*, and the C-Map inscription must be the $(21\frac{1}{4})^{\text{th}}$ *du* of *Dou*[8] in common. In addition, the sun's location at the autumnal equinox in the inscription of the C-Map, as well as those in *Songshu/Jinshu* should be corrected from *wu du shao ruo* (五度少弱) of *Jiao*[1] to *wu du ruo* (五度弱) of *Jiao*[1] and its missing values for the solstices in the inscription of the C-Map must have been *er shi yi du shao* (二十一度少 or the $(21\frac{1}{4})^{\text{th}}$ *du*) of *Dou*[8] and *er shi wu du ban qiang* (二十五度半強 or the $(25\frac{7}{12})^{\text{th}}$ *du*) of *Jing*[22], respectively.

5. Obsetvational Epochs

Park (1995) determined the reference epoch for the locations of equinoxes in the inscription of the C-Map. He implicitly presumed that the lodge angles in the inscription were given as the cardinal numbers. Hence, he thought that the sun was located at $14\frac{1}{3}$ *du* of *Kui*[15] on the vernal equinox and the sun was located at $5\frac{1}{6}$ *du* of *Jiao*[1] on the autumnal equinox. He obtained the observational epoch of the sun's locations to be 50 BCE. Following his calculation method, his results are checked below. The determinative star of *Kui*[23] is ζ And, and the determinative star of *Jiao*[1] is α Vir. The right ascensions of the vernal and autumnal equinoxes are denoted as $\alpha_{\zeta \text{ And}}$ and $\alpha_{\alpha \text{ Vir}}$, respectively. Then, according to Park (1995), the right ascensions at the reference epoch should satisfy the following equations: $\alpha_{\zeta \text{ And}} + 14\frac{1}{3}\ du = 0^{\text{h}}$ and $\alpha_{\alpha \text{ Vir}} + 5\frac{1}{6}\ du = 12^{\text{h}}$. Solving these equations gives us solutions $\alpha_{\zeta \text{ And}} = 23^{\text{h}}\,03^{\text{m}}\,29^{\text{s}}$ and $\alpha_{\alpha \text{ Vir}} = 11^{\text{h}}\,39^{\text{m}}\,38^{\text{s}}$. The PC planetarium software Stellarium is used to determine epochs at which the two stars had these right ascensions. The observational epoch the vernal equinox is obtained to be 25 February 45 BCE ($t_{\text{o}} = -43.8$) and that for the autumnal equinox is obtained to be 9 September 54 BCE ($t_{\text{o}} = -52.3$). Their mean epoch agrees with Park (1995).

Contrary to the assumption, the lodge angles in the inscription were given in the ordinal numbers. Hence, his results may have an error of approximately 80 years. Moreover, his results may be considered unreliable due to the following errors. In ancient Chinese calendrical systems, which did not adopt the precession of equinoxes, the 24 solar terms (二十四氣 or *er shi si qi*) are equally elapsed a little more than 15 days apart with reference to the winter solstice, assuming that the sun moves one *du* a day (Cullen 2017). On estimating the observational epoch of C-Map, the sun's location at the winter solstice is the primary element rather than equinoxes

Table 2. The epochs determined from the sun's locations at equinoxes and solstices. The observational epochs are obtained with reference to the determinative stars. The lodge angles are given in ordinal numbers. Here, the uncertainties in the epochs are estimated to be 43.0 years from the measurement errors of archaic determinative stars (Ahn 2020, 2023). Exceptionally, the uncertainty for the solstices in *Songshu* and *Jinshu* are estimated from the truncation errors because those are rounded-down values.

Calendars	W. solstice	Epoch	V. equinox	Epoch	S. solstice	Epoch	A. equinox	Epoch
Santongli	*Niu* 1	−449.4	*Lou* 4	−405.7	*Jing* 31	−368.4	*Jiao* 10	−354.5
Sifenli	*Dou* $21\frac{8}{32}$	−18.3	*Kui* $14\frac{10}{32}$	+38.0	*Jing* $25\frac{20}{32}$	−3.7	*Jiao* $4\frac{30}{32}$	+40.3
Jingchuli	*Dou* $21\frac{1}{4}$	−18.3	*Kui* $14\frac{1}{3}$	+36.3	*Jing* $25\frac{7}{12}$	−0.8	*Jiao* $4\frac{11}{12}$	+41.9
Song/*Jin*	*Dou* 21	−31.7 ± 39	*Kui* $14\frac{1}{3}$	+36.3	*Jing* 25	+4.9 ± 39	*Jiao* $5\frac{1}{6}$	+22.2
C-Map(1)	*Dou* $21\frac{1}{3}$	−21.0	*Kui* $14\frac{1}{3}$	+36.3	*Jing* $25\frac{3}{4}$	−12.2	*Jiao* $5\frac{1}{6}$	+22.2
C-Map(2)	*Dou* $21\frac{5}{12}$	−26.4	*Kui* $14\frac{1}{3}$	+36.3	*Jing* $25\frac{3}{4}$	−12.2	*Jiao* $5\frac{1}{6}$	+22.2

because sun's location at the winter solstice is the reference direction for azimuthal angle in the circular-chart coordinate system. Moreover, although a lodge angle represents the sun's location at a solar term, the lodge angle is not measured with reference to the determinative star. Still, it is calculated by considering the lodge spans of the lunar lodges between the winter solstice and the solar term, which has been shown in Equation (3) of Section 2 in this paper. Therefore, if the lodge angles of the solar terms are converted to the right ascensions only referring to the determinative stars, which has also been shown in Equation (1) of Section 2, the stars' positional error will be added to the results.

An ancient star catalogue called SSXJ (石氏星經) lists the coordinates of about 120 stars that is thought to have been observed during the Former *Han* dynasty. The catalogue of determinative stars inscribed on the C-Map is also thought to have originated from the Former *Han* dynasty. From these positional data, we can estimate their measurement errors. The measurement errors for the determinative stars in the C-Map catalogue are approximately 0.55° in lodge spans and 0.6° in polar distances (Ahn 2023), and the measurement error for the determinative stars in SSXJ is approximately 0.53° in polar distances (Ahn 2020).

For an object having a coordinate (α, δ) in the equatorial system, the rates of changes in its right ascension and declination due to the precession of equinoxes (Smart 1965) are given as

$$\dot{\alpha} = 46.1'' + 19.9'' \sin\alpha \tan\delta \text{ per year} \tag{4}$$

and

$$\dot{\delta} = 19.9'' \cos\alpha \text{ per year.} \tag{5}$$

Here, the rate of change of the ecliptic longitude due to precession is adopted $\dot{\lambda} = 50.2''$ per year and the obliquity of the ecliptic is adopted $\epsilon = 23.4°$.

In this paper, considering carefully these facts, the observational epoch is determined from the sun's location at the winter solstice. In *Santongli*, the sun's location at the winter solstice is given as *Qian niu chu du* (牽牛初度), which means the 1st *du* of *Niu*[9] or zero *du* of *Niu*[9]. The determinative star of *Niu*[9] is β Cap. Thus, at the observational epoch, the right ascension of β Cap should be 18^h. Using the PC planetarium software Stellarium, the epoch is determined to be $t_o = -449.4$ or August 451 BCE. This result is comparable with the result obtained by Pan (2009), who stated on Page 37 of his book that the sun's location of *Qian niu chu du* corresponds to 450 BCE.

In 85 CE, *Santongli* was replaced with *Houhan Sifenli*. *Jingchuli* began to be used from 237 CE in the *Wei* (魏) Kingdom at the end of the Latter *Han* dynasty. In *Jingchuli*, the sun's location at the winter solstice is given as the $(21\frac{1}{4})^{\text{th}}$ *du* of *Dou*[8]. This value had been succeeded by *Qianxiangli*, *Wang Fan* (*Songshu* and *Jinshu*), and the inscription in the C-Map.

Here, the $(21\frac{1}{4})^{\text{th}}$ *du* is equal to $20\frac{1}{4}$ *du*. The angle of $20\frac{1}{4}$ *du* is equivalent to $\Delta_\alpha \equiv 1^h\,19^m\,50^s$ in right ascension. The determinative star of *Dou* is identified with ϕ Sgr (HIP 92041). At the observational epoch, the right ascension of ϕ Sgr, denoted by $\alpha_{\phi Sgr}$, should satisfy the equation $\alpha_{\phi\ \mathrm{Sgr}} + \Delta_\alpha = 18^h$. Hence, the right ascension of ϕ Sgr should be $\alpha_{\phi \mathrm{Sgr}} = 16^h\,40^m\,10^s$ at the observational epoch. Using the PC planetarium software Stellarium, the observational epoch is determined to be $t_o = -18.3$ or August 20 BCE.

The reading error for *Houhan Sifenli*, adopting the 32-equipartition system, is thought to be $\frac{1}{64}$ *du*. The reading error for *Jingchuli* and its descendants, adopting the 12-equipartition system, is thought to be $\frac{1}{12}$ *du*. However, they are related to calculations of 24 solar terms rather than actual measurements. On the other hand, the measurement error in lodge spans (or right ascensions) for the 28 determinative stars in the C-Map catalogue is estimated to be approximately 0.55° (Ahn 2023). The measurement errors in polar distances (or declinations) for the stars in SSXJ or the C-Map catalogue are comparable to this value. Equation (4) shows that the precession rate of right ascension for the winter solstice along the equator is 46.1″ per year. So, the uncertainty in the observational epoch is estimated to be 43.0 years, and the observational epoch $t = -18.3 \pm 43.0$.

For other cases, the observational epochs are also calculated from the sun's locations at the winter solstice. Since the values in *Songshu* and *Jinshu* are rounded-down ones, their position can be estimated to be the $(21.5 \pm 0.5)^{\text{th}}$ *du* of *Dou*[8]. So, the observational epoch and uncertainty are obtained to be $t_o = -31.7 \pm 39.0$. For the case of the inscription on the

C-Map, the sun's locations at the solstices are estimated from those values at the equinoxes, presuming the mean motion rule: $t_o = -21.0$ and $t_o = -26.4$ for the allowed two values. All these results are shown in Table 2.

From the values for the summer solstice and the vernal/autumnal equinoxes, the reference epochs are obtained by calculating their right ascensions with reference to their determinative stars. Here, the determinative stars for *Kui*, *Jing*, and *Jiao* are ζ And, μ Gem, and α Vir, respectively. These results are shown in Table 2 for reference purposes.

Now, it will be demonstrated how significant additional errors will be if the epoch is determined solely from lodge angles with reference to the determinative stars. At the observational epoch $t_o = -18.3$ or 21 September 20 BCE, the right ascension of winter solstice should be $\alpha_{WS} = 18^h$. The angular span from the winter solstice to the vernal equinox is $91\frac{1}{3}$ *du* or $6^h\,0^m\,5^s$. Thus, the right ascension of the vernal equinox is $\alpha_{VE} = 24^h\,0^m\,5^s$.

On the other hand, the observational epoch is determined by analyzing the lodge angle with reference to the determinative star of *Kui*. The determinative star of *Kui* is ζ And, and its right ascension on 21 September 20 BCE (or $t_o = -18.3$) was $\alpha_{\zeta\ \mathrm{And}} = 23^h\,04^m\,45^s$. The lodge angle is $13\frac{1}{3}$ *du*, which is equivalent to $\Delta_\alpha = 0^h\,52^m\,34^s$. Thus, the right ascension of the vernal equinox is $\alpha_{VE} = \alpha_{\zeta\ \mathrm{And}} + \Delta_\alpha = 23^h\,57^m\,19^s$.

Comparing these two values, a difference of $02^m\,46^s$ or $2,490''$ is obtained. Since $\alpha = 0^h$ at the vernal equinox and $\delta = 0°$ along the equator, the rate of change of right ascensions along the equator due to the precession is $46.1''$ from Equation (4). Hence, the difference in right ascensions is equivalent to 54.0 years in the observational epoch. This fact can be confirmed in Table 2 because there is a 54.6-year difference between the epochs for the winter solstice and the vernal equinox for the case of *Jingchuli*. Therefore, if the observational epoch of a star is estimated by analyzing its lodge angle from its determinative star, then the observational epoch will have an additional error associated with the positional error of the determinative star.

6. Conclusions

6.1. Summary

It is confirmed that the sun's location at equinoxes, which are inscribed on the *Cheonsang Yeolcha Bunyajido* (天象列次分野之圖), were inherited from the astronomical treatises in the *Songshu* (宋書, History of the *Liu Song* dynasty) and *Jinshu* (晉書, History of the *Jin* dynasty). According to the treatises, while manufacturing an armillary sphere, *Wang Fan* (王蕃, 228–266 CE) referred to the sun's locations in *Qianxiangli* (乾象曆), a calendrical system used between 223 CE and 280 CE. The sun's location at the winter solstice in *Qianxiangli* was given as the $(21\frac{1}{4})^{\mathrm{th}}$ *du* of *Dou*[8], which was also shared with *Houhan Sifenli* (後漢四分曆, used from 85 CE to 237 CE) and *Jingchuli* (景初曆, used from 237 CE to 444 CE). Hence, it is argued that the sun's locations at solstices, which are missing in the inscription of *Cheonsang Yeolcha Bunyajido* and given as rounded-down values in *Songshu* and *Jinshu*, must have had the same value as those in *Jingchuli*, because they commonly adopted the 12-equipartition system of fractions, the mean motion rule (平氣法, the *ping qi* rule), and the common value of the sun's location at the winter solstice. As such, it is proposed that the sun's location at the autumnal equinox on the C-Map, as well as that in *Songshu* and *Jinshu*, should be corrected from *wu du shao ruo* (五度少弱) to *wu du ruo* (五度弱) of *Jiao*[1], following the value in *Jingchuli*.

The projection method of ancient circular star charts such as *Cheonsang Yeolcha Bunyajido* is a sort of polar projection. In this paper, we reconstruct the projection in detail by considering the following facts. (1) According to *Zu Chonzhi* (祖冲之), the *Zi* direction (子方), meaning the north, should be aligned with the lunar lodge of *Xu*[11], which corresponds to the equatorial sector *Xuanxiao* (玄枵). (2) According to the drawing method of a circular star chart in *Xin Tangshu* (新唐書), the sun's location at the winter solstice should be aligned with the center of a specific equatorial sector, which should be the *Xingji* (星紀) sector as it is the beginning of 12 equatorial sectors. (3) Since *Taichuli* (太初曆), which was promulgated in 104 BCE, the month having the winter solstice has been defined as the 11th month, so the first month begins with the solar term of *lichun* (立春) if the mean motion rule is adopted. (4) The 24 solar terms are marked clockwise on the circular star chart along the sun's apparent motion. (5) The 12 equatorial sectors, also called '*ci* (次) sectors, are defined clockwise beginning from the *Xingji* (星紀) sector. In contrast, the 12 *chen* (辰) sectors are defined counterclockwise beginning from the *Zi* (子) sector, whose center should be aligned with the due north. Figure 1 shows the reconstructed polar coordinate used in the circular star chart.

Notably, the sun's location at the winter solstice was a reference for azimuthal angles in the polar projection. Since ancient calendars, not adopting the precession, use the mean motion rule, solar terms can be marked by successively adding a constant increment of $15\frac{7}{32}$ *du*. The direction of the winter solstice is only useful in estimating the observational epoch, while those for the other solar terms are not useful to determine the observational epoch. In addition, converting lodge angles into right ascensions with reference to the determinative stars introduces an additional error in the observational epoch, except for the case of the winter solstice.

The observational epoch is determined to be $t_o = -18.3$ by analyzing the sun's location at the winter solstice or the $(21\frac{1}{4})^{\mathrm{th}}$ *du* of *Dou*[8], considering carefully the fact that the lodge angle was given in an ordinal number. Its uncertainty has been estimated to be 43.0 years from the measurement error of the archaic determinative stars (Ahn 2020, 2023).

6.2. Statistical Tests

The observational epoch estimated here, $t_o = -18.3 \pm 43.0$, indicates that the sun's location was measured either during the latter period of the Former *Han* dynasty (前漢, 206 BCE–8 CE) or during the early period of the Latter *Han* dynasty (後漢, 25–220 CE). Fortunately, the positions of 121 stars

Table 3. Results of hypothesis tests if the sun's location at the winter solstice was measured during the Former *Han* dynasty. We compare our result with the epochs of stars listed in *Shi Shi Xing Jing* (石氏星經) (Ahn 2020).

Groups	Epoch ($\langle t_2 \rangle$)	Epoch ($\langle t_1 \rangle$)	Z-score	p-values
Results for SSXJ (石氏星經) referred to Table 5 of Ahn (2020):				
Group A	-200 ± 45	-18.3 ± 43.0	2.919	.0035
Main group	-107 ± 16	-18.3 ± 43.0	1.933	.053
Determinatives25	-58.8 ± 26.9	-18.3 ± 43.0	0.798	.425
Group B	100 ± 35	-18.3 ± 43.0	2.134	.033

Table 4. Results of hypothesis tests if the sun's location at the winter solstice was measured at a historical occasion. The location was measured during the latter period of the Former *Han* dynasty.

Occasions	t_o	t_i	σ_c	Z	p-values	Possibly related occasions
202 BCE	−18.3	−201	43.0	4.249	.00002	Former *Han* establishment
104 BCE	−18.3	−103	43.0	1.970	.049	*Tang Du*(唐都), *Taichuli*(太初曆)
[78, 76] BCE	−18.3	−76	43.0	1.342	.180	*Xianyu Wangren*(鮮于妄人)'s observations
52 BCE	−18.3	−51	43.0	0.760	.447	*Geng Shouchang*(耿壽昌)'s circular instrument
85 CE	−18.3	85	43.0	2.402	.016	*Houhan Sifenli* (後漢 四分曆)
92 CE	−18.3	92	43.0	2.565	.010	*Jia Kui*(賈逵)'s memorial
103 CE	−18.3	103	43.0	2.821	.005	Observations with ecliptic armillary

are preserved in books such as *Kai Yuan Zan Jing* (開元占經) published during the *Kaiyuan* reign-period (718–726 CE). The star catalogue was originally contained in a book entitled *Shi Shi Xing Jing* (石氏星經, hereafter, SSXJ in abbreviation), where *Shi Shi* (石氏) was an astronomer of the 4th century BCE in the Warring States period. Ahn (2020) analyzed them to find that the stars in the catalogue are composed of four groups of stars: Group A of 8 stars seems to have been observed at $t_o = -200 \pm 45$, the main group of 63 stars seem to have been observed at $t_o = -107 \pm 16$, the determinative stars of 25 stars seem to have been observed at $t_o = -64 \pm 28$, and finally the group B of 13 stars seem to have been observed at $t_o = 100 \pm 35$.

In this paper, a check will be made if the sun's location at the winter solstice was measured simultaneously with any of these groups in SSXJ. There is insufficient statistical information on the measured values. Some groups of stars in SSXJ have insufficient numbers of stars. Despite these limitations, a two-sample z-test will be performed.

Suppose that μ_1 denotes the population mean of Sample 1 and μ_2 denotes that of Sample 2. Then, a null hypothesis $H_0 : \mu_1 = \mu_2$ and the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ are set. In this case, a two-sample two-sided z-test is performed by defining the standardized Z-statistics as

$$Z = \frac{|\langle t_1 \rangle - \langle t_2 \rangle|}{\sqrt{\sigma_1^2 + \sigma_2^2}}. \quad (6)$$

Here, $\langle t_1 \rangle$ is the sample average of Sample 1, and $\langle t_2 \rangle$ is the sample average of Sample 2. The p-values obtained for the samples are shown in Table 3. Here, the null hypothesis H_0 fails to be rejected for the case of the determinative stars in SSXJ at the significance level of 95%. This indicates that the sun's location at the winter solstice, adopted in the calendrical systems of the Latter *Han* dynasty, must have been measured simultaneously with the determinative stars in SSXJ. At least, it can be said that the location must have been measured during the latter period of the Former *Han* dynasty.

It may also be questioned which historical occasion the sun's location at the winter solstice was measured. According to the treatises on musical scales and astronomical systems (律曆志, *Lulizhi*) in *Hou Hanshu*, the locations of the determinative stars might have been measured on a few occasions during the Former and Latter *Han* dynasties: (1) the establishment of the Former *Han* dynasty in 202 BCE, (2) *Luoxia Hong*(落下閎)'s observations in 104 BCE, (3) *Xianyu Wangren*(鮮于妄人)'s observation in 78–76 BCE, (4) *Geng Shouchang*(耿壽昌)'s observation in 52 BCE, (5) the calendrical method called *Houhan Sifenli* began to be used in 85 CE, (6) the observations with the ecliptic armillary in 103 CE that might have been related with *Jia Kui*(賈逵)'s memorial to the throne on observations in 92 CE (Ahn 2020). The times for those historical occasions are denoted by t_i and regarded as the population means: $t_1 = -201$, $t_2 = -103$, $t_3 = [-77, -75]$, $t_4 = -51$, $t_5 = 85$, $t_6 = 92$ and $t_7 = 103$.

The observational epoch and its uncertainty of the sun's location at the winter solstice are determined to be $t_o \pm \sigma_o = -18.3 \pm 43.0$ in this paper. Then, z-tests are performed. The null hypothesis $H_0 : t_o = t_i$ and the alternative hypothesis $H_a : t_o \neq t_i$ are set. The single sample Z-score statistic is calculated as

$$Z = \frac{|t_o - t_i|}{\sigma_o}. \quad (7)$$

Then, the two-sided p-values are evaluated for the Z-scores.

The results of the hypothesis tests are shown in Table 4.

The null hypothesis H_0 fails to be rejected for the occasions of [78, 76] BCE and 52 BCE at the significance level of 95%. The observational epoch t_o has the 95% confidence interval $[-102.6, 66.0]$ or [104 BCE, 66 BCE]. Therefore, it is concluded that the sun's location at the winter solstice, which was commonly adopted in *Houhan Sifenli*, *Jinchuli*, and *Qianxiangli*, was observed on an occasion that happened during the latter period of the Former *Han* dynasty.

Last of all, it seems interesting to revisit the sentences given by *Zu Chongzhi* in *Songshu*. Their translations are shown in the Appendix section. According to him, during the reign period of the emperor *Wudi* (武帝) of the Former *Han* dynasty, the calendrical system was reformed to *Taichuli* (太初曆), when the sun was located at the 1st *du* (牛初 *niu chu*) or zero *du* of the lunar lodge *Niu*[9]. Interestingly, these words can be directly confirmed on the star chart of *Cheonsang Yeolcha Bunyajido*. The direction to the sun at the winter solstice was set to be *niu chu*.

According to *Zu Chongzhi*, the sun was located at the 21st *du* of Dou (斗二十一) in *Houhan Sifenli*. This value can be derived from the sun's positional values at equinoxes that are inscribed on *Cheonsang Yeolcha Bunyajido*, and it has been proved in this paper that its origin can be traced back to *Houhan Sifenli*. *Houhan Sifenli* was promulgated in 85 CE during the Latter *Han* dynasty. Therefore, in conclusion, the inscription of the C-Map is related to *Houhan Sifenli* of the Latter *Han* dynasty, while the star chart of the C-Map is related to *Taichuli* of the Former *Han* dynasty.

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Appendix A. *Zu Chongzhi*(祖沖之)'s Memorial on the Drift of the Winter Solstice

What has changed in the calendar: Firstly, there were seven intercalary months in 19 years or one cycle of Rule Years (章) in old methods. · · · Secondly, according to the *Yao* chapters (堯傳) in *Shujing* (書經), "The lunar lodge *Mao* (昴, the Pleiades) was culminating when the length of a day was shortest, so the lunar lodge became the culminating star of thjiang jie winter solstice." From this sentence, we see that the sun was located at the position left (east) from its present location by approximately 50 *du* during the era of *Yao* (堯). The early *Han* dynasty succeeded the calendar system of the *Qin* dynasty, in which the sun was located at the 6th *du* of *Qian Niu* (牽牛) at the winter solstice. The emperor *Han Wudi* (武帝) reformed the calendar into *Taichuli* (太初曆), in which the sun was located at *chu du* (初度, the first *du*) of the lunar lodge *Qian Niu* (牽牛). In the *Sifenli* of the Latter *Han* dynasty (後漢 四分曆), the sun was located at the 11th *du* of *Dou* (斗) at the winter solstice. During the *Jin* (晉) dynasty, by analyzing lunar eclipses, *Jiang Ji* (姜岌, fl.384 CE) estimated the sun's location to be the 17th *du* of *Dou* at the winter solstice. Presently, the lunar lodge of *Shen* (參, Orion's belt) becomes the culminating star (at the winter solstice), and from our analyses on lunar eclipses, we get to know that the sun's location at the winter solstice is the 11th *du* of *Dou* (斗). In conclusion, a difference of 2 *du* has occurred within less than one hundred years. [*Yuan jia li fa*, *Lulizhi*, *Songshu*]

改易者: 其一, 以舊法一章十九歲有七閏, · · · . 其二, 以堯典云: "日短星昴, 以正仲冬," 以此推之, 唐代冬至, 日在今宿之左五十許度. 漢代之初, 用秦曆, 冬至日在牽牛六度. 漢武改立太初曆, 冬至日在牛初. 後漢四分法, 冬至日在斗二十一. 晉時姜岌以月蝕檢日, 知冬至在斗十七. 今參以中星, 課以蝕望, 冬至之日, 在斗十一. 通而計之, 未盈百載, 所差二度. [元嘉曆法, 律曆志, 宋書]