



Dating the Stars in the Calendrical Method *Shoushili* of the Yuan Dynasty

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Abstract

Shoushili was the official calendrical method promulgated in 1280 CE by the Yuan dynasty. It contains a list of the angular spans in right ascensions for the 28 lunar lodges. They are known to have been measured by *Guo Shoujing* with his advanced instruments with an unprecedented precision or reading error of 5'. Such precise data are useful to determine their observational epoch with an error range which is narrow enough to pinpoint on which historical occasion they were observed. Using the precise SIMBAD data based on eDR3 of GAIA and carefully identified determinative stars and considering the precession of equinoxes and proper motions, we apply linear regression methods to those data and obtain the observational epoch of 1271 ± 16 CE and the measurement error of 4.1'. We also have polar distances corresponding to declinations written in another manuscript of the Ming dynasty. Since the two data sets have similar significant digits, they were suggested to have the same origin. However, we obtain their observational epoch of 1364 ± 5 CE and the measurement error of 5.7'. They must have been measured with different instruments and on a different occasion from the observations related to *Shoushili*. We review the history of the calendrical reform during the 13th century in the Yuan dynasty. We conclude that the observational epoch obtained from lodge spans in *Shoushili* agrees with the period of observations led by *Guo Shoujing* or 1276–1279 CE, which is also supported by the fact that the ecliptic lodge span values listed in *Shoushili* were calculated from the equatorial lodge spans.

Keywords: history and philosophy of astronomy — astrometry — catalogues — methods: statistical

1. Introduction

Guo Shoujing was an astronomer working in the court of the Yuan dynasty during the 13th century. Later in the 17th century, he was called ‘the Tycho Brahe of China’ by Adam Schall von Bell, who was a Jesuit astronomer working in the Court of the Qing dynasty (Engelfriet 1998). According to the Biography in the History of the Yuan dynasty (hereafter *Yuanshi*), he began to engage the calendrical reform in 1276 CE. He engineered new advanced instruments and led the observations to reform the official calendrical method, promulgated in 1280 CE as a title of *Shoushili*. He began to edit the formulae and tables from 1282 CE and published them into books in 1286 CE.

Throughout the calendar reform, he made astronomical observations to improve the precision of astronomical quantities, such as winter solstices, a tropical year, a sidereal year, and the obliquity of the Earth’s axis to the ecliptic plane. One of his achievements is the accurate measurements of the lodge spans of 28 lunar lodges. These data remain in the Calendri-

cal Treatises of *Yuanshi* and also in its Korean revised version titled *Chiljeongsan Naepyeon* meaning the Chinese part of the method to calculate the seven celestial objects. The Korean version was developed during the reign of King *Sejong* and promulgated in 1444 CE. The lodge spans are also written on the first page of the manuscript book written in the early Ming dynasty, titled *San Yuan Lie She Ru Xiu Qu Ji Ji* or ‘Lodge and Polar Distances for the Three Prefectures and Lunar Lodges’ (hereafter QJJ in abbreviation). Moreover, he measured the locations of unnamed and traditional Chinese stars and wrote books which are not extant. QJJ was once regarded as a copied version of his books (Chen 1986; Pan 1989, 2009), which is recently negated by Sun (1996) and Cao (2019). Therefore, the lodge spans left in *Shoushili* seem to be the only remaining data observed by *Guo Shoujing*.

Previous researchers have estimated the observational epoch of the lodge spans. Pan (1989, 2009) regarded the year of 1280 CE, when *Shoushili* was promulgated, as the observational epoch. Recently Nakamura (2018) and Takesako (2018) analysed the data and obtained the observational epoch

of 1277 ± 30 CE and 1272 ± 15 CE, respectively. According to historical chronicles, the calendar reform began in 1276, and the intensive observations were made by *Guo Shoujing* from the winter of 1276 to the year 1279 CE. Hence, the observational epochs estimated by Nakamura (2018) and Takesako (2018) are suggested to agree with the period of *Guo Shoujing*'s observations.

However, we can now use a more accurate star catalogue based on astrometric observations of mill-arcsecond accuracy. The proper motions of the determinative stars reach as precise as $0.1''\text{yr}^{-1}$, which may result in a difference of a couple of years in the observational epoch. Even more important is identifying determinative stars, reference stars of lunar lodges. Some determinative stars have been changed, which will significantly affect determining the epoch and its uncertainty. Thus, considering these aspects, we can improve the estimation of the observational epoch and its uncertainty.

However, the lodge spans are differential quantities because they are defined as the difference in the right ascensions of two neighbouring determinative stars. They are a slowly-varying function of time, and so they cause significant uncertainties in estimating the observational epoch. Recently Ahn (2023) has developed a linear regression method and applied it to the lodge spans inscribed on a Korean planisphere titled *Cheonsang Yeolcha Bunyajido* to obtain the uncertainty in the observational epoch of approximately 100 years and the measurement error of 0.4° . On the other hand, *Guo Shoujing*'s instruments are known to have had a reading error of 0.05° (Pan 1989, 2009), which was an unprecedented precision at the time. Roughly speaking, the lodge spans in *Shoushili* are eight times more precise than those in *Cheonsang Yeolcha Bunyajido*. Since the uncertainty σ_t is proportional to the measurement error (Ahn 2023), the observational epoch estimated from the lodge spans in *Shoushili* will have uncertainties eight times smaller or 13 years. This will help us to verify on which historical occasion the lodge spans were measured.

Section 2.1 describes the observed coordinate values and their identifications. Section 2.2 describes the analysis method. In Section 2.3, we apply the analysis method to the lodge spans in *Shoushili* to obtain the observational epochs and their uncertainties. In Section 2.4, we apply the analysis method to the polar distances in QJJ. Section 2.5 compares our results with the previous works. Then, in Section 3, we will discuss the historical occasions that may be related to the observations of the lodge spans in *Shoushili*.

2. Data Analyses

2.1. Data

Ancient Chinese astronomers used a celestial coordinate system similar to the equatorial coordinate system in modern astronomy to specify the locations of celestial objects. In the coordinate system, lodge angles correspond to right ascensions, and polar distances correspond to declinations. The lodge angle is defined as the eastward angular difference of a celestial object in right ascension with reference to the deter-

minative star of its nearest-to-the-west lunar lodge. The lodge span of the i -th lunar lodge is defined as the lodge angle of the determinative star of the i -th lunar lodge with respect to the determinative star of the $(i + 1)$ -th lunar lodges. The polar distance is the angular distance from the north celestial pole to the object. As such, the polar distance of i -th determinative star, P_i , can be converted into the declination δ_i by $\delta_i = 90^\circ - P_i$.

The lodge spans are listed in a table called 'Lodge spans along the equator (*Qi Dao Xiu Du*)' in the chapter of 'Concepts of *Shoushili* (*Shou Shi Li Yi*)' in *Yuanshi*. The table summarises the lodge spans which were used throughout Chinese histories, such as *Luo Xiaohong*'s observations in the Han dynasty, *Yixing*'s observation during the Tang dynasty, the *Huangyu*, *Yuanfeng*, and *Chongning* catalogues of the Song dynasties, and the *Zhiyuan* catalogue made by *Guo Shoujing*.

We present the lodge spans measured by *Guo Shoujing* in Table 1. We see in the table that all figures are presented up to hundredths of *du* or Chinese degree, but most of them are effective down to 0.1 *du*, and only four of them have angles down to 0.05 *du*. Therefore, the reading error must have been 0.05 *du*. We also see that the scale must have been engraved on the instruments so that 1 *du* was divided into ten equal parts. However, according to the Biography of *Guo Shoujing* in *Yuanshi*, at the completion of the calendar reform, *Guo Shoujing* sent a memorial to the throne stating that the 1 -*du* scale on his instruments was divided into 36 equal parts¹. However, *Guo Shoujing*'s instruments were demolished by the Jesuits around the end of the Ming dynasty, so we cannot verify *Guo Shoujing*'s statements. Only the simplified armilla and the celestial globe, duplicated during the Ming Dynasty, remain at the Purple Mount Observatory in Nanjing China. Scales are engraved on them such that 1 degree is divided into ten equal parts (Pan 1989, 2009).

Notably, the sum of the lodge spans is called a *zhou tian* corresponding to one sidereal year. For the case of *Shoushili*, one *zhou tian* is 365.2575 *du*. Based on this fact, we verify the correct lodge span for Xu[11] as 8.9575 *du* in *Shoushili*. We also convert Chinese degrees into Babylonian degrees by multiplying Chinese degrees by $360^\circ/365.2575$ *du*.

Polar distances in Table 1 are taken from a manuscript QJJ, which was known to have inherited the observations of *Guo Shoujing* (Pan 1989, 2009), but their observational epoch turned out to be 1375 CE (Sun 1996; Cao 2019). Although their epoch was suggested to differ from the time of *Guo*'s observations by almost one hundred years, we will analyse them as a cross-check.

The identification of stars is essential for determining their observational epoch. Although we can refer to previous identifications by researchers (Pan 1989; Sun & Kistemaker 1997; Nakamura 2018; Takesako 2018), we must be careful when

¹“In the *Daming* calendrical system, angles have fractional numbers down to a quarter *du*, but it seems that they are not observed numbers but just subjective imaginations. Presently, detailed scales are engraved on all the new instruments, and we divide every *du* into 36 equal parts. Moreover, by replacing the sighting tube with the sighting threads, we can observe the real fractional degree and do not lean on imaginations.” (*Guo Shoujing*'s Biography in *Yuanshi*)

Table 1. Lodge spans in *Shoushili* and Polar distances in *San Yuan Lie She Ru Xiu Qu Ji Ji* for the 28 lunar lodges.

No.	Lodges	Lodge Spans	Polar Distances	R.A.(J2000) [$^{\circ}$ $^{\prime}$ $^{\prime\prime}$]	Dec.(J2000) [$^{\circ}$ $^{\prime}$ $^{\prime\prime}$]	PM R.A. [mili-''yr $^{-1}$]	PM Dec. [mili-''yr $^{-1}$]	HIP Number	B	F	
1	Jiao	12.10	99.20	13 25 11.57937	-11 09 40.75010	-42.350	-30.670	65474	α	67	Vir
2	Kang	9.20	98.70	14 12 53.74538	-10 16 25.33400	7.250	139.880	69427	κ	98	Vir
3	Di	16.30	104.50	14 50 52.71309	-16 02 30.39550	-105.680	-68.400	72622	α^2	9	Lib
4	Fang	5.60	115.60	15 58 51.11324	-26 06 50.78860	-11.420	-26.830	78265	π	6	Sco
5	Xin	6.50	115.30	16 21 11.31571	-25 35 34.05148	-10.600	-16.280	80112	σ	20	Sco
6	Wei	19.10	128.20	16 51 52.22835	-38 02 50.63807	-10.451	-18.315	82514	μ^1		Sco
7	Ji	10.40	121.60	18 05 48.48810	-30 25 26.72346	-53.920	-180.900	88635	γ^2	10	Sgr
8	Dou	25.20	118.90	18 45 39.38610	-26 59 26.79444	51.610	1.220	92041	ϕ	27	Sgr
9	Niu	7.20	108	20 21 00.66660	-14 46 53.06737	44.133	0.360	100345	β^1	9	Cap
10	Nü	11.35	103	20 47 40.55260	-09 29 44.78771	33.980	-34.770	102618	ϵ	2	Aqr
11	Xu	8.9575	99.70	21 31 33.53171	-05 34 16.23201	18.770	-8.210	106278	β	22	Aqr
12	Wei	15.40	94.50	22 05 47.03593	-00 19 11.45677	18.250	-9.390	109074	α	34	Aqr
13	Shi	17.10	79.30	23 04 45.65345	+15 12 18.96170	60.400	-41.300	113963	α	54	Peg
14	Bi	8.60	79.60	00 13 14.15123	+15 11 00.93676	1.980	-9.280	1067	γ	88	Peg
15	Kui	16.60	69.90	00 47 20.32547	+24 16 01.84085	-101.170	-81.770	3693	ζ	34	And
16	Lou	11.80	73.40	01 54 38.41099	+20 48 28.91330	98.740	-110.410	8903	β	6	Ari
17	Wei	15.60	66.10	02 43 27.11428	+27 42 25.73787	8.502	-11.433	12719		35	Ari
18	Mao	11.30	69.10	03 44 52.53688	+24 06 48.01122	20.840	-46.060	17499		17	Tau
19	Bi	17.40	73.50	04 28 37.00026	+19 10 49.56314	107.526	-36.200	20889	ϵ	74	Tau
20	Zi	0.05	82.5	05 34 49.23804	+09 29 22.48781	0.270	-2.260	26176	ϕ^1	37	Ori
21	Shen	11.10	92.3	05 32 00.40009	-00 17 56.74240	0.640	-0.690	25930	δ	34	Ori
22	Jing	33.30	68.40	06 22 57.62686	+22 30 48.89790	56.390	-110.030	30343	μ	13	Gem
23	Gui	2.20	71	08 31 35.72996	+18 05 39.90542	-59.639	-56.615	41822	θ	31	Cnc
24	Liu	13.30	83.3	08 37 39.36747	+05 42 13.63594	-68.867	-7.551	42313	δ	4	Hya
25	Xing	6.30	97.5	09 27 35.24270	-08 39 30.95830	-15.230	34.370	46390	α	30	Hya
26	Zhang	17.25	103.5	09 51 28.69384	-14 50 47.77103	18.880	-21.850	48356	ν^1	39	Hya
27	Yi	18.75	106.5	10 59 46.46516	-18 17 55.63039	-462.303	128.614	53740	α	7	Crt
28	Zhen	17.30	105.50	12 15 48.37081	-17 32 30.94960	-158.610	21.860	59803	γ	4	Crv

The second column represents the names of lunar lodges; the third and fourth columns represent lodge spans and polar distances in the unit of Chinese degrees; the fifth and sixth columns represent the SIMBAD J2000.0 right ascensions and declinations; the seventh and eighth columns are the proper motions in the unit of mili-arcsec per year; the 9–12th columns represent the Hipparcos number, Bayers designation, Flamsteed number, and constellation name for each determinative star.

adopting them because the definitions of some determinative stars have varied over time. We verify those identifications by calculating their coordinates at the epoch relevant to the star catalogue. For the case of the stars in *Shoushili*, the lodge spans were measured precisely so that we could make robust identifications. For the case of the polar distances of the determinative stars in QJJ, we can also use the fact that the determinative stars are graphically marked in the star charts depicted in the book.

We show the identified stars in Table 1 and discuss some related issues shortly. First, the determinative star of Zi[20] is identified as ϕ^1 Ori. The star satisfies a condition that the westernmost star in the lunar lodge usually defines the determinative star of a lunar lodge. The lunar lodge of Zi[20] had a lodge span of 2 *du* during the Han Dynasties (202 BCE–220 CE), gradually decreased due to the precession, became zero around 1250 CE, and became negative after that. Calculations show that the lodge span of Zi[20] around 1280 CE was -0.025 *du* if ϕ^1 Ori was the determinative star of Zi[20]. Although the value is negative, we can accept the value within the reading error of 0.05 *du* in *Guo Shoujing's* instruments (Pan 1989,

2009). In the 17th century, the lodge span reached as large as -0.4 *du*, so the determinative star of Zi[20] was changed from ϕ^1 Ori to λ Ori during the reformation of calendars to the *Chongzhen* calendrical system from 1631 CE to 1635 CE. If λ Ori was the determinative star of Zi[20], the lodge span of Zi[20] in 1280 CE is calculated to be -0.076 *du*. Although this value must have been large enough to be detected with *Guo Shoujing's* instruments, he did not change the determinative star. We also consider that the determinative star is marked as ϕ^1 Ori in the star chart of QJJ. Therefore, we identify ϕ^1 Ori as the determinative star of Zi[20] in *Shoushili*. On the other hand, Nakamura (2018) identified λ Ori as the determinative star of Zi[20] when he analysed the lodge spans in *Shoushili*. Although his identification would not significantly affect the observational epoch, we will consider this case to quantify the difference.

Second, the determinative star of Liu[24] is identified as δ Hya. There had been confusion about whether the determinative star of Liu[24] was either σ Hya or δ Hya. The neighbouring determinative star of Gui[23] is sure to have been

θ Cnc. Calculations show that the lodge span of *Gui*[23] was 13.0 *du* (13.4 *du*) if σ Hya (δ Hya) was the determinative star of *Liu*[24]. Since the observed lodge span of 13.30 *du* is given in *Shoushili*, we identify δ Hya as the determinative star of *Liu*[24]. We note that the determinative star of *Liu*[24] is graphically defined as δ Hya in a star chart of QJJ.

The locations of the determinative stars at a particular time are calculated from the J2000 locations of the stars, obtained from SIMBAD², by considering the precession of equinoxes and proper motions and using the algorithms in Meeus (1998). We show the J2000 coordinates³ of the identified determinative stars in Table 1.

2.2. Methods

Ahn (2023) developed a linear regression method to analyse the lodge spans and polar distances. Here we cite his paper for the detailed derivations and describe the method shortly. We denote the observational epoch by t_o and an arbitrary time close to the epoch by t . The equatorial coordinates at time t_o and t for the determinative star of the i -th lunar lodge are written as (α'_i, δ'_i) and (α_i, δ_i) , respectively. The equatorial coordinates at time t_o and t for the determinative star of the $(i+1)$ -th lunar lodge are written as $(\alpha'_{i+1}, \delta'_{i+1})$ and $(\alpha_{i+1}, \delta_{i+1})$, respectively. Then, we obtain a relationship (Ahn 2023)

$$L'_i - L_i(t) = n(t)\Delta t (\sin \alpha_{i+1} \tan \delta_{i+1} - \sin \alpha_i \tan \delta_i). \quad (1)$$

Here $L'_i \equiv \alpha'_{i+1} - \alpha'_i$ is the observed lodge span of the i -th lunar lodge at time t_o , and $L_i(t) \equiv \alpha_{i+1} - \alpha_i$ is the calculated lodge span of the i -th lunar lodge at the time t . We write $L_i(t)$ to show that it is a function of time t .

Similarly to the case of lodge spans, the declination (or polar distance) of the determinative star for the i -th lunar lodge has a relationship (Ahn 2023)

$$\delta'_i - \delta_i(t) = n(t)\Delta t \cos \alpha_i(t). \quad (2)$$

In both equations, $\Delta t \equiv t_o - t$ and the secular precession component in the ecliptic longitude direction

$$n(t) = 20''.0431 - 0''.0085T, \quad (3)$$

where $T(t)$ is the elapsed time in Julian centuries since J2000.0 (Meeus 1998). We note that $n = 0.558453^\circ/\text{century}$ around 1280 CE.

In Equation (1), $L'_i - L_i(t)$ is linearly proportional to $(\sin \alpha_{i+1} \tan \delta_{i+1} - \sin \alpha_i \tan \delta_i)$. Likewise, in Equation (2), $\delta'_i - \delta_i(t)$ is linearly proportional to $\cos \alpha_i(t)$. Hence, we set $X_i \equiv (\sin \alpha_{i+1} \tan \delta_{i+1} - \sin \alpha_i \tan \delta_i)$ and $Y_i \equiv L'_i - L_i(t)$. Then, introducing the Y -intercept A and defining the proportional coefficient $B \equiv n(t)\Delta t$, we have $Y_i = A + BX_i$. Likewise, we set $X_i \equiv \cos \alpha_i(t)$ and $Y_i = \delta'_i - \delta_i(t)$. Then, we also have

$Y_i = A + BX_i$. Therefore, we have only to solve a fitting problem with a straight-line model $Y = A + BX$ for a set of N data $(X_1, Y_1), \dots, (X_N, Y_N)$ resulting from N independent observations. Here A includes the systematic error or bias and the coefficient $B = n(t)(t - t_o)$. Each independent measurement of Y_i has an error σ_i , and the X_i 's are precisely known. Then the fitting problem becomes weighted least squares fittings or weighted linear regressions, which gives us the Y -intercept A and the slope B (see also Press et al. (1988)). Furthermore, we can regard the measurement error for every observation as constant because either lodge spans or polar distances are sure to have been measured simultaneously with the same instruments. Since the lodge spans were measured independently of polar distances, we can assume that every lodge span has the same measurement error σ_L and that every polar distance has the same measurement error σ_δ . Then, the fitting problem becomes more straightforward.

If we find the time t when $B = 0$, then the time will be the observational epoch or $t = t_o$. With the parameters A and B at $t = t_o$, we can calculate the standard deviation of the residuals to obtain the measurement error σ_m , which can be either σ_L or σ_δ . Then we calculate the uncertainty in the Y -intercept $\sigma_A = (\sigma_m/\sqrt{N})(X^2/(X^2 - \bar{X}^2))$ and the uncertainty in the slope $\sigma_B = (\sigma_m/\sqrt{N})(1/(X^2 - \bar{X}^2))$ as shown in Ahn (2023). Here $X^2 \equiv \sum X_i^2/N$ and $\bar{X} \equiv \sum X_i/N$. From the definition of B , we see that $\sigma_B \approx n(t_o)\sigma_t$. Here σ_t is the uncertainty in the observational epoch, which can be $\sigma_t = \sigma_B/n(t_o)$. Here $n(t_o)$ is also given by Equation (3).

The Y -intercept can be understood as a bias related to the choice of the coordinate's origin. For the case of polar distances, the Y -intercept A can be interpreted as the misalignment of the instrument's axis to the celestial pole, as shown by Sun (1994) and Sun & Kistemaker (1997). On the other hand, since lodge spans are defined as differences in right ascensions with reference to neighbouring determinative stars, we expect $A \approx 0$.

In practice, we first obtain the temporal variations in the coefficients A and B by performing simple linear regressions, with the measurement errors, either σ_L or σ_δ , presumed to be 1° . Then, we can determine the observational epoch when $B = 0$ because the two parameters A and B can be obtained irrespective of the measurement errors. Then we calculate the measurement error from their residuals at the observational epoch, either σ_L or σ_δ . Then we obtain the uncertainties in the fitting parameters, σ_A and σ_B , which are multiples of either σ_L/\sqrt{N} or σ_δ/\sqrt{N} . Finally, σ_B is converted to the uncertainty in the observational epoch, σ_t .

2.3. Epoch of the Lodge Spans in *Shoushili*

In this subsection, we will apply the analysis method described in the previous subsection to determine the observational epoch of the lodge spans in *Shoushili*. We analyse all 28 lodge spans in *Shoushili* shown in Table 1. We present the results on the upper rows in the top part of the table denoted by 'Lodge $NN\phi\text{Ori}$ ' in Table 2, where NN means the number of data.

At first, the measurement error is arbitrarily set to be

²<https://simbad.u-strasbg.fr/simbad/sim-fid>

³SIMBAD locations and proper motions are referred to the ICRS. They are based on the GAIA eDR3 data as of December 2022. The reference epoch for Gaia eDR3 is 2016.0; SIMBAD coordinates are given as values converted to J2000. We note that the reference epoch of the Hipparcos catalogue is J1991.25.

Table 2. Results of linear regressions for the lodge spans in *Shoushili* and polar distances in *San Yuan Lie She Ru Xiu Qu Ji Ji* for 28 lunar lodges.

Data		Linear regression			Measurement errors		Coefficients' errors				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Name	N	t_o [year]	A [°]	B [°]	\bar{Y} [°]	\bar{X} [°]	σ_L [°]	σ_A [°]	σ_B [°]	$n(t_o)$ [°yr ⁻¹]	σ_t [year]
Lodge28phiOri	28	1272.14	0.000000	-0.000008	0.000000	-0.000000	0.086245	0.016299	0.111764	0.558471	20.01
Lodge27phiOri	27	1268.98	-0.007163	0.000013	-0.007163	-0.001951	0.079252	0.015253	0.102985	0.558479	18.44
Lodge26phiOri	26	1267.06	-0.013528	0.000007	-0.013528	-0.003385	0.073731	0.014463	0.095947	0.558483	17.18
Lodge25phiOri	25	1271.07	-0.019731	0.000010	-0.019731	-0.000120	0.068154	0.013631	0.089179	0.558474	15.97
Lodge28phiOri	28	1272.14	0.000000	-0.000008	0.000000	-0.000000	0.086245	0.016299	0.111764	0.558471	20.01
Lodge28phiOri*	28	1268.29	-0.000000	0.000004	-0.000000	-0.000000	0.094041	0.017772	0.121968	0.558480	21.84
Takesako	28	1272					0.09				15.3
Lodge28lamOri	28	1271.10	-0.000000	-0.000012	-0.000000	-0.000000	0.087782	0.016589	0.113679	0.558474	20.36
Lodge28lamOri*	28	1267.30	-0.000000	0.000020	-0.000000	-0.000000	0.095037	0.017960	0.123175	0.558483	22.06
Nakamura	28	1277					0.13				30
Name	N	t_o [year]	A [°]	B [°]	\bar{Y} [°]	\bar{X} [°]	σ_δ [°]	σ_A [°]	σ_B [°]	$n(t_o)$ [°yr ⁻¹]	σ_t [year]
Polar28phiOri	28	1362.90	0.070211	-0.000008	0.070211	0.004429	0.133107	0.025155	0.035809	0.558257	6.41
Polar26phiOri	26	1360.69	0.049252	-0.000020	0.049252	-0.015744	0.113902	0.022344	0.032089	0.558262	5.75
Polar25phiOri	25	1361.23	0.038684	0.000013	0.038684	-0.010268	0.102795	0.020561	0.028983	0.558261	5.19
Polar24phiOri	24	1363.88	0.029806	-0.000015	0.029805	0.023800	0.095289	0.019462	0.027644	0.558255	4.95
Polar28phiOri	28	1362.90	0.070211	-0.000008	0.070211	0.004429	0.133107	0.025155	0.035809	0.558257	6.41
Polar28phiOri*	28	1363.86	0.074012	0.000002	0.074012	0.004445	0.137914	0.026064	0.037104	0.558255	6.65
Takesako	28	1363					0.14				6.0
Polar28lamOri	28	1361.42	0.054214	-0.000033	0.054214	0.004388	0.177614	0.033567	0.047782	0.558261	8.56
Polar28lamOri*	28	1362.38	0.058029	0.000013	0.058029	0.004405	0.181660	0.034331	0.048873	0.558258	8.75
Nakamura	28	1370									9.76

The linear relationship has a functional form of $Y = A + BX$. The upper part shows the results of iterations for lodge spans, and the lower part shows those for polar distances. The first column is the names of datasets, and the second column is the number of data analysed. The numbers in the names of datasets are the numbers of data in the datasets. The asterisked names of datasets represent the results of neglecting proper motions. The names with phiOri or lamOri represent the results for data sets in which the determinative star of Zi[20] is identified as ϕ^1 Ori and λ Ori, respectively. In the third column, t_o is the observational epochs, and their uncertainties σ_t are shown in the 12th column. In the fourth and fifth columns are shown the fitting parameters A and B . Their errors, denoted by σ_A and σ_B , are shown in the ninth and tenth columns. The sixth and seventh columns represent the averages of X and Y used in the linear regression. In the eighth column, σ_L or σ_δ are measurement errors of lodge spans and polar distances. $n(t_o)$'s in the 11th column are the rate of changes of ecliptic longitude projected on the declination direction in a unit of $^\circ\text{century}^{-1}$.

$\sigma_L = 1^\circ$, and we perform a linear regression to a set of 28 points of $(X_i, Y_i) \equiv (\sin \alpha_{i+1}(t) \tan \delta_{i+1}(t) - \sin \alpha_i(t) \tan \delta_i(t), L'_i - L_i(t))$. Here the equatorial coordinates $(\alpha_i(t), \delta_i(t))$ and $(\alpha_{i+1}(t), \delta_{i+1}(t))$ for the i -th and $(i+1)$ -th determinative stars at a time t are calculated from the SIMBAD J2000.0 coordinates by using the equations in Meeus (1998). The lodge span for the i -th lunar lodge at a time t , $L_i(t)$, is calculated as $L_i(t) = \alpha_{i+1}(t) - \alpha_i(t)$. The observed lodge span of the i -th lunar lodge, L'_i , is given in *Shoushili*.

We obtain the fitting parameters A and B during an appropriate period with a time step of 0.01 year. Then, we find a time t_o when $B = 0$. As a result, we find $t_o = 1272.14$ when $B = -0.000008^\circ$ is closest to zero. Here, as shown in Table 2, we can verify that $A = \bar{Y}$ and that $A = 0$ for 28 lodge spans (Ahn 2023). We calculate the residuals by applying the fitting parameters to obtain their standard deviation which is regarded as the measurement error $\sigma_L = 0.086245^\circ = 5.2'$.

At this step, there is one outlier or the lodge span of the lunar lodge Wei[17] whose residual ($L' - L(t)$) outliers from

the average A by $2\sigma_L$. So we discard the outlier and perform a linear regression analysis for the remaining 27 lodge spans. We present the results on the row denoted by 'Lodge27phiOri' in Table 2.

We obtain the final results after repeating these procedures until there is no outlier. During the procedures, we remove outlying lodge spans for the lunar lodges such as Bi[14], Wei[17], and Jing[22]. We show the results for the successive iterations in the upper group of rows in the top part of Table 2, and the final results are summarised by labelling 'Lodge25phiOri'. Finally, we obtain $t_o = 1271.07$ when $B = 0.000010$. After obtaining the measurement error $\sigma_L = 0.068154 = 4.1'$, we calculate the uncertainty in the Y -intercept $\sigma_A \simeq 0.013631^\circ$ and the uncertainty in the slope $\sigma_B \simeq 0.089179^\circ$. From Equation (3), we calculate $n(t_o) = 0.558474^\circ\text{century}^{-1}$ when $t = t_o$. Then, from the uncertainty in the slope σ_B , we obtain the uncertainty in the observational epoch $\sigma_t = \sigma_B/n(t_o) = 15.97$ yrs. We also show the final fitting results in Figure 1, where the three out-

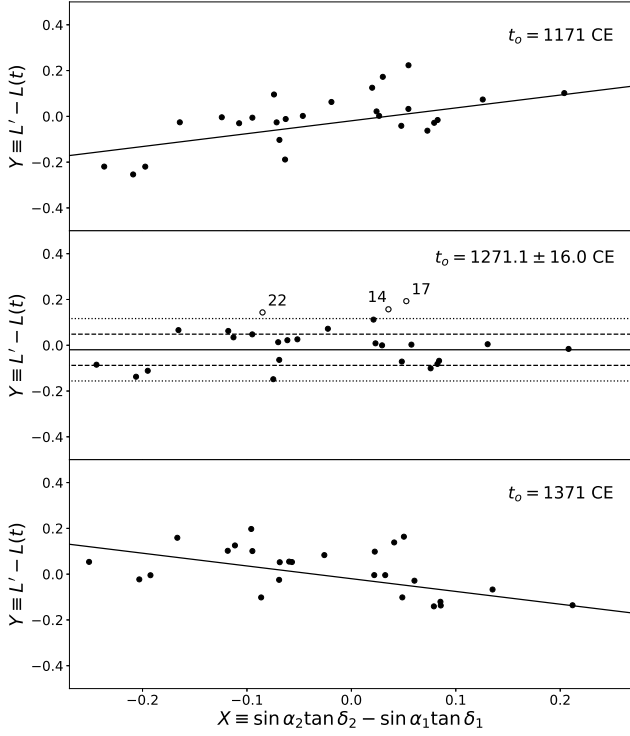


Figure 1. Results of linear regressions for the lodge spans in *Shoushili*. The equatorial circumference is assumed to be 365.2575 *du*. Discarding three outliers, we obtain the observational epoch $t_o = 1271.07 \pm 15.97$ and the measurement error in lodge spans $\sigma_L = 4.1'$. In the middle panel, $1\sigma_L$ and $2\sigma_L$ measurement-error bars are shown as dashed and dotted lines, respectively. The three outliers are plotted as open dots.

liers are depicted with open dots.

2.4. Epoch of the Polar Distances in *QJJ*

We also analyse the polar distances of the 28 determinative stars in *QJJ*. Following similar procedures to the case of lodge spans, we perform linear regressions that have been described in Section 2.2. We set the measurement error $\sigma_\delta = 1^\circ$ and apply a linear regression to a set of 28 points of $(X_i, Y_i) \equiv (\cos \alpha_i(t), \delta'_i - \delta_i(t))$ at a time t to obtain the Y -intercept A and the slope B . Here $\alpha_i(t)$ and $\delta_i(t)$ are calculated from the SIMBAD J2000.0 coordinate by considering its precession of equinoxes and the proper motion (Meeus 1998). δ'_i is calculated from the observed polar distance, P_i , of the i^{th} determinative star by using a formula $\delta'_i = 90^\circ - P_i$.

We analyse all the polar distances for the 28 determinative stars as shown in Table 1, where the determinative star of the lunar lodge $Zi[20]$ is identified as ϕ^1 Ori. We show the results in the upper group in the bottom part of Table 2 labelled by ‘Polar NN phiOri’, where NN represents the number of data. After discarding the outliers such as $Di[3]$, $Wei[6]$, $Ji[7]$, and $Kui[15]$, we obtain the final results labelled ‘Polar24phiOri’ in Table 2. The observational epoch $t_o \pm \sigma_t = 1363.88 \pm 4.95$, the measurement error $\sigma_\delta = 0.095289^\circ = 5.7'$, and the Y -intercept $A = 0.029806^\circ = 1.8'$. We show the fitting results in Figure 2. Note that the observational epoch we have obtained agrees with that of Takesako (2018), but a bit differs from

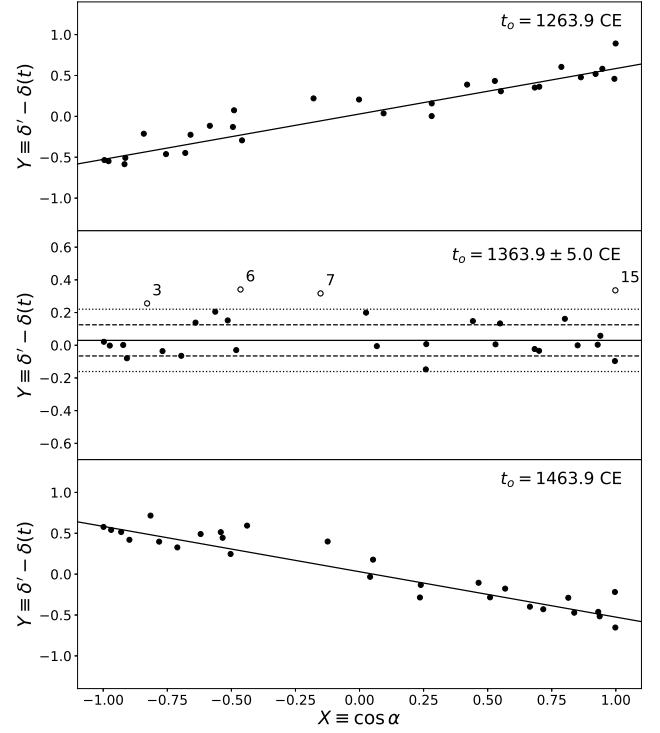


Figure 2. Results of linear regression analyses for the polar distances of the 28 determinative stars listed in *San Yuan Lie She Ru Xiu Qu Ji Ji*. The equatorial circumference is assumed to be 365.2575 *du*. Removing four outliers, we obtained the observational epoch $t_o \pm \sigma_t = 1363.88 \pm 4.95$, the measurement error $\sigma_\delta 5.7'$, and the Y -intercept or the misalignment error $A = 1.8'$. The dashed and dotted lines in the middle panel show $1\sigma_\delta$ and $2\sigma_\delta$ measurement-error bars, respectively. The four outliers are plotted with open dots.

Nakamura (2018). From the values listed in Table 2, we also confirm the properties of fitting parameters for polar distances (Ahn 2023): $A = \bar{Y}$, $\sigma_A \approx \sigma_\delta / \sqrt{N}$, and $\sigma_B \approx \sqrt{2} \sigma_\delta / \sqrt{N}$.

From these results, we can confirm an assertion that the observational epoch of the polar distances in *QJJ* differs from the period of *Guo Shoujing*’s observations around 1276 CE by one hundred years (Sun 1996; Nakamura 2018; Takesako 2018; Cao 2019), which is strengthened by a fact we have found that the measurement error of lodge angles in the *Shoushi* calendar ($\sigma_L = 4.1'$) is different from that of polar distances in *QJJ* ($\sigma_\delta = 5.7'$). We also see that the Y -intercept values or the misalignment errors range within the measurement errors. We conclude that the axis of the instrument used for measuring the polar distances was accurately aligned with the celestial poles.

2.5. Comparisons with Previous Researches

As stated in Section 1 and Section 2.1, Nakamura (2018) and Takesako (2018) also analysed the lodge spans in *Shoushili*. Their methods are considered a kind of χ^2 -minimisation scheme, similar to ours. We note that the χ^2 -minimization method is described in Ahn (2020) in detail. In terms of the descriptions of this paper, the χ^2 in Takesako (2018) can be defined as $\chi^2 \equiv \sum_{i=1}^N (L'_i + d - L_i(t))^2$, where L'_i ’s could have offset errors d . We find a time t_o when the χ^2 value minimises. Takesako (2018) defined the root square variance

$(\chi_{min}^2/(N-1))^{1/2}$. He called the quantity ‘residual’, which seems equivalent to the measurement error σ_L in this paper. He obtained the uncertainty in the observational epoch using the so-called simulation method. This method may correspond to the bootstrapping method introduced by Nakamura (2018) in estimating the population standard deviations. The bootstrapping method is described in detail in Ahn (2020).

Our research has several differences from these previous researches. First, the locational accuracies of star catalogues are different. Nakamura (2018) used the Yale Bright Star Catalogue⁴ (Hoffleit & Jaschek 1982) and Takesako (2018) used the Version 5 of SKY2000 Master Catalog⁵ (Myers et al. 2001). On the other hand, we use the SIMBAD coordinates based on the GAIA eDR3 data in this paper, whose coordinates and proper motions are much more precise and accurate than the other two.

Second, it is unclear whether the proper motions are considered in the previous works. So we perform the linear regressions neglecting the proper motions as well. The results for the lodge spans without considering proper motions are shown in the upper part of Table 2 by tagging asterisks on the labels of the data sets. The results for the polar distances without considering proper motions are shown in the lower part of Table 2 by tagging asterisks on the labels of the data sets. For the cases of lodge spans, neglecting proper motions causes a decrease of approximately four years in t_o and an increase of roughly two years in σ_t . For the cases of polar distances, neglecting proper motions causes an increase of approximately one year in t_o and an increase of 0.2–0.25 years in σ_t .

Third, the identification of one determinative star is different. The identifications of Takesako (2018) are the same as ours, whereas Nakamura (2018) identified the determinative star of the lunar lodge Zi[20] as λ Ori, instead of ϕ^1 Ori. Hence, to check the effects, we apply the linear regression method to the cases of identifying the determinative star of Zi[20] as λ Ori. We show our results labelled ‘lamOri’ in Table 2 for both lodge spans and polar distances. For the cases of lodge spans, the choice of λ Ori instead of ϕ^1 Ori causes a decrease of 1 year in t_o and an increase of 0.2–0.3 years in σ_t . For the cases of polar distances, the choice causes a decrease of approximately 1.5 years in t_o and a decrease of 2.1–2.3 years in σ_t . We note that ϕ^1 Ori and λ Ori have a slight difference in right ascensions but a more significant difference in declinations. Table 2 also shows the results obtained by Takesako (2018) and Nakamura (2018). Our results, labelled ‘Lodge28phiOri’ and ‘Polar28phiOri’, better agree with those of Takesako (2018).

3. Conclusions

To determine the observational epoch, we have applied a method of weighted linear regressions to the lodge spans for the 28 determinative stars listed in the official calendar of the Yuan dynasty titled *Shoushili*. We carefully identified the determinative stars and used the accurate locations and proper motions from the SIMBAD star catalogue based on the GAIA eDR3 data. After discarding three outliers, we have obtained the observational epoch $t_o \pm \sigma_t = 1271.07 \pm 15.97$ CE and the measurement error $\sigma_L = 4.1'$.

The polar distances of the determinative stars are listed in another manuscript book titled *San Yuan Lie She Ru Xiu Qu Ji Ji* (Lodge spans and polar distances of stars in Three Prefectures and Lunar lodges, QJJ in abbreviation). Chen (1986) and Pan (1989) asserted that the polar distances had been observed at the time of developing *Shoushili*. However, the assertion was negated in the following studies (Sun 1996; Nakamura 2018; Takesako 2018; Cao 2019). Hence, we have performed linear regressions to the polar distances to verify this assertion with the new analysis method (Ahn 2023). Discarding four outliers, we have obtained the observational epoch $t_o \pm \sigma_t = 1363.88 \pm 4.95$ CE, the measurement error $\sigma_\delta = 5.7'$, and the misalignment error 1.8'. The misalignment error is much smaller than the measurement error, so the instrument’s axis must have been aligned accurately with the celestial poles. These results show that the observational epoch is approximately one hundred years later than lodge spans in *Shoushili*. It is also remarkable that the measurement error of lodge spans differs from that of polar distances. Therefore, we confirm that the polar distances were measured on a different occasion from the lodge span measurements, ensuring the previous works.

Pan (1989) presented an additional data in *Ling Tai Mi Yuan* and asserted that they were also measured by *Guo Shoujing*. In a manuscript produced during the Ming dynasty, there left lodge angles and polar distances for six asterisms that make up for the omissions in the book. Pan (1989) suggested that they were also measured by *Guo Shoujing* or later. However, his suggestion is negated by our analyses, and details are shown in Appendix A.

It is known that the lodge spans in *Shoushili* were observed by a famous astronomer named *Guo Shoujing* during the 13th century. However, we cannot neglect the role of Muslim astronomers working around that time in the Court of the Yuan dynasty. Here we summarise the contribution of Muslims to the astronomical development during the Mongolian Empire.

As the Mongol empire expanded through the Eurasia continent during the reign of *Genghis Khan* (r.1206–1227) in the 13th century, the astronomical interaction between the Islamic world and the Chinese world enhanced. The first direct encounter was made by *Yelu Chucai*, who, developing a calendrical method called *Xi Zheng Jing Wu Yuan Li*, recognised that the Islamic calendar-making method is better for calculating the planetary motions, and so he developed the *Madafa* calendrical method adopting the Islamic knowledge (Yabuuchi

⁴<http://tdc-www.harvard.edu/software/catalogs/bsc5.html>

⁵<http://tdc-www.harvard.edu/catalogs/sky2k.html>

1967).

Hülegü Khan (r.1256–1265), a grandson of *Genghis Khan* and a brother of *Kublai Khan*, conquered the Persian area and founded the Ilkhanate. In 1259 CE, he ordered *Nasir al-Din al-Tusi* to construct a great observatory at *Maragheh*, where personnel and academic exchanges between Chinese and Islamic astronomy occurred (van Dalen 2002; Isahaya 2015). In another part of the Mongolian Empire, a Muslim astronomer called *'Isa* came to the capital of the Mongolian Empire during the reign period of *Güyük Khan* (r.1246–1248). In 1263 CE, *Kublai Khan* (r.1264–1294) established two offices for Muslim astronomy and medicine called *Xiyu Xinglisi* and *Xiyu Yiyaosi* in the new capital of the *Yuan* dynasty and appointed *'Isa* to their director (Yamada 1980; Li 2016; Chen 1996). In 1267 CE, the office for Muslim astronomy was elevated to the Muslim Observatory. It became under the jurisdiction of the Secretariat, an influential institution under the direct control of the emperor.

Jamal ad-Din also known as *Zhamaluding* came to the *Yuan* dynasty from the Ilkhanate in the 1250s (Yabuuchi 1967; Yamada 1980; van Dalen 2002; Isahaya 2015). In 1267 CE, he constructed Islamic-style astronomical instruments such as Ptolemaic-style ecliptic armillary spheres, Ptolemy's rulers, a celestial sphere, and an astrolabe (Tasaka 1957; Yabuuchi 1967; Chen 1996; Yabuuchi 1997). He also developed a calendrical method for Muslims called *Wannianli* meaning 'Ten Thousand Year Calendar' in the same year of 1267 CE. According to pieces of *Wannianli* left in other literature, the calendar used the Zodiac and Babylonian-degree system, so it is thought that the calendar was an Islamic-style one (Chen 1996, 2000; van Dalen 2002; Lee et al. 2018), which usually lists various astronomical tables including star catalogues in ecliptic coordinates.

In 1271 CE, *Jamal ad-Din* became the first director of the Muslim Astronomical Bureau called *Hui Hui Si Tian Tai*, an extension of the previous office of the Muslim astronomical agency. The Bureau continued to operate parallel to the Chinese Astronomical Bureau. In 1273 CE, the Muslim and Chinese Astronomical Bureaus were merged under the management of the Bureau of Imperial Secretariat. The Muslim Astronomical Bureau was succeeded by the *Ming* dynasty and closed in 1398 CE according to the records in the History of the *Ming* dynasty (*Mingshi*).

After the Mongol Empire annexed the Southern *Song* dynasty, the Chinese Astronomical Bureau began to revise the Imperial calendar at the order of *Kublai Khan*. Until 1276 CE, the *Yuan* dynasty still used the Revised *Damingli* inherited from the *Jin* dynasty. *Kublai Khan* established an Astronomical Service called *Tai Shi Ju* and appointed *Wang Xun* as its director, charging the calendrical reform. Soon both *Xu Heng* (1209–1281) and *Guo Shoujing* (1231–1316) were brought to the task of observations and calculations by leading the astronomers of both Islamic and Chinese Astronomical Bureaus. Since the inherited instruments were of little use, *Guo Shoujing* engineered two fundamental instruments such as the Simplified Armilla and the Tall Gnomon with wood. According to

Xu Heng's Biography in *Yuanshi*, their major observational tasks were measuring the length of the gnomon's shadow for three years from 1277 CE to 1279 CE to determine the winter solstice, the length of the tropical year, and the precession rate.

In 1279 CE, the Astronomical Service was reformed into the Astronomical Commission called *Tai Shi Yuan*. *Wang Xun* was designated as the commissioner, and *Guo Shoujing* was designate as the vice-commissioner. In 1280 CE, the new calendrical method was completed, and *Kublai Khan* endowed the name of *Shoushili*. *Guo Shoujing* sent a memorial at the throne, listing seven astronomical parameters revised by observations and five new methodologies for precise calculations. Among them, the sixth parameter was the lodge spans for the 28 lunar lodges. The lodge spans were measured to the nearest tenth of a Chinese degree for the first time in history. He made two innovations to achieve this precision: He adopted sighting threads instead of sighting tubes to aim at observational targets and developed finer graduations on observational instruments divided into 36 equal parts.

In 1282 CE, the commissioner *Wang Xun* passed away. Although the new calendar was promulgated at that time, the formulae and the tables were not compiled into books. Hence, *Guo Shoujing* compiled the documents into books. In 1286 CE, *Guo Shoujing* became the commissioner. He published several books related to calendrical reform. Among them, we can see one volume of "Newly Measured lodge and polar distances for 28 lunar lodges and other asterisms." and one volume of "Newly Measured Anonymous Stars". These books must have been star catalogues, so we believe that *Guo Shoujing* must have observed stars and made catalogues. Unfortunately, they are not extant now. However, a manuscript titled *San Yuan Lie She Ru Xiu Qu Ji Ji* meaning 'Lodge and polar distances for 28 lunar lodges and other asterisms' was discovered in the 1980s. Chen (1986) and Pan (1989) regarded this star catalogue as those measured by *Guo Shoujing*, but Sun (1996) negated this suggestion by finding the observational epoch of approximately 1380 CE. Recently Nakamura (2018) determined the observational epoch to be 1370 ± 10 CE, and Takesako (2018) also obtained the observational epoch of 1363 ± 6 CE. We note that the latter agrees with our results.

In this paper, we have determined the observational epoch of the lodge spans in *Shoushili* to be 1271.07 ± 15.97 CE. Accordingly, we can find two possible occasions for observing the lodge spans whose times coincide with our result. One occasion is the completion of *Wannianli* in 1267 CE by a Muslim astronomer *Jamal ad-Din*, and the other is the intensive observations made by *Guo Shoujing* and his colleagues from 1277 CE to 1279 CE.

The calendrical method developed by *Jamal ad-Din* seems to have been an Islamic-style calendar or *Zij*, and a *Zij* usually has star catalogues containing the locational data. Indeed, he manufactured seven large instruments, including the Ptolemaic-style armillary sphere and ruler. Hence, it is highly probable for them to be used to measure ecliptic coordinates of celestial objects instead of equatorial ones. On the other hand, *Guo Shoujing* engineered an advanced instrument called the

Simplified Armilla for measuring the equatorial coordinates. We know that the ecliptical lodge spans called *Huang Dao Xiu Du* written in *Shoushili* of *Yuanshi* were indeed converted from the equatorial lodge spans. This means that the observations related to *Shoushili* were made using equatorial instruments. Moreover, he told in his memorial sent to the throne that he made several innovations in astronomical instruments, including the adoption of sighting threads instead of sighting tubes and 10-times-finer graduations. This statement was verified by the fact we have found in this paper that the measurement error of the lodge spans is $4.1'$.

Therefore, although both occasions occurred during the observational epoch we have determined, we conclude that the lodge spans in *Shoushili* were measured by *Guo Shoujing* around 1277–1279 CE. However, indeed, the Muslim astronomers worked together with the Chinese astronomers, Islamic astronomical knowledge must have contributed to the observations of the lodge spans in some way.

Acknowledgments

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Appendix A. Six Stars in *Ling Tai Mi Yuan*

Ling Tai Mi Yuan is an astrology book written by *Yu Jicai* (516–603 CE) in the 6th century China. One manuscript produced during the Ming dynasty was discovered in the national library of China (Pan 1989), and there exist lodge angles and polar distances for six asterisms complemented to make up for the omissions in the book. Since the coordinates have the same precisions as those made by *Guo Shoujing*, Pan (1989) suggested that they were produced by *Guo Shoujing* or later. Here we show these data are not relevant to *Guo Shoujing*'s observations but to the star catalogue titled *San Yuan Lie She Ru Xiu Qu Ji Ji* (QJJ in abbreviation). This book is also a manuscript of the Ming dynasty and is included as one book in a collection titled '*Tian Wen Hui Chao*'. The observational epoch of the QJJ catalogue was estimated to be approximately 1375 CE (Sun 1996), 1364 CE (Cao 2019), and 1354 CE (Chu & Yang 2022). In conclusion, most of the coordinate values of the six stars are not from *Guo Shoujing*'s observations but probably from some other observations made during the later period of the *Yuan* dynasty or the early period of the *Ming* dynasty.

A.1. The Second Star of *Ligong*'s Northwestern Pair

Ligong consists of three pairs of stars distributed around the lunar lodge of *Shi*[13]. One pair is at the Northwest of *Shi*[13], another at the Southwest of *Shi*[13], and the other at the East of *Shi*[13]. The first entry in *Ling Tai Mi Yuan* is a star in the Northwestern pair whose lodge angle and polar distance are the same as those in QJJ. We will describe the analysing procedures in detail only for this first entry.

Its polar distance is 63.80 *du* which corresponds to the declination $\delta_{\text{star}} = 90^\circ - 63.80 \text{ du} \times (360^\circ/365 \text{ du}) = +27^\circ 4' 26''$. The Northwestern pair of *Ligong* consists of two stars, η Peg and o Peg. Using the PC planetarium software

Table A.1. Six stars inserted in *Ling Tai Mi Yuan* as complements to the omissions.

No.	Star	P.A.	Declination (at t_r)	Identified stars	Epoch (t_r)	Lunar Lodge	Determinative star	L.A. Measured	L.A. Calculated	Id.	QJJ
(1)	NW Ligong No.2	63.80	+27 04 26	η Peg	03 Mar 1386	Wei	α Aqr	10.20	10.22	O	yes
				o Peg	14 Oct 1566	"	"	10.20	9.64	X	
(2)	SW Ligong No.1	69.60	+21 21 12	μ Peg	12 Apr 1373	Wei	α Aqr	11.80	11.78	O	yes
				λ Peg	08 Jan 1575	"	"	11.80	10.70	X	
(3)	East Ligong No.1	70.50	+20 27 57	τ Peg	26 Jan 1395	Shi	α Peg	4	4.09	O	—
				ν Peg	17 Dec 1461	"	"	4	5.23	X	
(4)	Fa Xing	98.10	−06 45 22	ι Ori	09 Oct 1203	Shen	δ Ori	1.40	1.30	O	—
				θ^2 Ori	25 Jan 914	"	"	1.40	1.42	X	
				42 Ori	11 May 632	"	"	1.40	1.44	O	
(5)	Jisi	69.20	+21 44 53	Praecepte	07 Jan 1355	Gui	θ Cnc	1.60	2.15	O	yes
(6)	Taijun	42.90	+47 41 16	ψ UMa	13 Sep 1402	Yi	α Crt	1.20	1.09	O	—

Stellarium, we can calculate the declination of η Peg at the past times. The calculated declination of η Peg agrees with the observed one derived from the given polar distance on 3 Mar 1386. At this time, the right ascension of η Peg was $\alpha_{\text{star}} = 22^{\text{h}}14^{\text{m}}29^{\text{s}}$. The lodge angle of the star is given in the book as 10.20 *du* of *Wei*[12] whose determinative star is α Aqr. So we obtain the right ascension of α Aqr on 3 Mar 1386, $\alpha_{\text{ref}} = 21^{\text{h}}34^{\text{m}}09^{\text{s}}$. Thus, the lodge angle is calculated as $\alpha_{\text{star}} - \alpha_{\text{ref}} = 10.22$ *du*.

On the other hand, the declination of o Peg agrees with α_{star} on 14 Oct 1566. At this time, the right ascension of o Peg $\alpha_{\text{star}} = 22^{\text{h}}21^{\text{m}}33^{\text{s}}$, and that of the determinative star of *Wei*[12] (α Aqr) was $\alpha_{\text{ref}} = 21^{\text{h}}43^{\text{m}}30^{\text{s}}$. Thus, the lodge angle must be $\alpha_{\text{star}} - \alpha_{\text{ref}} = 9.64$ *du*.

Since the observed lodge angle of the star is 10.20 *du* of *Wei*[12], we conclude that the second star of *Ligong's* Northwestern pair is identified as η Peg and observed around 1386 CE. We note that Pan (1989) also identified this star in QJJ as η Peg.

A.2. The First Star of *Ligong's* Southwestern Pair

Its lodge and polar distances are the same as those in QJJ. Its polar distance is 69.60 *du*, and so its declination $\delta_{\text{star}} = +21^{\circ}21'12''$. The Southwestern pair of *Ligong* consists of two stars, λ Peg and μ Peg.

The declination of λ Peg, calculated using Stellarium, agrees with δ_{star} on 8 Jan 1575. The observed lodge angle of the star is given as 11.80 *du* of *Wei*[12]. The determinative star of *Wei* is α Aqr. On 8 Jan 1575, The right ascension of λ Peg is calculated to be $\alpha_{\text{star}} = 22^{\text{h}}26^{\text{m}}08^{\text{s}}$, and the right ascension of α Aqr, the determinative star of *Wei*[12], is calculated to be $\alpha_{\text{ref}} = 21^{\text{h}}43^{\text{m}}54^{\text{s}}$. Hence the calculated lodge angle is $\alpha_{\text{star}} - \alpha_{\text{ref}} = 10.70$ *du*.

On the other hand, the declination of μ Peg agrees with δ_{star} on 12 Apr 1373. Likewise, we obtain $\alpha_{\text{star}} = 22^{\text{h}}19^{\text{m}}58^{\text{s}}$ and $\alpha_{\text{ref}} = 21^{\text{h}}33^{\text{m}}29^{\text{s}}$ on 12 Apr 1373. Hence, the calculated lodge angle is $\alpha_{\text{star}} - \alpha_{\text{ref}} = 11.78$ *du*.

Since the lodge angle of μ Peg is closer to the observed value of 11.80, we conclude that the first star of *Ligong's* Southwestern pair is identified as μ Peg and observed around 1373 CE. We note that Pan (1989) also identified this star in QJJ as μ Peg.

A.3. The First Star of *Ligong's* Eastern Pair

The polar distance is given 70.50 *du* corresponding to $\delta_{\text{star}} = +20^{\circ}27'57''$. The Eastern pair of *Ligong* consists of τ Peg and ν Peg. One candidate τ Peg had δ_{star} on 26 Jan 1395, when we obtain the calculated lodge angle of 4.09 *du* for the determinative star of the lunar lodge *Shi*[13] or α Peg. Likewise, the other candidate ν Peg had δ_{star} on 17 Dec 1461, when we obtain the calculated lodge angle of 5.23 *du*. Since the lodge angle of the star was given in the book as 4 *du* of *Shi*[13], we conclude that the first star of *Ligong's* Eastern pair is identified as τ Peg and observed around 1395 CE. We note that the star is NOT listed in the star catalogue of QJJ.

A.4. *Fa Xing*

It is known well that *Fa Xing* lies in Orion and consists of three stars: ι Ori, θ^2 Ori, and 42 Ori. These stars have similar right ascensions, so the declination will be decisive in identification. The observed polar distance of the star is given as 68.10 *du*, which means the star lies north of the equator. However, *Fa Xing* is south of the equator. Considering the declinations of the three stars, we guess that the polar distance must be corrected as 98.10 *du*. The polar distance of 98.10 *du* corresponds to the declination $\delta_{\text{star}} = -6^{\circ}45'22''$. We find the observational epochs for the three stars by calculating their declinations, as shown in Table A.1. ι Ori had the same declination to δ_{star} on 9 Oct 1203, which is closest to the times of *Guo Shoujing*. On the other hand, 42 Ori had the same declination to δ_{star} on 11 May 632, which is closest to the times of the Northern *Zhou* dynasty (557–581 CE) when the book *Ling Tai Mi Yuan* was written. However, both values of the lodge and the polar

distances are given up to the first decimal place, which can not be expected in the 6th century, and also ι Ori is the brightest star among the three stars. ι Ori has a visual magnitude of 2.75, while θ^2 Ori and 42 Ori have visual magnitudes of 5.00 and 4.55, respectively. Therefore, we identify ι Ori as the star called *Fa Xing*. We note that these observed angles are different from those in QJJ. However, the declination value given in the catalogue is modified arbitrarily, which weakens our conclusion.

A.5. *Ji Shi*

The celestial object is known to be an open star cluster called Praesepe or Beehive cluster. The observed polar distance is given as 69.20 *du*, corresponding to the declination of $\delta_{\text{star}} = +21^\circ 44' 53''$. The calculated declination of Praesepe coincides with δ_{star} on 7 Jan 1355. Considering the determinative star of the lunar lodge *Gui*[23] is θ Cnc, we calculate its lodge angle to be 2.15 *du*, which can be compared with the observed value of 1.60 *du*. The error is relatively large, probably because the open cluster is a fuzzy object. We note that the coordinate values given in the book are the same as those in QJJ.

A.6. *Tai Zun*

The polar distance is given as 42.90 *du*, which is converted to the declination of $\delta_{\text{star}} = +47^\circ 41' 16''$. Its lodge angle is given as 1.20 *du* of the determinative star of the lunar lodge *Yi*[27] or α Crt. Searching any candidate star around this coordinate around 1300 CE, we can find only one candidate or ψ UMa. The star had the same declination on 13 Sep 1402, when its calculated lodge angle is 1.09 *du*. This calculated lodge angle is comparable to the observed lodge angle given in the book as 1.20 *du*. On the other hand, the lodge and polar distances for this star are given in QJJ as 4 *du* of *Yi*[27] and 49.00 *du*, respectively. They differ from those in *Ling Tai Mi Yuan*.