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SOME PROPERTIES FOR SPIRALLIKE FUNCTIONS INVOLVING GENERALIZED q-INTEGRAL OPERATOR

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Abstract. In this note, we establish a new subfamily of spirallike functions by making use of a generalized q-integral operator. We examine characterization rule for functions which are member of this subclass. We further obtain coefficient estimate, subordination results and integral mean inequalities for functions in this class. The Fekete-Szegö inequalities are also derived.

1. Introduction and Preliminaries

A complex-valued function f of a complex variable is named univalent in a domain D if it does not take the same value twice, that is, $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$ for $z_1, z_2 \in D$. Now, assume D is a simply connected domain and $z \in D$. A necessary condition for an analytic function $f \in D$ to be univalent is that its derivative does not vanish on D. Thus, if f is analytic and univalent in D with Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then we can suppose that D is the unit disk $\mathbb{D} := \{z : |z| < 1\}$. Indicate by $\mathcal{A}$, the family of analytic functions f in $\mathbb{D}$ and with the normalization $f(0) = f'(0) - 1 = 0$. In the case of univalency of f, we indicate the subfamily of $\mathcal{A}$ by $\mathcal{S}$. For a detailed survey, we refer to the recent study by Thomas et al. [16].

For analytic functions $f_1,\ f_2$ in $\mathbb{D}$, we describe that f_1 is subordinate to $f_2\ \ (f_1 \prec f_2)$ for a Schwarz function

$$\Upsilon(z) = \sum_{n=1}^{\infty} s_n z^n \quad (\Upsilon(0) = 0, |\Upsilon(z)| < 1),$$

analytic in $\mathbb{D}$ such that

$$f_1(z) = f_2(\Upsilon(z)) \qquad (z \in \mathbb{D}).$$

Now, we shall deal with a subfamily of $\mathcal{S}$ which is of special interest in its own right, namely the spirallike functions.

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Assume ϑ $\left(|\vartheta| < \frac{\pi}{2}\right)$ and $t \in \mathbb{R}$. Then, for a nonzero complex number ω_0, a logarithmic ϑ-spiral is a curve that $\omega = \omega_0 \exp(-e^{-i\vartheta} t)$. Here, 0-spirals are radial half-lines and there is a unique ϑ-spiral which connects ω to the origin [4]. A domain D comprising the origin is defined as ϑ-spirallike if $\omega \neq 0$ in D, the arc of the ϑ-spiral from ω to the origin lies entirely in D. An analytic univalent function is called ϑ-spirallike if its range is ϑ-spirallike. Analytically, $f \in \mathcal{A}$ belongs to $\mathcal{S}_\vartheta$ if and only if $\Re\left(e^{i\vartheta}\frac{zf'(z)}{f(z)}\right) > 0$ [14]. We note that $\mathcal{S}_\vartheta \subset \mathcal{S}$ and for $\vartheta = 0$, $\mathcal{S}_0 = \mathcal{S}^*$ of starlike functions. Afterwards, Libera [7] established the family $\mathcal{S}_\vartheta(\sigma)$ of functions ϑ-spirallike of order σ. Analytically,

$$\Re\left(e^{i\vartheta}\frac{zf'(z)}{f(z)}\right) > \sigma \cos\vartheta.$$

Clearly, $\mathcal{S}_\vartheta(\sigma) \subset \mathcal{S}_\vartheta$.

The quantum calculus presents developments in various fields. For example, the use of q-calculus in Geometric Function Theory (GFT) was partially provided by Srivastava [15]. Afterwards, the field has gained great attentions. The paper deals with the complex-valued function f for $0 < q < 1$.

Next, consider the q-derivative of f [5]:

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0 \end{cases}. \tag{1}$$

If f is differentiable at z, then $\lim_{q\to 1^-} D_q f(z) = f'(z)$. Further, consider the q-integral of f [6]:

$$\int_0^x f(u) d_q u = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),$$

provided the series converges. Here, q-gamma function is defined as

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(x+1) = [x]_q!, \tag{2}$$

and q-factorial is defined as

$$[x]_q! = \begin{cases} [x]_q[x-1]_q \dots [2]_q[1]_q, & x \geq 1 \\ 1, & x = 0. \end{cases}$$

If we set $q \to 1^-$, we find $\Gamma_q(x) \to \Gamma(x)$ [5].

The q-beta function can be defined by means of the q-integral representation

$$\Sigma_q(u,s) = \int_0^1 x^{u-1}(1-qx)_q^{s-1} d_q x, \quad (0 < q < 1; u, s > 0) \tag{3}$$

with a q-analogue of Euler's formula [6]

$$\Sigma_q(u,s) = \frac{\Gamma_q(u)\Gamma_q(s)}{\Gamma_q(u+s)}. \tag{4}$$

Next

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{5}$$

represent the Gauss q-binomial coefficients [3].

In a recent study [9], a generalized q-integral operator $\chi^{\alpha}_{\beta,q}f : \mathcal{A} \to \mathcal{A}$

$$\chi^{\alpha}_{\beta,q}f(z) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{z^{\beta}} \int_0^z \left(1 - \frac{qu}{z}\right)_q^{\alpha-1} u^{\beta-1} f(u) d_q u \quad (\alpha > 0, \beta > -1) \tag{6}$$

is introduced. From (2), (3), (4) and (5), they arrive

$$\chi^{\alpha}_{\beta,q}f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^n. \tag{7}$$

Next, from (1) and (7), we arrive

$$D_q\left(\chi^{\alpha}_{\beta,q}f(z)\right) = 1 + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)\,[n]_q}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^{n-1},$$

where $[n]_q = \dfrac{1-q^n}{1-q}$ is the $q-$number.

For some special values, we find the following previously known integral operators.

(i) For $\alpha = 1$, $\chi^{\alpha}_{\beta,q}f$ reduces to the q-Bernardi integral operator [11]

$$J_{\beta,q}f = \frac{[1+\beta]_q}{z^{\beta}} \int_0^z u^{\beta-1} f(u) d_q u = \sum_{n=1}^{\infty} \frac{[1+\beta]_q}{[n+\beta]_q} a_n z^n.$$

(ii) For $\alpha = 1, q \to 1^-$, we get the Bernardi integral operator [1]

$$J_{\beta}f(z) = \frac{1+\beta}{z^{\beta}} \int_0^z u^{\beta-1} f(u) du = \sum_{n=1}^{\infty} \frac{1+\beta}{n+\beta} a_n z^n.$$

(iii) For $\alpha = 1, \beta = 0, q \to 1^-$, we get the Alexander integral operator [13]

$$J_0 f(z) = \int_0^z \frac{f(u)}{u} du = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n.$$

By inserting the function $\chi^{\alpha}_{\beta,q}f$, we present the following definiton.

Definition 1.1. *A function $f \in \mathcal{A}$ belongs to the family $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$ if*

$$\Re\left(e^{i\vartheta}\frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)}\right) > \sigma\cos\vartheta,$$

where $|\vartheta| < \frac{\pi}{2}$, $0 \le \sigma < 1$.

Note that

1) Letting $q \to 1^-$ and $\alpha = 1$ in Definition 1.1, we arrive the family $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta) := \mathcal{ST}_{\vartheta}(\sigma,\beta)$ involving the Bernardi integral operator given in (ii).

2) Letting $q \to 1^-$, $\alpha = 1$ and $\beta = 0$ in Definition 1.1, we find that $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta) := \mathcal{ST}_{\vartheta}(\sigma)$ involving the Alexander integral operator given in (iii).

This paper deals with the new family of $\vartheta-$spirallike functions involving a generalized $q-$integral operator and its several properties. Firstly, we get membership condition, coefficient estimate. We also seek the subordination results, integral mean inequalities and Fekete-Szegö inequalities for such functions.

2. Membership Condition and Coefficient Estimate

Theorem 2.1. *Let $f \in \mathcal{A}$ and $\mathfrak{r}$ $(0 \le \mathfrak{r} < 1)$ be a real number. If*

$$\left| \frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)} - 1 \right| \le 1 - \mathfrak{r}, \tag{8}$$

then $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$ if $|\vartheta| \le \cos^{-1}\left(\frac{1-\mathfrak{r}}{1-\sigma}\right)$.

Proof. If we continue from (8), we can write

$$\frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)} = (1-\mathfrak{r})\,\Upsilon(z) + 1.$$

Further computations give

$$\begin{aligned} \Re\left\{ e^{i\vartheta}\frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)} \right\} &= \Re\left\{ e^{i\vartheta}(1-\mathfrak{r})\,\Upsilon(z) + e^{i\vartheta} \right\} \\ &= (1-\mathfrak{r})\,\Re\left\{ e^{i\vartheta}\Upsilon(z) \right\} + \cos\vartheta \\ &\ge -(1-\mathfrak{r})\left| e^{i\vartheta}\Upsilon(z) \right| + \cos\vartheta \\ &= \cos\vartheta - (1-\mathfrak{r}). \end{aligned}$$

Here, $\cos\vartheta - (1-\mathfrak{r}) \ge \sigma\cos\vartheta$ if $|\vartheta| \le \cos^{-1}\left(\frac{1-\mathfrak{r}}{1-\sigma}\right)$. □

Corollary 2.2. *Let $\mathfrak{r} = 1 - (1-\sigma)\cos\vartheta$. If*

$$\left| \frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)} - 1 \right| \le (1-\sigma)\cos\vartheta,$$

then $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$.

Theorem 2.3. *A function* $f \in \mathcal{A}$ *is in* $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$, *if*

$$\sum_{n=2}^{\infty} \left\{1-\sigma+\left([n]_q-1\right)\sec\vartheta\right\} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} |a_n| \leq 1-\sigma. \tag{9}$$

Proof. By Corollary 2.2, since

$$\left|\frac{zD_q\left(\chi^{\alpha}_{\beta,q} f(z)\right)}{\chi^{\alpha}_{\beta,q} f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}\left([n]_q-1\right) A_n a_n z^{n-1}}{1+\sum_{n=2}^{\infty} A_n a_n z^{n-1}}\right|$$

$$< \frac{\sum_{n=2}^{\infty}\left([n]_q-1\right) A_n |a_n|}{1-\sum_{n=2}^{\infty} A_n |a_n|},$$

we arrive

$$\sum_{n=2}^{\infty}\left([n]_q-1\right) A_n |a_n| \leq (1-\sigma)\cos\vartheta \left\{1-\sum_{n=2}^{\infty} A_n |a_n|\right\}$$

or

$$\sum_{n=2}^{\infty}\left\{\left([n]_q-1\right)\sec\vartheta + (1-\sigma)\right\} A_n |a_n| \leq 1-\sigma,$$

where $A_n = \dfrac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}$. □

Corollary 2.4. *If* $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$, *then*

$$|a_n| \leq \frac{1-\sigma}{E_n} \qquad (n \geq 2),$$

where

$$E_n = \left\{\left([n]_q-1\right)\sec\vartheta + (1-\sigma)\right\} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha+\beta+n)}.$$

For the sharpness, set $f_n(z) = z + \frac{1-\sigma}{E_n} z^n$.

3. Subordination outcomes

Definition 3.1. [17] *Stand by* K *the subfamily of* $\mathcal{S}$ *comprising convex functions*

$$k \in K \Leftrightarrow \Re\left\{1+\frac{zk''(z)}{k'(z)}\right\} > 0,$$

where $k(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Definition 3.2. [17] $\{t_n\}_{n=1}^{\infty}$ *is a subordinating factor complex numbers sequence if, whenever* k *is regular, univalent and convex in* $\mathbb{D}$, *we find*

$$\sum_{n=1}^{\infty} b_n t_n z^n \prec k(z) \qquad (z \in \mathbb{D}).$$

Lemma 3.3. [17] *$\{t_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1+2\sum_{n=1}^{\infty} t_n z^n\right\} > 0.$$

Theorem 3.4. *Assume $k \in K$ and $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$. Then*

$$\frac{E_2}{2(1-\sigma+E_2)}(f*k)(z) \prec k(z), \tag{10}$$

where

$$E_2 = \frac{\{1-\sigma+q\sec\vartheta\}(\beta+1)}{\alpha+\beta+1}$$

and

$$\Re\{f(z)\} > -\frac{1-\sigma+E_2}{E_2}. \tag{11}$$

The constant factor $\frac{E_2}{2(1-\sigma+E_2)}$ cannot be changed with a larger number.

Proof. By Definition 3.2, the subordination (10) will be satisfied if

$$\left\{\frac{E_2}{2(1-\sigma+E_2)}a_n\right\}_{n=1}^{\infty}$$

is a sequence with $a_1 = 1$. From Lemma 3.3, it is adequate to express

$$\Re\left\{1+2\sum_{n=1}^{\infty}\frac{E_2}{2(1-\sigma+E_2)}a_n z^n\right\} > 0.$$

Therefore, providing that $|z| = r < 1$, we have

$$\begin{aligned}\Re\left\{1+\frac{E_2}{(1-\sigma+E_2)}\textstyle\sum_{n=1}^{\infty} a_n z^n\right\} &\geq 1-\frac{E_2}{1-\sigma+E_2}\left|\textstyle\sum_{n=1}^{\infty} a_n z^n\right| \\ &= 1-\frac{E_2}{1-\sigma+E_2}r-\frac{1}{1-\sigma+E_2}\textstyle\sum_{n=2}^{\infty} E_n|a_n|r^n \\ &= 1-\frac{1-\sigma+E_2}{1-\sigma+E_2}r > 0.\end{aligned}$$

Hence, equation (10) holds. Now, by taking $k(z) = \frac{z}{1-z}$ in (10), we arrive inequality (11). That is,

$$\begin{aligned}\frac{E_2}{2(1-\sigma+E_2)}f(z) &\prec k(z) \\ \frac{E_2}{2(1-\sigma+E_2)}\Re\{f(z)\} &> \Re\{k(z)\} \\ \frac{E_2}{2(1-\sigma+E_2)}\Re\{f(z)\} &> -\frac{1}{2}.\end{aligned}$$

Consider a function

$$F(z) = z - \frac{1-\sigma}{1-\sigma+E_2} z^2$$

for the sharpness. Clearly $F \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$. By (10), since

$$\frac{E_2}{2(1-\sigma+E_2)} (F * k)(z) \prec \frac{z}{1-z},$$

we arrive

$$\min_{|z|\leq 1} \left[\Re \left\{ \frac{E_2}{2(1-\sigma+E_2)} F(z) \right\} \right] = -\frac{1}{2}.$$

Thus, the constant $\frac{E_2}{2(1-\sigma+E_2)}$ cannot be changed with any larger one. □

Next, we give integral mean inequalities. Thus, we need the following lemma which was proven by Littlewood [8] in 1925.

Lemma 3.5. *If f_1 and f_2 are analytic in $\mathbb{D}$ with*

$$f_1(z) \prec f_2(z),$$

then, for $\nu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f_1(z)|^\nu \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\nu \, d\theta. \tag{12}$$

Theorem 3.6. *Suppose that $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$, $\nu > 0$ and*

$$f_2(z) = z + \frac{1-\sigma}{E_2} z^2,$$

where $E_2 = \dfrac{\{q \sec\vartheta + (1-\sigma)\}(\beta+1)}{\alpha+\beta+1}$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\nu \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\nu \, d\theta.$$

Proof. For the function $f(z) = z + \sum_{n=2}^\infty a_n z^n$ $(a_n > 0)$ the relation (12) is equivalent, if

$$\int_0^{2\pi} \left| 1 + \sum_{n=2}^\infty a_n z^{n-1} \right|^\nu d\theta \leq \int_0^{2\pi} \left| 1 + \frac{1-\sigma}{E_2} z \right|^\nu d\theta.$$

By Lemma 3.5, it enough to show that

$$1 + \sum_{n=2}^\infty a_n z^{n-1} \prec 1 + \frac{1-\sigma}{E_2} z.$$

By setting

$$1 + \sum_{n=2}^\infty a_n z^{n-1} = 1 + \frac{1-\sigma}{E_2} \Upsilon(z).$$

and using (9), we obtain

$$\begin{aligned}|\Upsilon(z)| &= \left|\sum_{n=2}^{\infty} \frac{E_2}{1-\sigma} a_n z^{n-1}\right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{E_2}{1-\sigma} a_n \\ &\leq |z| \sum_{n=2}^{\infty} \frac{E_n}{1-\sigma} a_n \\ &\leq |z| < 1,\end{aligned}$$

where E_n given in Corollary 2.4. □

4. The Fekete-Szegö problem

For a function $f \in \mathcal{S}$, finding the maximum value of the functional $\Theta_\Psi(f) = \left|a_3 - \Psi a_2^2\right|$ $(\Psi \in [-1, 1])$ known as the Fekete-Szegö problem [2].

Lemma 4.1. [10] *For a Schwarz function* Υ,

$$|s_1| \leq 1, \quad |s_2| \leq 1 - |s_1|^2$$

and for any complex number Ψ

$$\left|s_2 - \Psi s_1^2\right| \leq \max\{1, |\Psi|\}.$$

If $\Upsilon(z) = z$ *and* $\Upsilon(z) = z^2$, *the outcome is sharp.*

Next, consider a function

$$P(z) = \frac{1 + \left(e^{-i\vartheta} - 2\sigma\cos\vartheta\right) e^{-i\vartheta} z}{1 - z} = 1 + \sum_{n=1}^{\infty} P_n z^n \quad (z \in \mathbb{D}), \tag{13}$$

which maps $\mathbb{D}$ onto $H = \left\{z \in \mathbb{C} : \Re\left(e^{i\vartheta} z\right) > \sigma\cos\vartheta\right\}$ [12].

Theorem 4.2. *Assume that $f \in \mathcal{A}$ belongs to $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$. Then*

$$\left|a_3 - \Psi a_2^2\right| \leq$$

$$\begin{cases} \dfrac{2(\alpha+\beta+2)(\alpha+\beta+1)(1-\sigma)\cos\vartheta}{q(1+q)(\beta+2)(\beta+1)} \\ \times\left[1+\dfrac{2(1-\sigma)}{q} - \Psi\dfrac{2(\alpha+\beta+1)(\beta+2)(1-\sigma)(1+q)}{q(\alpha+\beta+2)(\beta+1)}\right]; \quad \Psi \leq \varDelta_1 \\ \dfrac{2(\alpha+\beta+2)(\alpha+\beta+1)(1-\sigma)\cos\vartheta}{q(1+q)(\beta+2)(\beta+1)}; \qquad \varDelta_1 \leq \Psi \leq \varDelta_2 \\ \dfrac{2(\alpha+\beta+2)(\alpha+\beta+1)(1-\sigma)\cos\vartheta}{q(1+q)(\beta+2)(\beta+1)} \\ \times\left[\Psi\dfrac{2(\alpha+\beta+1)(\beta+2)(1-\sigma)(1+q)}{q(\alpha+\beta+2)(\beta+1)} - \dfrac{2(1-\sigma)}{q} - 1\right]; \quad \Psi \geq \varDelta_2 \end{cases},$$

where

(14) $$\varDelta_1 = \frac{(\alpha+\beta+2)(\beta+1)}{(1+q)(\alpha+\beta+1)(\beta+2)},$$

(15) $$\varDelta_2 = \frac{(1+q-\sigma)(\alpha+\beta+2)(\beta+1)}{(1+q)(1-\sigma)q(\alpha+\beta+1)(\beta+2)}.$$

and Ψ is a real number. All estimates are sharp.

Proof. Let $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$. By Definition 1.1 and (13), we arrive

$$\frac{zD_q\left(\chi^{\alpha}_{\beta,q}f(z)\right)}{\chi^{\alpha}_{\beta,q}f(z)} = P(\Upsilon(z)),$$

that is

$$1+qA_2a_2z+\left\{q(1+q)A_3a_3 - qA_2^2a_2^2\right\}z^2+\cdots = 1+P_1s_1z+\left(P_1s_2+P_2s_1^2\right)z^2+\cdots,$$

where

$$A_2 = \frac{\beta+1}{\alpha+\beta+1},$$

$$A_3 = \frac{(\beta+2)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+1)}.$$

Hence, we arrive

(16) $$a_2 = \frac{P_1}{qA_2}s_1$$

and

$$a_3 = \frac{P_1}{q\,(1+q)\,A_3}\left[s_2 + \left(1+\frac{P_1}{q}\right)s_1^2\right]. \tag{17}$$

Further computations give

$$\begin{aligned}\left|a_3-\Psi a_2^2\right| &\le \frac{|P_1|}{q\,(1+q)\,A_3}\left\{|s_2| + \left|1+\frac{P_1}{q}\left\{1-\Psi\frac{(1+q)\,A_3}{A_2^2}\right\}\right||s_1|^2\right\} \\ &\le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}\left[\left(1-|s_1|^2\right) + \left|1+Qe^{-i\vartheta}\cos\vartheta\right||s_1|^2\right],\end{aligned}$$

where

$$Q = \frac{2(1-\sigma)}{q}\left\{1-\Psi\frac{(1+q)\,A_3}{A_2^2}\right\}.$$

Or, equivalently

$$\left|a_3-\Psi a_2^2\right| \le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}\left[1+\left(\sqrt{1+Q(2+Q)\cos^2\vartheta}-1\right)|s_1|^2\right].$$

For $(x,y)\in[0,1]\times[0,1]$, set

$$\Theta(x,y) = 1+\left(\sqrt{1+Q(2+Q)x^2}-1\right)y^2,$$

where $x=\cos\vartheta$, $y=|\mathbf{c}_1|^2$.

Here, $\Theta(x,y)$ can not take a local maximum at any interior point of $(0,1)\times(0,1)$. Therefore, the maximum comes from the boundary. Since $\Theta(x,0)=1$, $\Theta(0,y)=1$ and $\Theta(1,1)=|1+Q|$, the maximum should be $\Theta(0,0)=1$ or $\Theta(1,1)=|1+Q|$. So,

$$\left|a_3-\Psi a_2^2\right| \le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}\max\left\{1,|1+Q|\right\}. \tag{18}$$

Case 1 If $\Psi\le\mathit{\Delta}_1$, where $\mathit{\Delta}_1$ is given by (14), then $Q\ge 0$ implies $|1+Q|\ge 1$. Now from (18) we find

$$\left|a_3-\Psi a_2^2\right| \le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}\left[1+\frac{2(1-\sigma)}{q}-\Psi\frac{2(1-\sigma)\,(1+q)\,A_3}{qA_2^2}\right].$$

Case 2 If $\Psi\ge\mathit{\Delta}_2$, where $\mathit{\Delta}_2$ is given by (15), then $Q\le -2$. Now from (18) we find

$$\left|a_3-\Psi a_2^2\right| \le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}\left[\Psi\frac{2(1-\sigma)\,(1+q)\,A_3}{qA_2^2}-\frac{2(1-\sigma)}{q}-1\right].$$

Case 3 If $\mathit{\Delta}_1\le\Psi\le\mathit{\Delta}_2$, then $|1+Q|\le 1$. Now from (18) we find

$$\left|a_3-\Psi a_2^2\right| \le \frac{2\,(1-\sigma)\cos\vartheta}{q\,(1+q)\,A_3}.$$

□

By Lemma 4.1, for $\Upsilon(z)=z$ and $\Upsilon(z)=z^2$, the outcomes are sharp.

Theorem 4.3. *Suppose that $f \in \mathcal{A}$ belongs to $\mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$ and Ψ is a complex number. Then*

$$\left|a_3 - \Psi a_2^2\right| \leq \frac{2(\alpha+\beta+2)(\alpha+\beta+1)(1-\sigma)\cos\vartheta}{q(1+q)(\beta+2)(\beta+1)} \max\{1, |\Omega|\},$$

where

$$\Omega = \frac{2e^{-i\vartheta}(1-\sigma)\cos\vartheta}{q}\left\{\Psi\frac{(1+q)(\alpha+\beta+1)(\beta+2)}{(\alpha+\beta+2)(\beta+1)} - 1\right\} - 1. \tag{19}$$

Proof. Suppose $f \in \mathcal{ST}_{\vartheta,q}(\sigma,\alpha,\beta)$. From (16) and (17), we get

$$\left|a_3 - \Psi a_2^2\right| \leq \frac{|P_1|}{q(1+q)A_3}\left|s_2 - \left[\frac{P_1}{q}\left\{\Psi\frac{(1+q)A_3}{A_2^2} - 1\right\} - 1\right]s_1^2\right|$$

$$\leq \frac{2(1-\sigma)\cos\vartheta}{q(1+q)A_3}\left|s_2 - \Omega s_1^2\right|,$$

where Ω is given by (19). The desired inequality follows by applying Lemma 4.1. □

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