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SOME DEFINITE INTEGRALS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract. The main object of this paper is to establish twenty definite integrals involving Struve functions and modified Srtuve functions associated with hypergeometric functions, which are presumably new and not present in the scientific literature.

1. Introduction

In [2], Yurry A. Brychkov (see p.199 (4.7.4.1)) has derived a valuable formula as given below:

$$\int_0^1 \upsilon\ \log \upsilon\ J_0(a\upsilon)\ d\upsilon = -\frac{1}{a^2}\big[J_0(a) - 1\big] \tag{1.1}$$

First kind Bessel function is denoted by $J_\upsilon(y)$, and defined as

$$J_\upsilon(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(k+\upsilon+1)} \left(\frac{y}{2}\right)^{2k+\upsilon} \tag{1.2}$$

where $\Gamma(\varpi)$ is the gamma function.

The first kind of modified Bessel function is defined as

$$I_\varsigma(z) = \iota^{-\varsigma} J_\varsigma(\iota z) = \sum_{i=0}^{\infty} \frac{1}{i!\ \Gamma(i+\varsigma+1)} \left(\frac{z}{2}\right)^{2i+\varsigma} \tag{1.3}$$

Struve functions are solutions of the non-homogeneous Bessel's differential equation:

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-\eta^2)y = \frac{4(\frac{t}{2})^{\eta+1}}{\sqrt{\pi}\ \Gamma(\eta+\frac{1}{2})} \tag{1.4}$$

and are defined as:

$$H_\eta(t) = \frac{2(\frac{t}{2})^\eta}{\Gamma(\eta+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin(t\cos\zeta)\sin^{2\eta}(\zeta)d\zeta \tag{1.5}$$

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Modified Struve functions are defined as:

$$L_\eta(t) = I_{-\eta}(t) - \frac{2(\frac{t}{2})^\eta}{\Gamma(\eta+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\infty \sin(t\mu)(1+\mu^2)^{\eta-\frac{1}{2}}d\mu \qquad (1.6)$$

Generalized hypergeometric function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is a function which can be described in the form of a hypergeometric series, i.e., a series for which the ratio of consecutive terms can be written as follows:

$$\frac{c_{m+1}}{c_m} = \frac{P(m)}{Q(m)} = \frac{(m+a_1)(m+a_2)...(m+a_p)}{(m+b_1)(m+b_2)...(m+b_q)(m+1)}\ z. \qquad (1.7)$$

Where $m+1$ in the denominator is present for documentary causes of notation (see [7], p.12(2.9)), and the developing generalized hypergeometric function is written as under:

$${}_pF_q\left[\begin{array}{lll} a_1,a_2,\cdots,a_p & ; & \\ & & z \\ b_1,b_2,\cdots,b_q & ; & \end{array}\right] = \sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m\cdots(a_p)_m z^m}{(b_1)_m(b_2)_m\cdots(b_q)_m m!} \qquad (1.8)$$

where the parameters $b_1,b_2,\cdots,b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$ (see [8], p.156(3)).

The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2, q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (see [6], pp.123-162).

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial, falling sequential product, or lower factorial) is defined as the polynomial (see, [10], p.8) and its mathematical expression is given below:

$$(z)_n = z(z-1)(z-2).......(z-n+1) = \prod_{k=1}^{n}(z-k+1) = \prod_{k=0}^{n-1}(z-k). \qquad (1.9)$$

2. Main Formulae of the Integration

In this section, we establish twenty definite integrals involving Struve functions and modified Srtuve functions associated with hypergeometric functions, as follows:

$$\int_0^1 v^1\ \log v\ H_1(av)\ dv = \frac{1}{2\pi}\left[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;-\frac{a^2}{4}\right)-1\right], \qquad (2.1)$$

$$\int_0^1 v^1\ \log v\ L_1(av)\ dv = -\frac{1}{2\pi}\left[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;\frac{a^2}{4}\right)-1\right], \qquad (2.2)$$

$$\int_0^1 v^3 \ \log v \ H_1(av) \ dv = -\frac{1}{8\pi a^2}\left[-4 \ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)+a^2+4\right], \tag{2.3}$$

$$\int_0^1 v^3 \ \log v \ L_1(av) \ dv = \frac{1}{8\pi a^2}\left[4 \ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4}\right)+a^2-4\right], \tag{2.4}$$

$$\int_0^1 v^5 \ \log v \ H_1(av) \ dv = -\frac{1}{72\pi a^4}\left[108 \ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\right)+4a^4+9a^2-108\right], \tag{2.5}$$

$$\int_0^1 v^5 \ \log v \ L_1(av) \ dv = \frac{1}{72\pi a^4}\left[108 \ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\right)+4a^4-9a^2-108\right], \tag{2.6}$$

$$\int_0^1 v^0 \ \log v \ H_1(av) \ dv = \frac{1}{3375\pi}\left[2a^2\left\{3a^2 \ {}_2F_3\left(1,\frac{5}{2};\frac{7}{2},\frac{7}{2},\frac{7}{2};-\frac{a^2}{4}\right)-125\right\}\right], \tag{2.7}$$

$$\int_0^1 v^0 \ \log v \ L_1(av) \ dv = -\frac{1}{3375\pi}\left[2a^2\left\{3a^2 \ {}_2F_3\left(1,\frac{5}{2};\frac{7}{2},\frac{7}{2},\frac{7}{2};\frac{a^2}{4}\right)+125\right\}\right], \tag{2.8}$$

$$\int_0^1 v^2 \ \log v \ H_1(av) \ dv = \frac{1}{11025\pi}\left[2a^2\left\{5a^2 \ {}_2F_3\left(1,\frac{7}{2};\frac{5}{2},\frac{9}{2},\frac{9}{2};-\frac{a^2}{4}\right)-147\right\}\right], \tag{2.9}$$

$$\int_0^1 v^2 \ \log v \ L_1(av) \ dv = -\frac{1}{11025\pi}\left[2a^2\left\{5a^2 \ {}_2F_3\left(1,\frac{7}{2};\frac{5}{2},\frac{9}{2},\frac{9}{2};\frac{a^2}{4}\right)+147\right\}\right], \tag{2.10}$$

$$\int_0^1 v^4 \ \log v \ H_1(av) \ dv = \frac{1}{178605\pi}\left[2a^2\left\{49a^2 \ {}_3F_4\left(1,\frac{9}{2},\frac{9}{2};\frac{5}{2},\frac{7}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\right)-1215\right\}\right], \tag{2.11}$$

$$\int_0^1 v^4 \ \log v \ H_1(av) \ dv = -\frac{1}{178605\pi}\left[2a^2\left\{49a^2 \ {}_3F_4\left(1,\frac{9}{2},\frac{9}{2};\frac{5}{2},\frac{7}{2},\frac{11}{2},\frac{11}{2};\frac{a^2}{4}\right)+1215\right\}\right], \tag{2.12}$$

$$\int_0^1 v^7 \ \log v \ H_1(av) \ dv = -\frac{1}{288\pi a^6}\left[-6480 \ {}_3F_4\left(1,1,1;-\frac{5}{2},-\frac{3}{2},2,2;-\frac{a^2}{4}\right)+9a^6+16a^4-108a^2+6480\right], \tag{2.13}$$

$$\int_0^1 v^7\ \log v\, L_1(av)\, dv = -\frac{1}{288\pi a^6}\left[-6480\, {}_3F_4\left(1,1,1;-\frac{5}{2},-\frac{3}{2},2,2;\frac{a^2}{4}\right)-9a^6+16a^4+108a^2+6480\right], \tag{2.14}$$

$$\int_0^1 v^{11}\ \log v\, H_1(av)\, dv = -\frac{1}{7200\pi a^{10}}\left[-357210000\, {}_3F_4\left(1,1,1;-\frac{9}{2},-\frac{7}{2},2,2;-\frac{a^2}{4}\right)+\right.$$
$$\left.+100a^{10}+144a^8-675a^6+18000a^4-1417500a^2+357210000\right], \tag{2.15}$$

$$\int_0^1 v^{11}\ \log v\, L_1(av)\, dv = -\frac{1}{7200\pi a^{10}}\left[-357210000\, {}_3F_4\left(1,1,1;-\frac{9}{2},-\frac{7}{2},2,2;\frac{a^2}{4}\right)-\right.$$
$$\left.-100a^{10}+144a^8+675a^6+18000a^4+1417500a^2+357210000\right], \tag{2.16}$$

$$\int_0^1 v^{99}\ \log v\ H_1(av)\ dv = -\frac{1}{15606\pi}\left[a^2\ {}_3F_4\left(1,51,51;\frac{3}{2},\frac{5}{2},52,52;-\frac{a^2}{4}\right)\right], \tag{2.17}$$

$$\int_0^1 v^{99}\ \log v\ L_1(av)\ dv = -\frac{1}{15606\pi}\left[a^2\ {}_3F_4\left(1,51,51;\frac{3}{2},\frac{5}{2},52,52;\frac{a^2}{4}\right)\right], \tag{2.18}$$

$$\int_0^1 v^{4445}\ \log v\, H_1(av)\, dv = -\frac{1}{29677056\pi}\left[a^2\, {}_3F_4\left(1,2224,2224;\frac{3}{2},\frac{5}{2},2225,2225;-\frac{a^2}{4}\right)\right], \tag{2.19}$$

and

$$\int_0^1 v^{4445}\ \log v\, L_1(av)\, dv = -\frac{1}{29677056\pi}\left[a^2\, {}_3F_4\left(1,2224,2224;\frac{3}{2},\frac{5}{2},2225,2225;\frac{a^2}{4}\right)\right], \tag{2.20}$$

provided that each member of the assertions (2.1) to (2.20) exists.

Proof. We first prove our assertion (2.1). In order to prove the assertion (2.1), we make use the properties of the Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^1\ \log v\ H_1(av)\ dv = \frac{1}{2\pi}\left[v^2\left\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;-\frac{a^2v^2}{4}\right)-\right.\right.$$
$$\left.\left.-2\log v\ {}_2F_3\left(1,1;\frac{1}{2},\frac{3}{2},2;-\frac{a^2v^2}{4}\right)+2\log v-1\right\}\right]_0^1$$
$$=\frac{1}{2\pi}\left[\left\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;-\frac{a^2}{4}\right)-1\right\}-\left\{0\right\}\right]$$

$$= \frac{1}{2\pi}\left[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;-\frac{a^2}{4}\right)-1\right].$$

Hence we have established the first assertion (2.1) of the theorem.

Next, to prove our second assertion (2.2), we make use the properties of the modified Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^1 \ \log v \ L_1(av)\ dv = -\frac{1}{2\pi}\left[v^2\left\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;\frac{a^2v^2}{4}\right)-\right.\right.$$

$$\left.\left.-2\log v\ {}_2F_3\left(1,1;\frac{1}{2},\frac{3}{2},2;\frac{a^2v^2}{4}\right)+2\log v-1\right\}\right]_0^1$$

$$= -\frac{1}{2\pi}\left[\left\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;\frac{a^2}{4}\right)-1\right\}-\left\{0\right\}\right]$$

$$= -\frac{1}{2\pi}\left[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{3}{2},2,2;\frac{a^2}{4}\right)-1\right].$$

Hence we have established the second assertion (2.2) of the theorem.

Further, to prove our third assertion (2.3), we make use the properties of the Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^3 \ \log v \ H_1(av)\ dv = \frac{1}{8\pi a^2}\ v^2\left[4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2v^2}{4}\right)-\right.$$

$$\left.-8\log v\ {}_2F_3\left(1,1;-\frac{1}{2},\frac{1}{2},2;-\frac{a^2v^2}{4}\right)-a^2v^2+4a^2v^2\log v+8\log v-4\right]_0^1$$

$$= \frac{1}{8\pi a^2}\left[\left\{4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)-a^2-4\right\}-\left\{0\right\}\right]$$

$$= \frac{1}{8\pi a^2}\left[4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)-a^2-4\right]$$

$$= -\frac{1}{8\pi a^2}\left[-4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)+a^2+4\right]$$

Hence we have established our third assertion (2.3) of the theorem.

Now we proceed, to prove our fourth assertion (2.4), we make use the properties of the modified Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^3 \ \log v \ L_1(av)\ dv = \frac{1}{8\pi a^2}\ v^2\left[4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;\frac{a^2v^2}{4}\right)-\right.$$

$$\left.-8\log v\ {}_2F_3\left(1,1;-\frac{1}{2},\frac{1}{2},2;\frac{a^2v^2}{4}\right)+a^2v^2-4a^2v^2\log v+8\log v-4\right]_0^1$$

$$= \frac{1}{8\pi a^2}\left[\left\{4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4}\right)+a^2-4\right\}-\left\{0\right\}\right]$$

$$= \frac{1}{8\pi a^2}\left[4\ {}_3F_4\left(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4}\right)+a^2-4\right]$$

Hence we have established our fourth assertion (2.4) of the theorem.

Again we proceed, to prove our fifth assertion (2.5), we make use the properties of the Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^5\ \log v\ H_1(av)\ dv = \frac{1}{72\pi a^4}\ v^2\Bigg[-108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;-\frac{a^2v^2}{4}\right)+$$

$$+216\log v\ {}_2F_3\left(1,1;-\frac{3}{2},-\frac{1}{2},2;-\frac{a^2v^2}{4}\right)-4a^4v^4+24a^4v^4\log v-$$

$$-9a^2v^2+36a^2v^2\log v-216\log v+108\Bigg]_0^1$$

$$= \frac{1}{72\pi a^4}\left[\left\{-108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\right)-4a^4-9a^2+108\right\}-\left\{0\right\}\right]$$

$$= -\frac{1}{72\pi a^4}\left[108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\right)+4a^4+9a^2-108\right]$$

Hence we have established our fifth assertion (2.5) of the theorem.

Further, to prove our sixth assertion (2.6), we make use the properties of the modified Struve function concerning hypergeometric functions, definite integral, and apply little algebra, we have

$$\int_0^1 v^5\ \log v\ L_1(av)\ dv = \frac{1}{72\pi a^4}\ v^2\Bigg[108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;\frac{a^2v^2}{4}\right)-$$

$$-216\log v\ {}_2F_3\left(1,1;-\frac{3}{2},-\frac{1}{2},2;-\frac{a^2v^2}{4}\right)+4a^4v^4-24a^4v^4\log v-$$

$$-9a^2v^2+36a^2v^2\log v+216\log v-108\Bigg]_0^1$$

$$= \frac{1}{72\pi a^4}\left[\left\{108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\right)+4a^4-9a^2-108\right\}-\left\{0\right\}\right]$$

$$= \frac{1}{72\pi a^4}\left[108\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\right)+4a^4-9a^2-108\right]$$

Hence we have established our sixth assertion (2.6) of the theorem.

The analogous proofs of the assertions (2.7) - (2.30) may be left as an exercise for the interested readers. We thus have completed our proof of the above theorem. □

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